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Weakly and Almost T-ABSO Fuzzy Submodules

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Abstract: Let, $\dot{M}$ be a unitary R-module over R be a commutative ring with identity and let X be a fuzzy module of an R-module $\dot{M}$. In this study, we present two concepts: the frist concept is a weakly T-ABSO fuzzy submodule where a proper fuzzy submodule A of fuzzy module X of an R-module $\dot{M}$ is called a weakly T-ABSO fuzzy submodule of X if whenever fuzzy singletons a_s, b_l of R, $x_v \subseteq X$, $\forall s, l, v \in L$ and $0_1 \neq a_s b_l x_v \subseteq A$ then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. And the second concept is an almost T-ABSO fuzzy submodule where let R be an integral domain, X be fuzzy module of an R-module $\dot{M}$ and A a proper fuzzy submodule of X. A is called an almost T-ABSO fuzzy submodule of X if for fuzzy singletons a_s, b_l of R and $x_v \subseteq X$ with $a_s b_l x_v \subseteq A-(A:_R X)A$, then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. We study some basic properties and characterizations of weakly T-ABSO fuzzy submodules and almost T-ABSO fuzzy submodules. We present almost T-ABSO fuzzy submodules of X as a new generalization of T-ABSO fuzzy and weakly T-ABSO fuzzy submodules and relationships between them concepts are given.

Key words: T-ABSO fuzzy submodules, weakly T-ABSO fuzzy ideals, weakly T-ABSO fuzzy submodules, almost T-ABSO fuzzy ideal, almost T-ABSO fuzzy submodule, characterizations

INTRODUCTION

A prime submodule which play an important turn in the module theory over a commutative ring. This concept was generalized to prime fuzzy submodule by Rabi (2001). Sonmez *et al*. (2017) presented the concept of 2-absorbing fuzzy ideal which is a generalization of prime fuzzy ideal. Darani and Soheilnia (2012) presented the concept of 2-absorbing submodule where "a proper submodule N of $\dot{M}$ is called 2-absorbing submodule of if whenever a, b$\in$R, m$\in \dot{M}$ and ab m$\in$N, then a m$\in$N or b m$\in$N or ab$\in$(N: $\dot{M}$)". Hatam (2001) expand this concept where "let X be fuzzy module of an R-module $\dot{M}$.

A proper fuzzy submodule A of X is called T-ABSO fuzzy submodule if whenever a_s, b_l be F. Singletons of R and $x_v \subseteq X$, $\forall s, l, v \in L$ such that $a_s b_l x_v \subseteq A$ then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$ (Hatam, 2001). Presented the concept of a weakly prime fuzzy ideal while Badawi and Darani (2013) were studied the concept of a weakly 2-absorbing ideal where A proper ideal I of a commutative ring R is called a weakly 2-absorbing ideal of R if whenever a, b, c$\in$R and 0$\neq$abc$\in$I, then either ab$\in$I or ac$\in$I or bc$\in$I (Badawi and Darani, 2013). A weakly prime submodule were presented by Atani and Farzalipour (2007) where "A proper submodule N of an R-module $\dot{M}$ is called a weakly prime if for a$\in$R and m$\in \dot{M}$ with 0$\neq$am$\in$N, then either m$\in$N or a$\in$(N:$_R$ $\dot{M}$)". "Darani and Soheilnia (2012) were generalized of weakly prime submodule to weakly 2-absorbing submodule where" A proper submodule N of an R-module $\dot{M}$ is called a weakly 2-absorbing of M if whenever a, b$\in$R, m$\in \dot{M}$ and 0$\neq$ab m$\in$N, then either ab$\in$ (N:$_R$ $\dot{M}$) or am$\in$N or bm$\in$N" (Darani and Soheilnia, 2012). "A proper ideal I of R is said to be almost prime provided that a b$\in$R with ab$\in$I-I^2 imply that a$\in$I or b$\in$I" (Bhatwadekar and Sharma, 2005) while A proper ideal I of R is said to be almost 2-absorbing ideal if whenever a, b, c$\in$R with abc$\in$I-I^2, then either ab$\in$I or ac$\in$I or bc$\in$I (Mohammad and Abu-Dawwas, 2016). Almost prime submodule studied by Khashan (2012) where "A proper submodule N ofan R-module $\dot{M}$ is called an almost prime submodule of $\dot{M}$ if whenever r$\in$R and m$\in \dot{M}$ such that rm$\in$N−(N: $\dot{M}$) N, then either m$\in$N or r$\in$(N: $\dot{M}$)". Mohammad and Abu-Dawwas (2016) were generalization this notionto the almost 2-absorbing submodules where "let R be an integral domain, M be an R-module and N a proper submodule of $\dot{M}$. N is called an almost 2-absorbing submodule of $\dot{M}$ if a, b$\in$R and m$\in \dot{M}$ with a bm$\in$N-(N: $\dot{M}$) N then either ab$\in$(N: $\dot{M}$) or am$\in$N or bm$\in$N" (Mohammad and Abu-Dawwas, 2016).

In our study, we present the concepts of weakly prime fuzzy submodule, T-ABSO fuzzy submodule, weaky T-ABSO fuzzy ideal, weakly T-ABSO fuzzy submodule, almost T-ABSO fuzzy ideal and almost

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T-ABSO fuzzy submodule and present a new basic properties, characterizations of these concepts and relationships between these concepts.

This study be composed of two sections: in section 1, we presentand study the concept of weakly T-ABSO fuzzyideal, weakly T-ABSO fuzzy submodule and we give many properties, characterizations and relationships between prime fuzzy submodule, weakly prime fuzzy submodule, T-ABSO fuzzy submodule and weakly T-ABSO fuzzy submodule.

In section 2, we present the concepts of almost prime fuzzy ideal, almost T-ABSO fuzzy ideal, almost prime fuzzy submodule and almost T-ABSO fuzzy submodule, so, many properties, characterizations and relationships between almost 2-absorbing submodule, T-ABSO fuzzy submodule and weakly T-ABSO fuzzy submodule are given. Note that, we denote to Fuzzy: F., Module: M., submodule: subm., [0,1]: L and otheroiwse: o.w.

WEAKLY T-ABSO F. SUBM

In this section, we shall expand the concepts of weakly prime subm., weakly 2-absorbing ideal and weakly 2-absorbing subm. to weakly prime F. subm., weakly T-ABSO F. ideal, T-ABSO F. subm. and weakly T-ABSO F. subm and search some properties, characterizations and relations of weakly T-ABSO F. subm. with other concepts of F. subm. First, we shall fuzzify those concepts as follows:

Definition 2.1: A proper F. subm. A of FM X of an R-M $\dot{M}$ is called weakly prime F. subm. if for F. singleton r_k of R and $x_v \subseteq X$ with $0_1 \neq r_k x_v \subseteq A$, then either $r_k \subseteq (A:_R X)$ or $x_v \subseteq A$ where:

$$0_1(y) = \begin{cases} 1 & y = 1 \\ 0 & y \neq 1 \end{cases}$$

The proposition specificates weakly prime F. subm. in terms of its level subm is given:

Proposition 2.2: Let A be F. subm. of FMX of an R-M $\dot{M}$. Then A is a weakly F. subm. of X iff the level A_v is a weakly prime subm. of X_v, $\forall_v \in L$.

Proof: ($\Rightarrow$) let $0 \neq ax \in \forall_v$ for each $a \in R$, $x \in X_v$, $\forall v \in L$, then $A(ax) \geq v$, hence $(ax)_v \subseteq A$, so that, $a_s x_k \subseteq A$ where $v = \min\{s, k\}$. But A is a weakly prime F. subm., then either $a_s \subseteq (A:_R X)$ or $x_k \subseteq A$, implies $a \in (A_v:_R X_v)$ or $x \in A_v$ where $(A:_R X)_v = (A_v:_R X_v)$ (Hatam, 2001). Thus A_v is a weakly prime subm. of X_v.

($\Leftarrow$) let $0_1 \neq a_s x_k \subseteq$ Afor F. singleton a_s of R and $x_v \subseteq X$, $\forall$s, k$\in$L, then $0_1 \neq (ax)_v \subseteq A$ where $v = \min\{s, k\}$, hence, $A(ax) \geq v$, so that, $ax \in A_v$. But A_v is a weakly prime subm., then either $a \in (A_v:_R X_v)$ or $x \in A_v$, hence, $a_s (A:_R X)$ or $x_k \subseteq A$, thus, A is weakly prime F. subm. of X.

Definition 2.3: A proper F. ideal I of acommutative Ring R is called weakly T-ABSO F. ideal if for F. singletons, a_s, b_l, r_k of R, $\forall$s, l, k$\in$L such that $0_1 \neq a_s b_l x_v \subseteq I$, then either $a_s b_l \subseteq I$ or $a_s r_k \subseteq I$ or $b_l r_k \subseteq I$. The proposition specificates T-ABSO F. subm. in terms of its level subm. is given:

Proposition 2.5: Let A be T-ABSO F. subm. of F. M. X of an R-M. $\dot{M}$, if f the level subm. A_v is T-ABSO subm. of X_v, $\forall v \in L$ (Khalaf and Hannon, 2018).

Definition 2.6: A proper F. subm. A of F.M.X of an R-M. $\dot{M}$ is called a weakly T-ABSO F. subm. of X if whenever F. singletons a_s, b_l of R, $x_v \subseteq X$, $\forall$s, l, v$\in$L and $0_1 \neq a_s b_l x_v \subseteq A$, then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. The proposition specificates weakly T-ABSOF. subm. in terms of its level subm is given:

Proposition 2.7: Let A be F. subm. of F. M. X of an R-M. $\dot{M}$. Then A is a weakly T-ABSO F. subm. of X if f the level A_v is a weakly T-ABSO subm. of X_v, $\forall_v \in L$.

Proof: By a similar on way to proof of proposition (2.5).

Remarks and examples 2.8:

- Prime F. subm.→ weakly prime F. subm.→T-ABSO F. subm.
- Weakly prime F. subm.→ weakly T-ABSO F. subm.
- T-ABSO F. subm.→weakly T-ABSO F. subm.

However, the converse incorrect, for example:
Let $X:Z_8 \rightarrow L$ such that:

$$X(y) = \begin{cases} 1 & \text{if } y \in Z_8 \\ 0 & \text{o.w.} \end{cases}$$

It is obvious that X be F. M. of Z-M.Z_8. Let, $A:Z_8 \rightarrow L$ such that:

$$A(y) = \begin{cases} v & \text{if } y \in (\bar{0}) \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$$

It is obvious that A is F. subm. of X. Now, $A_v = (\bar{0})$ and $X_v = Z_8$ as Z-M. where $A_v = (\bar{0})$ is not T-ABSO subm. since, $2.2.(\bar{2}) = (\bar{0})$ but $2.(\bar{2}) \neq (\bar{0})$ and $2.2 \notin (A_v:_Z X_v) = 8_Z$ while A_v is a weakly T-ABSO subm., so that, A is not T-ABSO F. subm. and it is weakly T-ABSO F. subm. (4) AF. subm. A is weakly prime F. subm. if f A is T-ABSO F. subm. and $(A:_R X)$ is a prime F. ideal. However, if A is T-ABSO F. subm. and $(A:_R X)$ is not a prime F. ideal then A is not necessary weakly prime F. subm. for example: Let X:Z→L such that $X(y) = \begin{cases} 1 & \text{if } y \in Z \\ 0 & \text{o.w.} \end{cases}$ It is obvious that X be FM of Z-M Z. Let A:Z→L such that:

$$A(y) = \begin{cases} v & \text{if } y \in 4Z \\ 0 & \text{o.w.} \end{cases} \quad \forall v \in L$$

It is obvious that A is F. subm. of X . Now, A_v = 4Z and X_v = Z as Z-M. where A_v = 4Z is T-ABSO subm. since, 2.21∈A_v = 4Z and 2.2∈A_v but A_v is not weakly prime subm. since, 0≠2.2∈A_v but 2∉A_v . So that, A is T-ABSO F. subm. but it is not weakly prime F. subm.

Theorem 2.9: Let R be a commutative ring and let X be F.M. of an R-M. $\dot{M}$. Then the intersection of each pair of distinct weak prime F. subm. of X is weakly T-ABSO F. subm.

Proof: Let A and B be two distinct weak prime F. subm. of X. Suppose that F. singletons a_s, b_l of R and $x_v \subseteq X$ such that $0_1 \neq a_s b_l x_v \subseteq A\cap B$ but $0_1 \neq a_s x_v \not\subseteq A\cap B$ and $0_1 \neq b_l x_v \not\subseteq A\cap B$. Then $0_1 \neq a_s x_v \not\subseteq A$, $0_1 \neq b_l x_v \not\subseteq A$, $0_1 \neq a_s x_v \not\subseteq B$ and $0_1 \neq b_l x_v \not\subseteq B$ these are impossible, since, A and B are weak prime F. subm. So, supoose that $0_1 \neq a_s x_v \not\subseteq A$ and $0_1 \neq b_l x_v \not\subseteq B$. Since, $0_1 \neq a_s b_l x_v \subseteq A$ and $0_1 \neq a_s b_l x_v \subseteq B$, then $b_l \subseteq (A:_R X)$ and $a_s \subseteq (B:_R X)$. So that, $a_s b_l \subseteq (A:_R X) \cap (B:_R X) = (A\cap B:_R X)$. Thus, A∩B is a weakly T-ABSO F. subm. of X.

Theorem 2.10: Let R be a commutative ring, X be F.M. of an R-M. $\dot{M}$ andA be a weakly T-ABSO F. subm. of X. If A is not T-ABSO F. subm. then $(A:_R X)^2 A = 0_1$.

Proof: Suppose that $(A:_R X)^2 A \neq 0_1$. We will show that A is T-ABSO F. subm. Let $a_s b_l x_v \subseteq A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. If $a_s b_l x_v \neq 0_1$ then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$, since, A is a weakly T-ABSO F. So, suppose that $a_s b_l x_v = 0_1$. Let $a_s b_l A \neq 0_1$, say $a_s b_l y_h \neq 0_1$ for some F. singleton $y_h \subseteq A$. Hence, $0_1 \neq a_s b_l y_h = a_s b_l (x_v + y_h) \subseteq A$. Since, A is weakly T-ABSO F. subm., we have $a_s b_l \subseteq (A:_R X)$ or $a_s(x_v + y_h) \subseteq A$ or $b_l(x_v + y_h) \subseteq A$. Then $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Thus, we may assume that $a_s b_l A = 0_1$. If $a_s x_v (A:_R X) \neq 0_1$, then there exists $r_k \subseteq (A:_R X)$ such that $a_s r_k x_v \neq 0_1$. Hence, $0_1 \neq a_s r_k x_v = a_s(b_l + r_k) x_v \subseteq A$. Since, A is weakly T-ABSO F. subm., we get $a_s(b_l + r_k) \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $(b_l + r_k) x_v \subseteq A$. Thus, $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. So, we can assume that $a_s x_v (A:_R X) = 0_1$. By a similar on way, we can assume that $b_l x_v (A:_R X) = 0_1$. Since, $(A:_R X)^2 A \neq 0_1$, there exist c_n, $d_m \subseteq (A:_R X)$ and $z_u \subseteq A$ with $c_n d_m z_u \neq 0_1$. If $a_s d_m z_u \neq 0_1$, then $0_1 \neq a_s d_m z_u = a_s(b_l + d_m)(x_v + z_u) \subseteq A$, hence, $a_s(b_l + d_m) \subseteq (A:_R X)$ or $a_s(x_v + z_u) \subseteq A$ or $(b_l + d_m)(x_v + z_u) \subseteq A$. So that, $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Then, we can assume that $a_s d_m z_u = 0_1$. By in a similar way, we can assume that $c_n d_m x_v = 0_1$ and $c_n b_l z_u = 0_1$. Hence, from $0_1 \neq c_n d_m z_u = (a_s + c_n)(b_l + d_m)(x_v + z_u) \subseteq A$, we have $(a_s + c_n)(b_l + d_m) \subseteq (A:_R X)$ or $(a_s + c_n)(x_v + z_u) \subseteq A$ or $(b_l + d_m)(x_v + z_u) \subseteq A$. Thus, $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Therefore, A is T-ABSO F. subm.

Recall that "A subm. N of an R-M $\dot{M}$ is called a nilpotent subm. if $(N:_R M)^k N = 0$ for some k∈Z+" (Ali, 2008). We shall fuzzify this concept as follows:

Definition 2.11: A F. subm. A of FM X of an R-M $\dot{M}$ is called a nilpotent F. subm. if $(A:_R X)^n A = 0_1$ for some n∈Z^+.

Corollary 2.12: Let R be a commutative ring and X be F. M. of an R-M. $\dot{M}$. Suppose that A be a weakly T-ABSO F. subm. of X that is not T-ABSO F. subm., then:

- A is a nilpotent F. subm.
- If X is a multiplication FM then $A^3 = 0_1$

The definitions of multiplication F.M. (Hatam, 2001), faithful F.M. (Badawi and Darani, 2013), finitely generated and cancellation FM (Hadi and Hamil, 2011).

Lemma 2.13: Let Abe F. subm. of a finitely generated faithful multiplication (and so cancellation) F.M.X of an R-M $\dot{M}$. Then, we have $(IA:_R X) = I(A:_R X)$ for every F. ideal I of R.

Proof: Since, X is a multiplication FM then $I(A:_R X)X = IA = (IA:_R X)X$. So that, $(IA:_R X) = I(A:_R X)$, since, X is cancellation FM.

Proposition 2.14: Let, X be a faithful multiplication FM of an R-M $\dot{M}$ and let A be a weakly T-ABSO F. subm. of X. If A is not T-ABSO F. subm. then $A \subseteq X\text{-}R(0_1)$.

Proof: Assume that A is not T-ABSO F. subm. By theorem (2.10), $(A:_R X)^2 A = 0_1$. By lemma (2.13), then $(A:_R X)^3 \subseteq ((A:_R X)^2 A:_R X) = (0_1:_R X) = 0_1$, since, X is faithful, hence $(A:_R X)^3 = 0_1$. If $r_k \subseteq (A:_R X)$, then $^3_k \subseteq 0_1$ and so, $r_k \subseteq \sqrt{0_1}$. Hence $(A:_R X) \subseteq \sqrt{0_1}$. Thus, $A = (A:_R X)X \subseteq \sqrt{0_1}\, X = \text{-}R(0_1)$. The definition of a cyclic F.M. (Hatam, 2001).

Proposition 2.15: Let, R be a commutative ring, X be a faithful cyclic FM of an R-M $\dot{M}$ andA be a weakly T-ABSO F. subm. of X then A is a weakly T-ABSO F. subm. of X if f $(A:_R X)$ is a weakly T-ABSO F. ideal of R.

Proof: (⇒) Let, $0_1 \neq a_s b_l r_k \subseteq (A:_R X)$ for F. singletons a_s, b_l, r_k of R. Suppose that $a_s b_l \not\subseteq (A:_R X)$ and $b_l r_k \not\subseteq (A:_R X)$. Hence, $0_1 \neq a_s b_l r_k x_v \subseteq A$ for F. singleton $x_v \subseteq X$. If $a_s b_l r_k x_v = 0_1$, then $a_s b_l r_k \subseteq (0_1:_R X) = 0_1$ this is impossible. Since, A is a weakly T-ABSO F. subm and $a_s b_l \not\subseteq (A:_R X)$ and $b_l r_k \not\subseteq (A:_R X)$, then $a_s r_k \subseteq (A:_R X)$. Thus, $(A:_R X)$ is a weakly T-ABSO F. ideal of R.

(⇐) suppose that $(A:_R X)$ is a weakly T-ABSO F. ideal of R and let $0_1 \neq a_s b_l x_v \subseteq A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. Since, X is a cyclic F. M., then there exists F. singleton r_k of R with $x_v = r_k y_h$ for each F. singleton $y_h \subseteq X$. Hence, $0_1 \neq a_s b_l r_k y_h \subseteq A$, then $0_1 \neq a_s b_l r_k \subseteq (A:_R y_h) = (A:_R X)$. Since, $(A:_R X)$ is a weakly T-ABSO F. ideal, then either $a_s b_l \subseteq (A:_R X)$ or $a_s r_k \subseteq (A:_R X)$ or $b_l r_k \subseteq (A:_R X)$. Therefore, $a_s b_l \subseteq (A:_R X)$ or $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Thus, A is weakly T-ABSO F. subm. New, we give two lemmas which are needed in the next theorem.

Lemma 2.16: Let, A be a weakly T-ABSO F. subm. of F.M.X of an R-M. $\dot{M}$ and F. singletons a_s, b_l of R. If for some F. subm. B of X $a_s b_l B \subseteq A$ and $0_1 \neq 2\ a_s\ b_l B$, then $a_s b_l \subseteq (A:_R X)$ or $a_s B \subseteq A$ or $b_l B \subseteq$.

Proof: Put $(A:_R X) = K$ and assume that $a_s b_l \not\subseteq K$. Then it is sufficent to prove that $B \subseteq (A:_X a_s) \cup (A:_X b_l)$. Let r_k be an arbitrary F. singleton of B. If $0_1 \neq a_s b_l r_k$ and $a_s b_l \not\subseteq K$, then either $a_s r_k \subseteq A$ or $b_l r_k \subseteq A$, since, A is a weakly T-ABSO F. subm. So that, $r_k \subseteq (A:_X a_s) \cup (A:_X b_l)$. Now, let $0_1 = a_s b_l r_k$. Since, $0_1 \neq 2 a_s b_l B$, then, $0_1 \neq 2 a_s b_l x_v$ for some F. singleton $x_v \subseteq B$ and hence $0_1 \neq a_s b_l x_v \subseteq A$. Since, A is a weakly T-ABSO F. subm. and $a_s b_l \not\subseteq K$, then either $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Put $y_h = x_v + r_k$. Hence, $0_1 \neq a_s b_l y_h \subseteq A$ and since, $a_s b_l \not\subseteq K$, then either $a_s y_h \subseteq A$ or $b_l y_h \subseteq A$. Now, we meditation three cases:

Case (1): $a_s x_v \subseteq A$ and $b_l x_v \subseteq A$. Note that, $a_s y_h \subseteq A$ or $b_l y_h \subseteq A$ and so, either $a_s r_k \subseteq A$ or $b_l r_k \subseteq A$.

Case (2): $a_s x_v \subseteq A$ and $b_l x_v \not\subseteq A$. On the contrary let $a_s r_k \not\subseteq A$. Hence, $a_s y_h \not\subseteq A$ and so, $b_l y_h \subseteq A$. Thus, $a_s(y_h + x_v) \not\subseteq A$ and $b_l(y_h + x_v) \not\subseteq A$. Since, A is a weakly T-ABSO F. subm. and $a_s b_l \not\subseteq K$, then $0_1 = a_s b_l (y_h + x_v) = 2 a_s b_l x_v$ this is impossible. Thus, $a_s r_k \subseteq A$.

Case (3): $a_s x_v \not\subseteq A$ and $b_l x_v \subseteq A$. Then, proof in a similar way case (2).

Lemma 2.17: Let I be F. ideal of R and A , B two F. subm. of F.M. X of an R-M such that $a_s IB \subseteq A$ where a_s be F. singleton of R. If A is a weakly T-ABSO F. subm. and $0_1 \neq 4\ a_s IB$, then $a_s I \subseteq (A:_R X)$ or $a_s B \subseteq a$ or IB.

Proof: Let $a_s I \not\subseteq (A:_R X) = K$. Then $a_s b_l \not\subseteq K$ for some F. singleton $b_l \subseteq I$. We claim that there exists $r_h \subseteq I$ such that $0_1 \neq 4\ a_s r_h B$ and $a_s r_h \not\subseteq K$. Since, $0_1 \neq 4\ a_s IB$, then $0_1 \neq 4\ a_s c_n B$ for some F. singleton $c_n \subseteq I$. If $a_s c_n \not\subseteq K$ or $0_1 \neq 4\ a_s r_h B$, then by putting $r_h = c_n$ or $r_h = b_l$, we get the outcome. Therefore, let $a_s c_n \subseteq K$ and $4\ a_s b_l B = 0_1$. Then, $0_1 \neq 4\ a_s(c_n + b_l) B \subseteq A$ and $a_s(c_n + b_l) \not\subseteq K$. So that, $r_h \subseteq I$ such that $0_1 \neq 4\ a_s r_h B$ and $a_s r_h \not\subseteq K$. Then, $0_1 \neq 2\ a_s r_h B$ and by lemma (2.16), we get $B \subseteq (A:_X a_s) \cup (A:_X r_h)$. If $a_s B \subseteq A$ there is nothing to prove. Therefore, suppose that $a_s B \not\subseteq A$ and hence, $r_h B \subseteq A$. Now, we show that $I \subseteq (K:_R a_s) \cup (A:_R B)$. Let F. singleton $u_m \subseteq I$. If $0_1 \neq 2 a_s u_m B$, then by lemma (2.16), $a_s u_m \subseteq K$ or $a_s B \subseteq A$ or $u_m B \subseteq A$. But, we assumed $a_s B \subseteq A$, then $u_m \subseteq (K:_R a_s) \cup (A:_R B)$.

Now, suppose that $2 a_s u_m B = 0_1$. Hence, $0_1 \neq 2 a_s(r_h + u_m) B \subseteq A$ and by lemma (2.16), then either $a_s(r_h + u_m) \subseteq K$ or $a_s B \subseteq A$ or $(r_h + u_m) B \subseteq A$. Since, $a_s B \not\subseteq A$, so that, $(r_h + u_m) \subseteq (K:_R a_s) \cup (A:_R B)$. If $(r_h + u_m) \subseteq (A:_R B)$, then $u_m \subseteq (A:_R B)$ because $r_h \subseteq (A:_R B)$. Therefore, let $(r_h + u_m) \subseteq (K:_R a_s)/(A:_R B)$.

Meditation $2 a_s(r_h + u_m + r_h) B = 4 a_s r_h B \neq 0_1$ and $2 a_s(r_h + u_m + r_h) B \subseteq A$. Since, $a_s r_h \not\subseteq K$ and $a_s(r_h + u_m) \subseteq K$, then $a_s(r_h + u_m + r_h) \not\subseteq K$. Then by lemma (2.16), $B \subseteq (A:_X a_s) \cup (A:_X r_h + u_m + r_h)$. But, since, $r_h + u_m \not\subseteq (A:_R B)$ and $r_h \subseteq (A:_R B)$, then $(r_h + u_m + r_h) \not\subseteq (A:_R B)$, hence, $B \subseteq (A:_X a_s)$ this is impossible. Thus, $r_h + u_m \subseteq (A:_R B)$ and since $r_h \subseteq (A:_R B)$, then $u_m \subseteq (A:_R B)$. Then $I \subseteq (K:_R a_s) \cup (A:_R B)$. So that, $IB \subseteq A$, since, $a_s I \not\subseteq K$. The following theorem gives a characterization of weakly T-ABSO F. subm.

Theorem 2.18: Let I_1, I_2 be F. ideals of R and A , B be F. subm. of F. M. X of an R-M. $\dot{M}$. If A is aweakly T-ABSO F. subm., $0_1 \neq 1_1 I_2\ B \subseteq A$ and $0_1 \neq 8(I_1 I_2 + (I_1 + I_2)(A:_R X))\ (B + A)$, then either $I_1 I_2 \subseteq (A:_R X)$ or $I_1 B \subseteq A$ or $I_2 B \subseteq A$.

Proof: Note that $0_1 \neq 8(I_1 I_2 + (I_1 + I_2)\ (A:_R X))\ (B + A) = 8 I_1 I_2 B + 8 I_1 I_2 A + 8 I_1 (A:_R X) B + 8\, I_2 (A:_R X) B + 8 I_1 (A:_R X) A + 8 I_2 (A:_R X) A$. Therefore, one of the following various types is satisfied, $0_1 \neq 8 I_1 I_2 B$.

Hence, for some F. singleton $a_s \subseteq I_2$, we have $0_1 \neq 8 a_s I_1 B$. Thus, $0_1 \neq 4 a_s I_1 B$ and by lemma (2.17), then either $a_s I_1 \subseteq (A:_R X) = K$ or $a_s B \subseteq A$ or $I_1 B \subseteq A$. If $I_1 B \subseteq A$, then, we get the outcome. Therefore, we assume that $I_1 B \not\subseteq A$ and so, $a_s \subseteq (K:_R I_1) \cup (A:_R B)$. Now, we prove that $I_2 \subseteq (K:_R I_1) \cup (A:_R B)$. Let, $b_l \subseteq I_2$. If $0_1 \neq 4 b_l I_1 B$, then by lemma (2.17) and since, $I_1 B \not\subseteq A$, we have $b_l \subseteq (K:_R I_1) \cup (A:_R B)$. Now, let $4 b_l I_1 B = 0_1$, then $0_1 \neq 4(a_s + b_l) I_1 B \subseteq A$. By lemma (2.17) and since, $I_1 B \not\subseteq A$ then $(a_s + b_l) \subseteq (K:_R I_1) \cup (A:_R B)$. We meditation the the following four cases:

Case 1: $(a_s + b_l) \subseteq (K:_R I_1)$ and $a_s \subseteq (K:_R I_1)$. Then $b_l \subseteq (K:_R I_1)$.

Case 2: $(a_s + b_l) \subseteq (A:_R B)$ and $a_s \subseteq (A:_R B)$. Then, $b_l \subseteq (A:_R B)$.

Case 3: $a_s \subseteq (K:_R I_1)/(A:_R B)$ and $(a_s + b_l) \subseteq (A:_R B)/(K:_R I_1)$. Then $(a_s + b_l + a_s) \not\subseteq (K:_R I_1)$ and $(a_s + b_l + a_s) \not\subseteq (A:_R B)$. So that, $(a_s + b_l + a_s) \not\subseteq (K:_R I_1) \cup (A:_R B)$. We meditation $4(a_s + b_l + a_s) I_1 B = 8 a_s I_1 B \neq 0_1$. By lemma (2.17) and since, $I_1 B \not\subseteq A$, then $(a_s + b_l + a_s) \subseteq (K:_R I_1) \cup (A:_R B)$ this is impossible. Since, $a_s \subseteq (K:_R I_1) \cup (A:_R B)$ and $(a_s + b_l) \subseteq (K:_R I_1) \cup (A:_R B)$, one of the following holds:

- $a_s \subseteq (A:_R B)$ and $(a_s + b_l) \subseteq (A:_R B)/(K:_R I_1)$. Then $b_l \subseteq (A:_R B)$
- $a_s \subseteq (K:_R I_1)/(A:_R B)$ and $(a_s + b_l) \subseteq (K:_R I_1)$. Then $b_l \subseteq (K:_R I_1)$

Case 4: $(a_s+b_l)\subseteq(K:_RI_1)/(A:_RB)$ and $a_s\subseteq(A:_RB)/(K:_RI_1)$. By in a similar way of case (3), we have $b_l\subseteq(K:_RI_1)\cup(A:_RB)$. Thus, $I_2\subseteq(K:_RI_1)\cup(A:_RB)$.

If $0_1\neq 8I_1I_2A$ and $8I_1I_2B = 0_1$, then $0_1\neq 8I_1I_2(B+A)\subseteq A$ and hence by part (1), $I_1I_2\subseteq(A:_RX)$ or $I_1(B+A)\subseteq a$ or $I_2(B+A)\subseteq A$. So that, $I_1I_2\subseteq(A:_RX)$ or $I_1B\subseteq A$ or $I_2B\subseteq A$.

Let, $0_1\neq 8I_2(A:_RX)$ B and $8I_1I_2$ B = 0_1. Hence, $8I_2(I_1+(A:_RX))B = 8I_2(A:_RX)B\neq 0_1$. By part (1), then either $I_2(I_1+(A:_RX))\subseteq(A:_RX)$ or $I_2B\subseteq A$ or $(I_1+(A:_RX))B\subseteq A$, hence, $I_1I_2\subseteq(A:_RX)$ or $I_1B\subseteq A$ or $I_2B\subseteq A$. By in a similar way if, $0_1\neq 8I_1(A:_RX)$ B, we get the outcome. Let $0_1\neq 8I_2(A:_RX)A$ and $8I_1I_2B = 8I_1I_2A = 8I_2(A:_RX)B = 8I_1(A:_RX)B = 0_1$.

Hence, $8I_2(I_1+(A:_RX))(B+A) = 8I_2(A:_RX)A\neq 0_1$ and by part (1), then $I_2(I_1+(A:_RX))\subseteq(A:_RX)$ or $I_2(B+A)\subseteq A$ or $(I_1+(A:_RX))(B+A)\subseteq A$. So that, $I_1I_2\subseteq(A:_RX)$ or $I_1B\subseteq A$ or $I_2B\subseteq A$. Obvious if $0_1\neq 8I_1(A:_RX)$, we get the outcome.

ALMOST T-ABSO F. SUBM.

In this section, we shall expand the concepts of almost prime subm, almost 2-absorbing ideal and almost 2-absorbing subm. to almost prime F. subm. almost T-ABSO F. ideal and almostT-ABSO F. subm.We present an almost T-ABSO F. subm. as a generalization of T-ABSO F. subm. and weakly T-ABSO F. subm. and study some basic properties, characterizations and relationships of almost T-ABSO F. subm., T-ABSO F. subm. and weakly T-ABSO F. subm. We shall fuzzify these concepts as follows:

Definition 3.1: A proper F. ideal I of R is said to be almost prime F. if whenever F. singletons a_s, b_l of R such that $a_sb_l\subseteq I-I^2$, then either $a_s\subseteq I$ or $b_l\subseteq I$.

Definition 3.2: A proper F. subm. Aof F.M. X of an R-M is called an almost prime F. subm. of X if whenever F. singletons a_s of R and $x_v\subseteq X$ such that $a_sx_v\subseteq A-(A:_RX)A$ then either $a_s\subseteq(A:_RX)$ or $x_v\subseteq A$.

Definition 3.3: A proper F. ideal I of R is said to be almost T-ABSO F. ideal if whenever F. singletons a_s, b_l, r_k of R such that $a_sb_lr_k\subseteq I-I^2$, then either $a_sb_l\subseteq I$ or $a_sr_k\subseteq I$ or $b_lr_k\subseteq I$.

Definition 3.4: Let R be an integral domain, X be F.M. of an R-M $\dot{M}$ and A a proper F. subm. of X. A is called an almost T-ABSO F. subm. of X if for F. singletons a_s, b_l of R and $x_v\subseteq X$ with $a_sb_lx_v\subseteq A-(A:_RX)A$, then either $a_sb_l\subseteq(A:_RX)$ or $a_sx_v\subseteq A$ or $b_lx_v\subseteq A$. The proposition specificates an almost T-ABSO F. subm. in terms of its level subm. is given:

Proposition 3.5: Let A be almost T-ABSO F. subm. of F.M. X of an R-M. $\dot{M}$., iff the level subm. A_v is almost T-ABSO subm. of X_v, $\forall v\in$ L.

Proof: ($\Rightarrow$) let $abx\in A_v-(A_v:_RX_v)A_v$ for each a, b$\in$R and $x\in X_v$, hence, $abx\in(A-(A:_RX)A)_v$ then $(A-(A:_RX)A)(abx)\geq v$, so, $(abx)_v\subseteq A-(A:_RX)A$ implies that where $v = \min\{s, l, k\}$. Since, A be almost T-ABSO F. subm., then either $a_sb_l\subseteq(A:_RX)$ or $a_sx_k\subseteq A$ or $b_lx_k\subseteq A$. Hence, $(ab)_v\subseteq(A:_RX)$ or $(ax)_v\subseteq A$ or $(bx)_v\subseteq A$, so that, $ab\in(A_v:_RX_v)$ or $ax\in A_v$ or $bx\in A_v$ where $(A:_RX)_v = (A_v:_RX_v)$ by Hatam, (2001). Thus, A_v is T-ABSO subm. of X_v.

($\Leftarrow$) Let $a_sb_lx_k\subseteq A-(A:_RX)A$ for F. singletons a_s, b_l of R and $x_k\subseteq X$, $\forall$s, l, k$\in$L, hence, $(abx)_v\subseteq A-(A:_RX)A$ where $v = \min\{s, l, k\}$, so that, $(A-(A:_RX)A)(abx)\geq v$, implies $abx\in A_v-(A_v:_RX_v)A_v$ but A_v is almost T-ABSO subm., then either $ab\in(A_v:_RX_v)$ or $ax\in A_v$ or $bx\in A_v$, since, $(A_v:_RX_v) = (A:_RX)_v$, hence, $ab\in(A:_RX)_v$. Hence, either $(ab)_v\subseteq(A:_RX)$ or $(ax)_v\subseteq A$ or $(bx)_v$, implies either $a_sb_l\subseteq(A:_RX)$ or $a_sx_k\subseteq A$ or $b_lx_k\subseteq A$. Thus, A be almost T-ABSO F. subm. of X.

Remark 3.6: Every T-ABSO F. subm. is weakly T-ABSO F. subm. and every weakly T-ABSO F. subm. is almost T-ABSO F. subm. However the converse incorrect, for example:

Let $X:Z_{24}\to L$ such that $X(y)=\begin{cases}1 & \text{if } y\in Z_{24}\\ 0 & \text{o.w.}\end{cases}$

It is obvious that X be F. M. of Z-M.Z_{24}.

Let $A:Z_{24}\to L$ such that $A(y)=\begin{cases}v & \text{if } y\in\langle\overline{12}\rangle\\ 0 & \text{o.w.}\end{cases}\ \forall v\in L$

It is obvious that A is F. subm. of X. Now, $A_v=\langle\overline{12}\rangle$ and $X_v = Z_{24}$ as Z-M., then $(A_v:_RX_v)A_v = 12\,Z\langle\overline{12}\rangle=\langle\overline{12}\rangle$. So that, A_v is an almost T-ABSO F. subm. but A_v is not T-ABSO F. subm. since, 2.2., $\langle\overline{3}\rangle\in A_v$ 2. $\langle\overline{3}\rangle\notin A_v$ and 2.2 $\notin(A_v:_ZX_v)$ and hence, A_v is not weakly T-ABSO F. subm. Thus, A is almost T-ABSO F. subm. but it is not T-ABSO F. subm. and it is not weakly T-ABSO F. subm.

Proposition 3.7: Let X be FM of an R-M. $\dot{M}$ and A be F. subm. of X. Then the following expressions are equipollent:

- A is an almost T-ABSO F. subm. of X.
- For F. singletons a_s, b_l of R with $a_sb_l\subseteq R-(A:_RX)$, $(A:_X\langle a_sb_l\rangle)=(A:_X\langle a_s\rangle)\cup(A:_X\langle b_l\rangle)\cup((A:_RX)A:(A:_X\langle a_sb_l\rangle)$
- For F. singletons a_s, b_l of R with $a_sb_l\subseteq R-(A:_RX)$, $(A:_X\langle a_sb_l\rangle) = (A:_X\langle a_s\rangle)$ or $(A:_X\langle a_sb_l\rangle) = (A:_X\langle b_l\rangle)$ or $(A:_X\langle a_sb_l\rangle = ((A:_RX)A:_X\langle a_sb_l\rangle)$

Proof: (1)$\Rightarrow$(2) If $a_sb_l\subseteq R-(A:_RX)$ and $x_v\subseteq(A:_X\langle a_sb_l\rangle)$, then $a_sb_lx_v\subseteq A$. But if $a_sb_lx_v\not\subseteq(A:_RX)A$, then $a_sb_lx_v\subseteq A-(A:_RX)$ A. Hence, $a_sx_v\subseteq A$ or $b_lx_v\subseteq A$. Thus, $x_v\subseteq(A:_X\langle a_s\rangle)$ or $x_v\subseteq(A:_X\langle b_l\rangle)$. (2)$\Rightarrow$(3) straight forward since, if F. subm. equals to the union of two F. subm. then it is one of them. (3)$\Rightarrow$(1) Let $a_sb_lx_v\subseteq A-(A:_RX)A$ for F. singletons a_s, b_l of R and $x_v\not\subseteq X$. Suppose that $a_sb_l\not\subseteq(A:_RX)$, we prove that $a_sx_v\subseteq A$ or $b_lx_v\subseteq A$. By (3) $(A:_X\langle a_sb_l\rangle) = (A:_X\langle a_s\rangle)$ or $(A:_X\langle a_sb_l\rangle) = (A:_X\langle b_l\rangle)$ or $(A:_X\langle a_sb_l\rangle) = ((A:_RX)A:_X\langle a_sb_l\rangle)$. Since,

$a_s b_l x_v \not\subseteq (A:_R X)A$, then $x_v \not\subseteq ((A:_R X)A:_X \langle a_s b_l \rangle)$. Thus, $x_v \not\subseteq (A:_X \langle a_s \rangle)$ or $x_v \not\subseteq (A:_X \langle b_l \rangle)$. Hence, $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$. Recall "If N is a subm. of R-M. $\dot{M}$ and $r \in R$ then a subm. N_r of $\dot{M}$ is defined by $N_r = (N:r) = \{m \in :rm \in N\}$". (Ashour *et al*., 2016). We shall fuzzify this concept as follows:

Definition 3.8: Let A be F. subm. of F.M. X of an R-M. $\dot{M}$ and F. singleton a_s of R, then F. subm. A_{a_s} of X is defined by $A_{a_s} = (A:_X a_s) = \{x_v \subseteq X : a_s a_v \subseteq A\}$.

Theorem 3.9: Let X be F.M. of an R-M. $\dot{M}$ and A be a proper F. subm. of X. The following expressions are equipollent:

- A is an almost T-ABSO F. subm
- For F. singletons $a_s b_l$ of R such that $a_s b_l \subseteq (A:_R X)$, $A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A:_R X)A) a_s b_l$

Proof: (1)⇒(2) Let A be an almost T-ABSO F. subm. and suppose that $a_s b_l \subseteq (A:_R X)$, let F. singleton $x_v \subseteq a_s b_l$, then $a_s b_l x_v \subseteq A$. If $a_s b_l x_v \not\subseteq (A:_R X)A$, hence, $a_s x_v \subseteq A$ or $b_l x_v \subseteq A$, so that, $x_v \subseteq A_{a_s}$ or $x_v \subseteq A_{b_l}$. If $a_s b_l x_v \subseteq A-(A:_R X)A$, hence, $x_v \subseteq ((A:_R X)A)_{a_s b_l}$. So that, $A_{a_s b_l} \subseteq A_{a_s} \cup A_{b_l} \cup ((A:_R X)A)_{a_s b_l}$. Since, $A_{a_s} \cup A_{b_l} \cup ((A:_R X)A)_{a_s b_l} \subseteq A_{a_s b_l}$. Then, $A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A:_R X)A)_{a_s b_l}$.

(2)⇒(1) Let F. singletons $a_s b_l$ of R and $x_v \subseteq X$ such that $a_s b_l x_k \subseteq A-(A:_R X)A$. Suppose that $a_s b_l \not\subseteq (A:_R X)$, then $x_v \subseteq A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A:_R X)A)_{a_s b_l}$ but $a_s b_l x_v \not\subseteq (A:_R X)A$, so that, $x_v \subseteq A_{a_s}$ or $x_v \subseteq A_{b_l}$. Then $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. Thus, A be an almost T-ABSO F. subm. of X.

Proposition 3.10: Let X be FM of an R-M. $\dot{M}$ and A be a proper F. subm. of X, then A is an almost T-ABSO F. subm. in X if f for any F. singletons a_s, b_l of R and F. subm. B of X such that $a_s b_l B-\{0_1\} \subseteq A-(A:_R X)A$, implies that $a_s b_l \subseteq (A:_R X)$ or $a_s B \subseteq A$ or $b_l B \subseteq A$.

Proof: (⇒) Suppose that $a_s b_l \not\subseteq (A:_R X)A$, hence, $B \subset A_{a_s b_l} = A_{a_s} \cup A_{b_l} \cup ((A:_R X)A)_{a_s b_l}$ but $a_s b_l B \not\subseteq (A:_R X)A$, so that, $B \subseteq A_{a_s}$ or $B \subseteq A_{b_l}$. Then $a_s B \subseteq A$ or $b_{sl} B \subseteq A$.

(⇐) Assume that $a_s b_l x_k \not\subseteq A-(A:_R X)$ for F. singletons a_s, b_l of R and $x_k \subseteq X$. Hence, $a_s b_l (x_k)-\{0_1\} \subseteq A-(A:_R X)A$, then $a_s b_l \subseteq (A:_R X)$ or $a_s (x_k) \subseteq A$ or $b_l (x_k) \subseteq A$. So that, $a_s b_l \subseteq (A:_R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$, thus, A is an almost T-ABSO F. subm. of X.

Theorem 3.11: Let X be a finitely generated faithful multiplication of an R-M. $\dot{M}$ and A be a proper subm. of X. The following expressions are equipollent:

- A is almost T-ABSO F. subm. in X
- $(A:_R X)$ is almost T-ABSO F. ideal in R
- A = IX for some almost T-ABSO F. ideal I of R

Proof: (1)⇒(2) Suppose A is almost T-ABSO F. subm. and let $a_s b_l x_k \subseteq A-(A:_R X)-(A:_R X)^2$ for F. singletons a_s, b_l, r_k of R. Hence, $a_s b_l r_k X-\{0_1\} \subseteq A-(A:_R X)A$. If $a_s b_l r_k X \subseteq (A:_R X)A$, then by lemma (2.13), $a_s b_l r_k \subseteq ((A:_R X)A:_R X = (A:_R X)^2$ this is impossible. Since, A is almost T-ABSO F. subm., then either $a_s b_l \subseteq (A:_R X)$ or $a_s r_k X \subseteq A$ or $b_l r_k X \subseteq A$, so that, $a_s b_l \subseteq (A:_R X)$ or $a_s r_k \subseteq (A:_R X)$ or $b_l r_k \subseteq (A:_R X)$. Thus, $(A:_R X)$ is almost T-ABSO F. ideal in R.

(2)⇒(1) Assume that $(A:_R X)$ is almost T-ABSO F. ideal in R and let $a_s b_l x_v \subseteq A-(A:_R X)A$ for F. singletons a_s, b_l of R and $x_v \subseteq X$. Hence, $a_s b_l ((x_v):_R X) \subseteq ((a_s b_l x_v):_R X \subseteq (A:_R X)$. Also $a_s b_l ((x_v):_R X) \not\subseteq (A:_R X)^2$ because if $((a_s b_l x_v):_R X) \subseteq (A:_R X)^2 \subseteq ((A:_R X)A:_R X)$, hence, $a_s b_l (x_v) = a_s b_l ((x_v):_R X)X$ this is impossible. Since, $(A:_R X)$ is almost T-ABSO F. ideal, then either $a_s b_l \subseteq (A:_R X)$ or $a_s ((x_v):_R X) \subseteq (A:_R X)$ or $b_l ((x_v):_R X) \subseteq (A:_R X)$. If $a_s ((x_v):_R X) \subseteq (A:_R X)$, then $(a_s x_v) \subseteq a_s (x_v) = a_s ((x_v):_R X)X \subseteq (A:_R X)X = A$, hence, $a_s x_v \subseteq A$. By in a similar way researchers get $b_l x_v \subseteq A$. So that, A is almost T-ABSO F. subm. in X. (2)⇔(3) if we choose $I = (A:_R X)$, we get the outcome. The definition of maximal F. subm (Saifur, 2016).

Recall "A nonzero R-M.M is calld local if it has a largest proper subm (namely Rad (M)) that is its unique subm has a to be the radical (where Rad (M) is an intersection of all maximal subm of M) (Clark *et al*., 2006). Now, we shall expand this concept to local F.M. as follows:

Defination 3.12: A F.M. $X \neq 0_1$ of an R-M. $\dot{M}$ is called local F.M. if has a largest proper F.subm (namely F-R (X)) that is its unique maximal F.subm has to be the radical (where F-R (X)) is an intersection of all maximal F.subm of X).

Propostion 3.13: Let X be a local multiplication F.M. subm of an R-M. $\dot{M}$ with a unique maximal F.subm. K and $(K:_R X) K = 0_1$ then any proper F. subm of X is almost T-ABSO F. subm iff it is weekly T-ABSO F. subm.

Proof: (⇒) for any proper F. subm. A of X, $A \subseteq K(A:_R X) A = 0_1$, since, $(K:_R X) K = 0_1$. Suppose that $a_s b_l x_v \subseteq A-(A:_R X)$, A for F. Singletons a_s, b_l of R and $x_v \subseteq X$, then $0_1 \neq a_s b_l x_v \subseteq A$. Since, A is almost T-ABSO F. subm, then either $a_s b_l \subseteq (A:_R X)$ or $a_s x_k \subseteq A$ or $b_l x_k \subseteq A$. So that, it is weakly T-ABSO F. subm. of X.

(⇐) it is petty, since, every weakly T-ABSO F. Subm. Is almost T-ABSO F. subm. Now, we give two lemmas which are needed in the next theorem.

Lemma 3.14: Let X be F.M. of an R-M Ṁ A be an almost T-ABSO F. subm. of X and F. singletons a_s, b_l of R. If B is F. subm. of X such that a_s, b_l B⊆($A:_R X$) A and 2 $a_s b_l$ B⊄($A:_R X$), then either $a_s b_l$ ⊆($A:_R X$) or a_sB⊆A or b_lB⊆ A.

Proof: By in a similar way to proof of lemma (2.16) but we replace $0_1 \neq a_s b_l r_k$ by $a_s b_l\ r_k \not\subseteq (A:_R X)A$ where $r_k \subseteq B$.

Lemma 3.15: Let, X be F.M. of an R-M. Ṁ A be an almost T-ABSO F.subm. Of X and F.singletons a_s,b_l of R. If I is F. ideal of R and B is F. subm. of X such that a_sIB⊆A and 4a_sIB⊆($A:_R X$) A then either a_sI⊆($A:_R X$) or a_s B⊆A or l B⊆A.

Proof: By in a similar way to proof of lemmaa (21.7) but we replace $0_1 \neq 4\ a_s$ IB by 4a_s IB⊄($A:_R X$) A. The following theorem gives a chareacterization of almost T-ABSO F. subm.

Theorem 3.16: Let I_1, I_2 be F. ideals of R and A, B F. subm. of F.M. X of an R-M. Ṁ . If A is an almost T-ABSO F. subm. of X such that $I_1 I_2$B⊆A-($A:_R X$)A and 8($I_1 I_2$+(I_1+I_2)($A:_R X$))(B+A)⊄($A:_R X$) or I_1B⊆A or I_2B⊆A.

Proof: By in s similar way to proof of theoeum (2.18) but we replace $0_1 \neq 8(I_1 I_2)+(I_1+I_2)(A:_R X))(B+A)$ by 8 ($I_1 I_2$) ($A:_R X$)) (B+A)⊄($A:_R X$)A.

The product AB = IJX where I, J are F. ideal of R and A, B are F. subm. of a multiplication FM of an R-M. Ṁ such that A = IX and B = JX (Atani and Farzalipour, 2007). By using this defination of products of F.subm, we give the following characterization of almost T-ABSO F. subm. under classes a finitely generated faithful multiplication FM.

Theorem 3.17: Let X be a finitely generated faithful multiplication F.M. of and R-M. Ṁ and A be a proper F.subm of X, then A is almost T-ABSO S.subm. in X iff whenever B, K and H are F.subm. of X such that BKH-{0_1}⊆A-($A:_R X$)A, then either BK⊆A or BH⊆or KH⊆A.

Proof: (⇒) Assume that A is almost T-ABSO F.subm. in X. by theorem (3.11), then ($A:_R X$) is a almost T-ABSO F. ideal in R. We have B = ($B:_R X$)X, K = ($K:_R X$) X and H = ($H:_R X$) X. Hence, BKH = ($B:_R X$) ($K:_R X$) ($H:_R X$) X. Assume that BKH-{0_1}⊆A-($A:_R X$) A but BK⊄A, BH⊄A and KH⊄A.

So that, ($B:_R X$) ($K:_R X$)⊄($A:_R X$), ($B:_R X$) ($H:_R X$)⊄($A:_R X$) and ($K:_R X$) ($H:_R X$)⊄($A:_R X$). Since, ($A:_R X$) is almost T-ABSO F. ideal in R, then ($B:_R X$) ($K:_R X$) ($H:_R X$)⊄($A:_R X$) or ($B:_R X$) ($K:_R X$) ($H:_R X$)⊆$(A:_R X)^2$.

If ($B:_R X$) ($K:_R X$) ($H:_R X$)⊄($A:_R X$), then BKH = ($B:_R X$) ($K:_R X$) ($H:_R X$) X⊄BKH = ($B:_R X$) ($K:_R X$) ($H:_R X$) X⊄$(A:_R X)^2$ then BKH = ($B:_R X$) ($K:_R X$)($H:_R X$)X⊆$(A:_R X)^2$ X = ($A:_R X$) A this is impossible. Now, if ($B:_R X$) ($K:_R X$) ($H:_R X$)⊆$(A:_R X)^2$, then thus, BK⊆A or BH⊆A or KH⊆A.

(⇐) To show that A is almost T-ABSO F. subm. in X, by theorem (3.11), it is sufficoent to show that ($A:_R X$) is almost T-ABSO F. ideal in R. Let $a_s b_t r_k$⊆($A:_R X$)-$(A:_R X)^2$ for F. singletons a_s,b_t,r_k of R. Hence, $a_s b_t r_k$X-{0_1}⊆A-($A:_R X$) A. Put a_sX = B, b_tX = K and r_k X = H, we have BKH-{0_1}⊆A-($A:_R X$) A. By assumption, BK⊆A or BH⊆A or KH⊆A, so that, $a_s b_t$ X = K⊆A or $a_s r_k$X⊆A or $b_t r_k$X⊆A. Then $a_s b_t$⊆($A:_R X$) or $a_s r_k$⊆($A:_R X$) or $b_t r_k$⊆($A:_R X$). Thus, ($A:_R X$) is almost T-ABSO F. ideal in R, so that, A is almost T-ABSO F. subm. in X.

Corollary 3.18: Let A be a proper F. subm. of a finitely generated faithful multiplication F.M. X of an R-M. Ṁ, then A is almost T-ABSO F. subm. in X iff whenever F. singletons x_v, y_h, z_m such that $x_v y_h$ z_m⊆A-($A:_R X$) A, then either x_v, y_h⊆A or x_v, z_m⊆A or $y_h z_m$⊆A.

Theorem 3.19: Let X be F.M. of an R-M. Ṁ and a be an almost T-ABSO F. subm. of X. Assume that F. singletons a_s, b_t of R and x_v⊆X such that $a_s b_t x_v$⊆($A:_R X$) A, $a_s b_t$ ⊄($A:_R X$), $a_s x_v$⊄A, $b_t x_v$⊄ A. Then, $a_s b_t$A⊆($A:_R X$)A.

Proof: Assume that $a_s b_t$A⊄($A:_R X$) A. Hence, there exists y_h⊆A such that $a_s b_t y_h$⊄($A:_R X$) A, so that, $a_s b_t$ (x_v+y_h)⊆A-($A:_R X$) A. Since, A is an almost T-ABSO F. subm., then $a_s b_t$A⊆($A:_R X$) or a_s(x_v+y_h) or b_l(x_v+y_h)⊆A, hence, a_s b_l⊆($A:_R X$) or a_s x_v⊆A or $b_t x_v$⊆A this is a discrepancy. Thus, $a_s b_t$A⊆($A:_R X$) A.

Proposition 3.20: Let X be F.M. of an R-M Ṁ, Y be an any F.M. of an R-M. M′ and A be F. subm. of X. Then, A is an almost T-ABSO F. subm. of X iff A⊕Y is an almost T-ABSO F. subm. of X⊕Y.

Proof: (⇒) Assume that A is an almost T-ABSO F. subm. of X. Let F. singletons $a_s b_t$ of R and (x_v, y_h)⊆X⊕Y such that $a_s b_t$ (x_v, y_h)⊆(A⊕Y)-(A⊕Y$:_R$X⊕Y) (A⊕Y), hence, $a_s b_t x_v$⊆A-($A:_R X$)A by (A⊕Y$:_R$X⊕Y) = ($A:_R X$). Since, A is an almost T-ABSO F. subm. of X, then $a_s b_t$⊆($A:_R X$) or $a_s x_v$⊆A or $b_t x_v$⊆A, so that, $a_s b_t$⊆(A⊕Y$:_R$X⊕Y) or a_s (x_v, y_h)⊆A⊕Y or b_t (x_v, y_h)⊆(A⊕Y). Thus, A⊕Y is an almost T-ABSO F. subm. of X⊕Y.

(⇐) Assume that A⊕Y is an almost T-ABSO F. subm. of X⊕Y. Let f. singletons $a_s b_t$ of R and x_v⊆X such that $a_s b_t x_v$⊆A-($A:_R X$)A. Hence, $a_s b_t$ (x_v, 0_1)⊆(A⊕Y)-(A⊕Y$:_R$X⊕Y) (A⊕Y). Since, A⊕Y is an almost T-ABSO F. subm. of X⊕Y that $a_s b_t$⊆(A⊕Y$:_R$X⊕Y) or a_s(x_v, 0_1)⊆A⊕Y or b_t(x_v, 0_1)⊆A⊕Y that is $a_s b_t$⊆($A:_R X$) or $a_s x_v$⊆A or $b_t x_v$⊆A. Thus, A is an almost T-ABSO F. subm. of X.

CONCLUSION

Through our research, we concluded to the concept F. prime subm. lead to the concept weakly F. prime subm. and through this we reached the concept weakly T-ABSO F. subm. One of the most important conclusions is the theorem (2.18). The other conclusion reached is almost T-ABSO F. subm. which was reached if A is a proper F. subm. of F. M. X then A is almost T-ABSO F. subm. in X iff is almost T-ABSO F. ideal and A = IX for some almost T-ABSO F. ideal I of R.

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