Journal of Engineering and Applied Sciences 14 (Special Issue 8): 10452-10458, 2019
ISSN: 1816-949X

Differential Subordination and Superordination for Univalent Meromorphic Functions Involving Cho-Kwon-Srivastava Operator

Asraa Abdul Jaleel Husien
Technical Institute of Al-Diwaniya, Al-Furat Al-Awsat Technical University, Diwaniya, Iraq

Abstract: Using a Cho-Kwon-Srivastava operator, we introduce and study some differential subordination and superordination results for univalent meromorphic functions in the punctured open unit disk U^*.

Key words: Univalent meromorphic function, subordination, super ordination, Cho-Kwon-Srivastava operator, admissible functions, investigating classes

INTRODUCTION

Let, D(U) be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}: |z|<1\}$ and D[σ, n] be the subclass of D(U) consisting of functions of the form $f(z)=\sigma+\sigma_n z^n+\sigma_{n+1}z^{n+1}+....$, with D_0 = D[0, 1] and D_1 = D[1,1]. Let f and g members of D(U). The function f is said to be subordinate to g in U written $f\prec g$ or $f(z)\prec g(z)$ ($z\in U$), if there exists a Schwarz function w(z) is analytic in U with w(0) = 0 and |w(z)|<1 in U such that $f(z) = g(w(z))$ ($z\in U$) (Bulboaca, 2005; Miller and Mocanu, 2000). In particular, if the function g is univalent in U, then, we have the following equivalence:

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \text{ and } f(U)\subset g(U)$$

In order to prove the results, we shall need the following definition and theorem.

Definition (1.1); Miller and Mocanu (2000): Denote by Q the set of all functions that are analytic and injective on $\bar{U}\backslash E(q)$ where:

$$E(q)=\left\{\zeta\in\partial U: \lim_{z\to\zeta} q(z)=\infty\right\}$$

And are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(q)$. Further let the subclass of Q for which q(0) = σ be denoted by Q(σ), $Q(0)\equiv Q_0$ and $Q1(1)\equiv Q_1$:

Definition (1.2); Miller and Mocanu (2000): Let Ω be a set in $\mathbb{C}\in Q$ and let n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z)\notin\Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$ and:

$$\operatorname{Re}\left\{\frac{t}{s}+1\right\}\geq k\operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}$$

$z\in U$, $\zeta\in\partial U\backslash E(q)$, $k\geq n$. We write $\psi_1[\Omega, q]$ as $\psi[\Omega,q]$. In particular when $q(z)=M\frac{Mz+\sigma}{M+\bar{\sigma}z}$ with M>0 and |σ|<M, then $q(U) = U_M = \{w:|w|<M\}$, q(0) = σ, E(q) = φ and $q\in Q(\sigma)$. In this case, we set $\psi_n[\Omega, M, \sigma] = \psi_n[\Omega, q]$ and in the special case when $\Omega=U_M$ the class is simply denoted by $\psi_n[M, \sigma]$.

Definition (1.3); Miller and Mocanu (2003): Let Ω be a set in $\mathbb{C}$, $q\in D[\sigma, n]$ with $q'(z) = 0$. The class of admissible functions $\psi'_n[\Omega, q]$ consists of those functions $\psi:\mathbb{C}^3\times\bar{U}\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta)\in\Omega$ whenever $r=q(z), s=\frac{zq'(z)}{m}$ and:

$$\operatorname{Re}\left\{\frac{t}{s}+1\right\}\leq\frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\}$$

$z\in U$, $\zeta\in\partial U$, $m\geq n\geq 1$. We write $\psi'_1[\Omega, q]$ as $\psi'[\Omega, q]$.

Theorem (1.4); Miller and Mocanu (2000): Let $\psi\in\Psi_n[\Omega,q]$ with q(0) = σ. If the analytic function $p(z) = \sigma+\sigma_n z^n+\sigma_{n+1}z^{n+1}+...(z\in U)$ satisfies the following inclusion relationship:

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right)\in\Omega$$

Then:

$$p(z)\prec q(z)(z\in U)$$

Theorem (1.5); Miller and Mocanu (2003): Let $\psi\in\Psi'_n[\Omega,q]$ with q(0) = σ. If $p(z)\in Q(\sigma)$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ is univalent U then:

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z)\}$$

Implies:

$$p(z) \prec q(z)(z \in U)$$

Let $\sum$ denote the class of meromorphic functions f (z) normalized by:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \sigma_n z^n \quad (1)$$

Which are analytic in the punctred unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. For function $f(z) \in \sum$ given by (Eq. 1) and $g(z) \in \sum$ defined by:

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (2)$$

The Hadamard product (or convolution) of f (z) and g (z) is given by:

$$(f * g)(z) = (g * f)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \sigma_n a_n z^n \quad (3)$$

Let us define the function $\tilde{\phi}(\alpha, \beta : z)$ by:

$$\tilde{\phi}(\alpha, \beta : z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left| \frac{(\alpha)n+1}{(\beta)n+1} \right| z^n \quad (4)$$

For $\beta \neq 0, -1, 12, \ldots$ and $\alpha \in \mathbb{C}/\{0\}$ where $\lambda_n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that:

$$\tilde{\phi}(\alpha, \beta : z) = \frac{1}{z} {}_2F_1(1, \alpha, \beta; z)$$

Where:

$${}_2F_1(1, \alpha, \beta; z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Is the well-known Gaussian hypergeometric function. Let us put:

$$q_{\lambda,\mu}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} z^n, (\lambda > 0, \geq 0)$$

Corresponding to the functions $\tilde{\phi}(\alpha, \beta; z)$ and $q_{\lambda,\mu}(z)$ and using the Hadamard product for $f(z \in \sum)$ we defined a linear operator $L(\alpha, \beta, \lambda, \mu)$ (Ghanim and Darus, 2011a, b) on $\sum$ by:

$$L(\alpha, \beta, \lambda, \mu) f(z) = \left(f(z) * \tilde{\phi}(\alpha, \beta : z) * q_{\lambda,\mu}(z) \right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left| \frac{(\alpha)n+1}{(\beta)n+1} \right| \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} \sigma_n z^n \quad (5)$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava (2002, 2003); Liu (2001; 2003); Liu and Srivastava (2001; 2004a, b); Cho and Kim (2007). For a function $f(z) \in L(\alpha, \beta, \lambda, \mu)$ we define:

$$I_{\alpha,\beta,\lambda}^{\mu,0} = L(\alpha, \beta, \lambda, \mu) f(z)$$

And for k = 1, 2 , ..., :

$$I_{\alpha,\beta,\lambda}^{\mu,0} = z\left(I^{K-1} L(\alpha, \beta, \lambda, \mu) f(z) \right)' + \frac{2}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left| \frac{(\alpha)n+1}{(\beta)n+1} \right| \left(\frac{\lambda}{n+1+\lambda} \right)^{\mu} \sigma_n z^n \quad (6)$$

Not that if n = β, k = 0 the operator $I_{\alpha,n,\lambda}^{\mu,0}$ have been introduced by Cho *et al.* (2004) for $\mu \in N_0 = N \cup 0$. It was known that the definition of the operator $I_{\alpha,n,\lambda}^{\mu,0}$ was motivated essentially by Choi *et al.* (2002) for analytic functions which includes a simpler integral operator studied earlier by Noor (1999) and others Liu (2001); Liu and Noor (2002). Note also the operator $I_{\alpha,\beta}^{0,k}$ have been recently introduced and studied by Ghanim and Darus (2010a, b; 2011). To our best knowledge, the recent work regarding $I_{\alpha,n,\lambda}^{\mu,0}$ was charmingly studied by Piejko and Sokol (2008). In the same direction, we will study for the operator $I_{\alpha,\beta,\lambda}^{\mu,k}$ given in Eq. 6. Now, it follows from Eq. 5 and 6 that:

$$z\left(I_{\alpha,\beta,\lambda}^{\mu+1,k} f(z) \right)' = \lambda I_{\alpha,\beta,\lambda}^{\mu,k} f(z) - (\lambda+1) I_{\alpha,\beta,\lambda}^{\mu+1,k} f(z) \quad (7)$$

$$z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z) \right)' = \alpha I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z) - (\alpha+1) I_{\alpha,\beta,\lambda}^{\mu+1,k} f(z) \quad (8)$$

SUBORDINATION RESULTS INVOLVING THE OPERATOR

Definition (2.1): Let Ω be a set in $\mathbb{C}$, $q(z) \in Q_1 \cap D[q(0), 1]$. The class of admissible functions $\phi_n[\Omega, q]$ consists of those function $\phi: \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

Whenever:

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta)+(\lambda+1)q(\zeta)}{\lambda}$$
$$\operatorname{Re}\left\{\frac{\lambda^2 w\text{-}(\lambda+1)^2 u}{\lambda v\text{-}(\lambda+1)u}\text{-}2\lambda\right\} \geq k\operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}$$
$$z \in U, \zeta \in \partial U \setminus E(q), k \geq 1$$

Theorem (2.2): Let $\phi \in \phi_n[\Omega, q]$. If $f(z) \in A$ satisfies:

$$\left\{\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right) : z \in U\right\} \subset \Omega \quad (9)$$

Then:

$$I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \prec q(z)$$

Proof: Define the analytic function J(z) in U by:

$$J(z) = I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \quad (10)$$

In view of relation:

$$z\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)\right)' = \lambda I^{\mu,k}_{\alpha,\beta,\lambda}f(z)\text{-}(\lambda+1)I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \quad (11)$$

From Eq. 10 we have:

$$I^{\mu,k}_{\alpha,\beta,\lambda}f(z) = \frac{1}{\lambda}\left[zJ'(z)+(\lambda+1)J(z)\right] \quad (12)$$

Further, a simple computation shows that:

$$I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z) = \frac{1}{\lambda^2}\left[z^2J''(z)+(2\lambda+3)zJ'(z)+(\lambda+1)^2 J(z)\right] \quad (13)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$by:

$$u(r,s,t) = r,\ v(r,s,t) = \frac{1}{\lambda}\left[s+(\lambda+1)r\right]$$

$$w(r,s,t) = \frac{1}{\lambda^2}\left[t+(2\lambda+3)s+(\lambda+1)^2 r\right] \quad (14)$$

Let:

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(\begin{array}{l} r, \frac{1}{\lambda}\left[s+(\lambda+1)r\right], \\ \frac{1}{\lambda^2}\left[t+(2\lambda+3)s+(\lambda+1)^2 r\right]; z \end{array}\right) \quad (15)$$

The proof shall make use of theorem (Eq. 4). Using (Eq. 10, 12 and 13), from (Eq. 15), we obtain:

$$\begin{array}{l}\psi\left(J(z), zJ'(z), z^2J''(z); z\right) = \\ \phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right)\end{array} \quad (16)$$

Hence, (19) becomes:

$$\begin{array}{l}\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right) = \\ \psi\left(J(z), zJ'(z), z^2J''(z); z\right) \in \Omega\end{array} \quad (17)$$

Note that:

$$\frac{t}{s}+1 = \frac{\lambda^2 w\text{-}(\lambda+1)^2 u}{\lambda v\text{-}(\lambda+1)u}\text{-}2\lambda$$

And since, the admissibility condition for $\phi \in \Phi_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in definition (1.2), hence, $\psi \in \Psi[\Omega, q]$ and by theorem (1.4) then $J(z) \prec q(z)$ or $I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \prec q(z)$.

Theorem (2.3): Let $\phi \in \Phi_n[h, q]$ with q(0) = 1. If f(z)∈A satisfies:

$$\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right) \prec h(z) \quad (18)$$

Then:

$$I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \prec q(z)(z \in U)$$

Our next result is an extension of theorem (2.2) to the case the behavior of q (z) on ∂U is not known.

Corollary (2.4): Let $\Omega \subset \mathbb{C}$ and let q(z) be univalent in U with q(0) = 1. Let $\phi \in \Phi_n[\Omega, q_p]$ for some p∈(0,1) where $q_p(z) = q(pz)$. If f(z)∈A and:

$$\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right) \in \Omega$$

Then:

$$I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \prec q(z)(z \in U)$$

Proof: By theorem (2.2), we get $I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z) \prec q(z)$. The result is now deduced from the following subordination relationship:

$$q_p(z) \prec q(z)(z \in U)$$

In the particular case q(z) = Mz, M>0 and in view of definition (2.1), the class of admissible functions $\Phi_n[\Omega, q]$ denoted by $\Phi_n[\Omega, M]$ is described below.

Definition (2.5): Let Ω be a set in $\mathbb{C}$ and M>0. The class of admissible functions $\Phi_n[\Omega, M]$ consists of those functions $\varphi: \mathbb{C}^3 \times U \to \mathbb{C}$ such that:

$$\phi\begin{pmatrix} Me^{i\theta}, \frac{1}{\lambda}[k+\lambda+1]Me^{i\theta}, \\ \frac{1}{\lambda^2}\left[1+[k(2\lambda+3)+(\lambda+1)^2\right]Me^{i\theta}; z \end{pmatrix} \notin \Omega \quad (19)$$

Whenever $z\in U$, $k\geq 1$, $\mathrm{Re}(Le^{-i\theta})\geq(k-1)kM$ and $\theta\in R$:

Corollary (2.6): Let $\phi\in\Phi_n[\Omega, M]$. If $f(z)\in A$ satisfy the following inclusion relationship:

$$\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right)\in\Omega$$

Then:

$$I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)\prec Mz(z\in U)$$

In the special case $\Omega = q(U) = \{w:|w|<M\}$ the class $\Phi_n[\Omega, M]$ is simply denoted by $\Phi_n[M]$.

Corollary (2.7): Let $\phi\in\Phi_n[M]$. If $f(z)\in A$ satisfies the following inclusion relationship:

$$\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z), I^{\mu,k}_{\alpha,\beta,\lambda}f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z); z\right)<M$$

Then:

$$\left|I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)\right|<M$$

Definition (2.8): Let Ω be a set in $\mathbb{C}$, $q(z)\in Q\cap D\,[q(0), 1]$. The class of admissible functions $\Phi_{n,1}[\Omega, q]$ consists of those functions $\phi:\mathbb{C}^3\times U\rightarrow\mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z)\notin\Omega$ whenever:

$$u=q(\zeta), v=\frac{1}{\lambda}\left\{\frac{k\zeta q'(\zeta)}{q(\zeta)}+\lambda q(\zeta)\right\}, (q(\zeta)\neq 0)$$
$$\mathrm{Re}\left\{\frac{v\left[2(\lambda+1)+\lambda(w-v)\right]}{v-u}+\lambda(v-2u)\right\}\geq k\,\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}$$
$$z\in U, \zeta\in\partial U\backslash E(q),\ \ k\geq 1$$

Theorem (2.9): Let $\phi\in\Phi_{n,1}[\Omega, q]$. If $f(z)\in A$ satisfies:

$$\left\{\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}; z\right): z\in U\right\}\subset\Omega \quad (20)$$

Then:

$$\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}\prec q(z)$$

Proof: Define the analytic function J(z) in U by:

$$J(z)=\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}\prec q(z) \quad (21)$$

Then, by using (1.7), we get:

$$\frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}=\frac{1}{\lambda}\left\{\frac{zJ'(z)}{J(z)}+\lambda J(z)\right\} \quad (22)$$

Differentiating logarithmically (Eq. 22) and further computations show that:

$$\frac{I^{\mu-2,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}=\frac{1}{\lambda}\left\{\begin{array}{l} -2(\lambda+1)\frac{zJ'(z)}{J(z)}+\lambda J(z)+\dfrac{1}{\frac{zJ'(z)}{J(z)}+\lambda J(z)} \\ \left[\frac{zJ'(z)}{J(z)}+\frac{z^2J''(z)}{J(z)}-\left(\frac{zJ'(z)}{J(z)}\right)^2+\lambda zJ'(z)\right]\end{array}\right\} \quad (23)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by:

$$u(r,s,t)=r,\quad v(r,s,t)=\frac{1}{\lambda}\left\{\frac{s}{r}+\lambda r\right\} w(r,s,t)=$$
$$\frac{1}{\lambda}\left\{-2(\lambda+1)+\frac{s}{r}+\lambda r+\frac{1}{\frac{s}{r}+\lambda r}\left[\frac{s}{r}+\frac{t}{r}-\left(\frac{s}{r}\right)^2+\lambda s\right]\right\} \quad (24)$$

Let:

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$

$$\phi\left(r, \frac{1}{\lambda}\left\{\frac{s}{r}+\lambda r\right\}, \frac{1}{\lambda}\left\{\begin{array}{l}-2(\lambda+1)+\frac{s}{r}+\lambda r+ \\ \dfrac{1}{\frac{s}{r}+\lambda r}\left[\frac{s}{r}+\frac{t}{r}-\left(\frac{s}{r}\right)^2+\lambda s\right]\end{array}\right\}; z\right) \quad (25)$$

Using (Eq. 21, 22 and 23), from (Eq. 25), it follows that:

$$\psi\left(J(z), zJ'(z), z^2J''(z); z\right)=$$
$$\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}; z\right)$$

Hence, (Eq. 20) $\psi(J(z), zJ'(z), z^2J''(z); z)\in\Omega$. The proof is complete if it can be shown that the admissibility condition for $\phi\in\Phi_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in definition (1.2). For this purpose, note that:

$$\frac{t}{s}+1=\frac{v\left[2(\lambda+1)+\lambda(w-v)\right]}{v-u}+\lambda(v-2u)$$

Hence, $\psi\in\Psi[\Omega, q]$ and by theorem (1.4), we have $J(z)\prec q(z)$ or:

$$\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}\prec q(z)$$

In the case $\Omega\neq\mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω, the class $\Phi_{n,1}[h(U), q]$ is written as $\Phi_{n,1}[h, q]$. The following result is an immediate consequence of theorem (2.9).

Theorem (2.10): Let $\phi\in\Phi_{n,1}[h(U),q]$ with $q(0) = 1$, if $f(z)\in A$ satisfies the following inclusion relationship:

$$\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)},\frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)},\frac{I^{\mu-2,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)};z\right)\prec h(z)$$

Then:

$$\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}\prec q(z)$$

In the particular case $q(z) = Mz$, $M>0$. The class of admissible function $\Phi_{n,1}[\Omega, q]$, denoted by $\Phi_{n,1}[\Omega, M]$ is described below.

Definition (2.11): Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_{n,1}[\Omega, M]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ such that:

$$\phi\left(\begin{matrix}Me^{i\theta},\frac{1}{\lambda}\{k+\lambda Me^{i\theta}\},\frac{1}{\lambda}\\ \left\{\begin{matrix}-2(\lambda+1)+k+\lambda Me^{i\theta}+\frac{1}{kM+\lambda M^2e^{i\theta}}\\ \left[Le^{-i\theta}+kM+\lambda kM-k^2M\right]\end{matrix}\right\};Z\end{matrix}\right)\notin\Omega \qquad (26)$$

Whenever and.

Corollary (2.12): Let $\phi\in\Phi_{n,1}[\Omega,M]$. If $f(z)\in A$ satisfies:

$$\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)},\frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)},\frac{I^{\mu-2,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)};z\right)\in\Omega$$

Then:

$$\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}\prec Mz(z\in U)$$

In the special case $\Omega = q(U) = \{w:|q|<M\}$, the class $\Phi_{n,1}[\Omega, M]$ is defined by $\Phi_{n,1}[M]$ and corollary (2.12) takes follows for:

Corollary (2.13): Let $\phi\in\Phi_{n,1}[M]$. If $f(z)\in A$ satisfies:

$$\left|\phi\left(\frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)},\frac{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)},\frac{I^{\mu-2,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z)};z\right)\right|<M$$

Then:

$$\left|\frac{I^{\mu,k}_{\alpha,\beta,\lambda}f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)}\right|<M$$

SUPERORDINATION RESULTS INVOLVING THE OPERATOR $I^{\mu,k}_{\alpha,\beta,\lambda}f(z)$

In this section we obtain differential super ordination for the operator $I^{\mu,k}_{\alpha,\beta,\lambda}f(z)$. For this purpose the class of admissible functions is given in the following definition.

Definition (3.1): Let Ω be a set in $\mathbb{C}$, $q(z)\in D[q(0), 1]$ with $zq'(z)\neq0$. The class of admissible functions $\Phi'_n[\Omega, q]$ consists of those function $\phi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u,v,w;\zeta)\in\Omega$$

Whenever:

$$u=q(z),\quad v=\frac{zq'(z)+m(\lambda+1)q(z)}{m\lambda}$$

$$\operatorname{Re}\left\{\frac{\lambda^2w-(\lambda+1)^2u}{\lambda v-(\lambda+1)u}\right\}\leq\frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\}$$

$$z\in U,\zeta\in\partial U,m\geq1$$

Theorem (3.2): Let $\phi\in\Phi'_n[\Omega,q]$. If $f(z)\in A$, $I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)\in Q_1$ and $\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z),I^{\mu,k}_{\alpha,\beta,\lambda}f(z),I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z);z\right)$ is univalent in then:

$$\Omega\subset\left\{\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z),I^{\mu,k}_{\alpha,\beta,\lambda}f(z),I^{\mu-1,k}_{\alpha,\beta,\lambda}f(z);z\right):z\in U\right\} \qquad (27)$$

Implies:

$$q(z)\prec I^{\mu+1,k}_{\alpha,\beta,\lambda}f(z)$$

Proof : Let $J(z)$ define by (2.2) and $\psi(z)$ define by (2.8). Since, $\phi\in\Phi'_n[\Omega,q]$ from (2.2) and (3.1) we have:

$$\Omega\subset\left\{\psi\left(J(z),zJ'(z),z^2J''(z);z\right):z\in U\right\}$$

From (2.7), we see that the admissibility condition for $\phi\in\Phi'_n[\Omega,q]$ is equivalent to the admissibility condition for

ψ as given in definition (1.2). Hence, $\psi \in \Psi'[\Omega, q]$ and by and by theorem (1.4), we have $q(z) \prec J(z)$ or $q(z) \prec I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)$.

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω, the class $\Phi'_{n,1}[h(U), q]$ is written as $\Phi'_{n,1}[h, q]$. The following result is an immediate consequence of theorem (3.2).

Theorem (3.3): Let $q(z) \in D$, $h(z)$ is analytic on U and $\phi \in \Phi'_n[\Omega, q]$. If $f(z) \in A, I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z) \in Q_1$ and:

$$\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z), I^{\mu,k}_{\alpha,\beta,\lambda} f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z); z\right)$$

Is univalent in then:

$$h(z) \prec \phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z), I^{\mu,k}_{\alpha,\beta,\lambda} f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z); z\right) \quad (28)$$

Implies:

$$q(z) \prec I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)$$

Combining theorem (2.3) and (3.3), we obtain the following sandwich theorem.

Corollary (3.4): Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $h_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_n[h_2, q_2] \cap \Phi_n[h_1, q_1]$. If $f(z) \in A, I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z) \in D \cap Q_1$ and:

$$\phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z), I^{\mu,k}_{\alpha,\beta,\lambda} f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z); z\right)$$

Is univalent in U then:

$$h_1(z) \prec \phi\left(I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z), I^{\mu,k}_{\alpha,\beta,\lambda} f(z), I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z); z\right) \prec h_2(z)$$

Implies:

$$q_1(z) \prec I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z) \prec q_2(z)$$

Definition (3.5): Let Ω be a set in $\mathbb{C}$, $q(z) \in D[q(0), 1]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{n,1}[\Omega, q]$ consists of those function $\phi: \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \notin \Omega$$

Whenever:

$$u = q(z),\ v = \frac{1}{\lambda}\left\{\frac{kzq'(z)}{mq(z)} + \lambda q(z)\right\}, (q(z) \neq 0)$$

$$\operatorname{Re}\left\{\frac{v[2(\lambda+1)+\lambda(w-v)]}{v-u} + \lambda(v-2u)\right\} \le \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\}$$

$$z \in U,\ \zeta \in \partial U \setminus E(q),\ m \ge 1$$

Now, we will give the dual result of theorem (2.9) for differential superordination.

Theorem (3.6): Let $\phi \in \Phi_{n,1}[\Omega, q]$. If $f(z) \in A, I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z) \in Q_1$ and:

$$\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}; z\right)$$

Is univalent in U, then:

$$\Omega \subset \left\{\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}; z\right) : z \in U\right\} \quad (29)$$

Implies:

$$q(z) \prec \frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}$$

Proof: Let $J(z)$ define by (2.13) and $\psi(z)$ define by. Since, $\phi \in \Phi'_{n,1}[\Omega, q]$ from (Eq. 25) and (29) we have:

$$\Omega \subset \left\{\psi\left(J(z), zJ'(z), z^2 J''(z); z\right) : z \in U\right\}$$

From (Eq. 25), we see that the admissibility condition for $\phi \in \Phi'_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in definition (1.3). Hence, $\psi \in \Psi'[\Omega, q]$ and by and by theorem (1.5), we have $q(z) \prec J(z)$ or:

$$q(z) \prec \frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}$$

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω, the class $\Phi'_{n,1}[h(U), q]$ is written as $\Phi'_{n,1}[h, q]$. The following result is an immediate consequence of theorem (3.6).

Theorem (3.7): Let $q(z) \in D$, $h(z)$ is analytic on U and $\phi \in \Phi'_{n,1}[\Omega, q]$. If $f(z) \in A, I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z) \in Q_1$ and:

$$\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}; z\right)$$

Is univalent in U, then:

$$h(z) \prec \phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}; z\right) \quad (30)$$

Implies:

$$q(z) \prec \frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}$$

Combining theorem (2.10) and (3.7), we obtain the following sandwich-type theorem.

Corollary (3.8): Let $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z)\in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{n,1}[h_2, q_2] \cap \Phi_{n,1}[h_1, q_1]$. If $f(z) \in A, \frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)} \in D \cap Q_1$ and:

$$\phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}; z\right)$$

Is univalent in U then:

$$h_1(z) \prec \phi\left(\frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}, \frac{I^{\mu-2,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu-1,k}_{\alpha,\beta,\lambda} f(z)}; z\right) \prec h_2(z)$$

Implies:

$$q_1(z) \prec \frac{I^{\mu,k}_{\alpha,\beta,\lambda} f(z)}{I^{\mu+1,k}_{\alpha,\beta,\lambda} f(z)} \prec q_2(z)$$

CONCLUSION

These results are obtained by investigating classes of admissible functions. Some of the results in this study would provide extensions of those given in earlier works.

REFERENCES

Bulboaca, T., 2005. Differential Subordinations and Superordinations: Recent Results. Casa Cartii De Stiinta S.R.L., Cluj-Napoca, Romania, ISBN: 9789736867774, Pages: 297.

Cho, N.E. and I.H. Kim, 2007. Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function. Applied Math. Comput., 187: 115-121.

Cho, N.E., O.S. Kwon and H.M. Srivastava, 2004. Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations. J. Math. Anal. Appl., 300: 505-520.

Choi, J.H., M. Saigo and H.M. Srivastava, 2002. Some inclusion properties of a certain family of integral operators. J. Math. Anal. Appli., 276: 432-445.

Dziok, J. and H.M. Srivastava, 2002. Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function. Adv. Stud. Contemp. Math. Kyungshang, 5: 115-125.

Dziok, J. and H.M. Srivastava, 2003. Certain subclasses of analytic functions associated with the generalized hypergeometric function. Trans. Spec. Funct., 14: 7-18.

Ghanim, F. and M. Darus, 2010a. Some properties of certain subclass of meromorphically multivalent functions defined by linear operator. J. Math. Stat., 6: 34-41.

Ghanim, F. and M. Darus, 2010b. Some results of p-valentmeromorphic functions defined by a linear operator. Far East J. Math. Sci., 44: 155-165.

Ghanim, F. and M. Darus, 2011a. A new class of meromorphically analytic functions with applications to the generalized hypergeometric functions. Abstr. Applied Anal., Vol. 2011, 10.1155/2011/159405

Ghanim, F. and M. Darus, 2011b. Certain subclasses of meromorphic functions related to Cho-Kwon-Srivastava operator. Far East J. Math. Sci., 48: 159-173.

Liu, J., 2001a. The Noor integral and strongly starlike functions. J. Math. Anal. Appl., 261: 441-447.

Liu, J.L. and H.M. Srivastava, 2001b. A linear operator and associated families of meromorphically multivalent functions. J. Math. Anal. Applied, 259: 566-581.

Liu, J.L. and H.M. Srivastava, 2004a. Certain properties of the Dziok-Srivastava operator. Applied Math. Comput., 159: 485-493.

Liu, J.L. and H.M. Srivastava, 2004b. Classes of meromorphically multivalent functions associated with the generalized hypergeometric function. Math. Comput. Model., 39: 21-34.

Liu, J.L. and K.I. Noor, 2002. Some properties of Noor integral operator. J. Nat. Geom., 21: 81-90.

Liu, J.L., 2003. A linear operator and its applications on meromorphic p-valent functions. Bull. Inst. Math. Acad. Sin., 31: 23-32.

Miller, S.S. and P.T. Mocanu, 2000. Differential Subordinations: Theory and Applications. Marcel Dekker, New York, USA., Pages: 456.

Miller, S.S. and P.T. Mocanu, 2003. Subordinants of differential superordinations. Complex Var., 48: 815-826.

Noor, K.I., 1999. On new classes of integral operators. J. Natur. Geom., 16: 71-80.

Piejko, K. and J. Sokol, 2008. Subclasses of meromorphic functions associated with the Cho-Kwon-Srivastava operator. J. Math. Anal. Appl., 337: 1261-1266.