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A Closed U-BG-Filter and Completely Closed U-BG-Filter of a U-BG-BH-Algebra

Husein Hadi Abbass and Alaa Abdulla Hamza
Department of Mathematics, Faculty of Education for Girls, University of Kufa, Iraq

Abstract: In this study, we introduce the notion of closed U-BG-filter and completely closed U-BG-filter in U-BG-BH-algebra and observed that every closed and completely closed filter of a U-BG-BH-algebra is a closed and completely closed U-BG-filter. A necessary and sufficient condition is derived for every closed and completely closed U-BG-filter of U-BG-BH-algebra to become a closed or completely closed filter. Some properties of closed and completely closed U-BG-filter are studied with respect to homomorphism, Cartesian products and quotient U-BG-BH-algebra.

Key words: BH-algebra, U-BG-BH-algebra, filter, U-BG-filter, closed filter, completely closed filter, homomorphism, Cartesian products and quotient U-BG-BH-algebra

INTRODUCTION

Deeba (1980) introduced the notion of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via. a filter. Also, Deeba and Thaheem (1990) studied a filters in BCK-algebra in 1990. Hoo (1991) was presented the filters in BCI-algebra. Meng (1996) introduced the notion of BCK-filter in BCK-algebra. Abbass and Dahham (2016) discussed the concept of completely closed filter of a BH-algebra and completely closed filter with respect to an element of BH-algebra. The notion of U-BG-BH-algebra was introduced and extensivelys studied by Abbass and Mahdi (2014). This class of U-BG-BH-algebra was introducsd as a combination of the classes of BH-algebra and BG-algebra. Abbass and Hamza (2017) introduced the notion of U-BG-filter of U-BG-BH-algebra. In this study, the notion of closed U-BG-filter and completely closed U-BG-filter of U-BG-BH-algebra are introduced. Some researchers have studied the filters in a practical way different from what we study in our research, for example, by Hameed and Purushothaman. Also, by Jeyachitra and Manickam, researchers proposed and developed a simple and new reconfigurable millimeter-wave photonic transversal filter featuring high quality windowing property.

MATERIALS AND METHODS

In this study, some basic concepts about a BG-algebra, BH-algebra, associative BH-algebra, BH-ideal, regular subset of X, U-BG-BH-algebra, filter, U-BG-filter, subalgebra, normal subset and quotient U-BG-BH-algebra are given.

Definition 1; Kim and Kim (2008): A BG-algebra is a non-empty set X with a constant 0 and a binary operation "*" satisfying the following axioms:

- $x*x = 0$, for all $x\in X$
- $x*0 = x$, for all $x\in X$
- $(x*y)*(0*y) = x$, for all $x, y\in X$

Lemma 1; Kim and Kim (2008): Let (X,*, 0) be a BG-algebra. Then:

- The right cancellation law holds in X, i.e., $x*y = z*y$ implies $x = z$,
- $0*(0*x) = x$, for all $x\in X$
- If $x*y = 0$, then $x = y$, for all $x, y\in X$
- If $0*x = 0*y$, then $x = y$ for all $x, y\in X$
- $(x*(0*x))*x = x$ for all $x\in X$

Definition 2; Jun *et al.* (1998): A BH-algebra is a nonempty set X with a constant 0 and a binary operation "*" satisfying the following conditions:

- $x*x = 0$, for all $x\in X$
- $x*y = 0$ and $y*x = 0$ imply $x = y$, for all $x, y\in X$
- $x*0 = x$, for all $x\in X$

Definition 3; Abbass and Mhadi (2014): A U-BG-BH-algebra is defined to be a BH-algebra X in which there exists a proper subset U of X such that:

- $0\in U$, $|U| = 2$
- U is a BG-algebra

Corresponding Author: Husein Hadi Abbass, Department of Mathematics, Faculty of Education for Girls, University of Kufa, Iraq

Definition 4; Baik and Park (2010): A nonempty subset S of a BH-algebra X is called a BH-subalgebra or subalgebra of X if $x*y \in S$ for all $x, y \in S$.

Definition 5; Abbass and Dahham (2012c): Let, X be a BH-algebra, a non-empty subset N of X is said to be normal of X if $(x*a)*(y*b) \in N$ for any $x*y$ and $a*b \in N$, for all $x, y, a, b \in X$.

Theorem 1; Abbass and Dahham (2012c): Every normal subset N of a BH-algebra X is a subalgebra of X.

Definition 6; Abbass and Dahham (2012a): A BH-algebra X is called an associative BH-algebra if $(x*y)*z = x*(y*z)$, For all $x, y, z \in X$.

Theorem 2; Abbass and Dahham (2014): Let, X be an associative BH-algebra. Then the following proposition are hold:

- $0*x = x$, for all $x \in X$
- $x*y = y*x$, for all $x, y \in X$
- $x*(x*y) = y$, for all $x, y \in X$
- $(z*x)*(z*y) = x*y$, for all $x, y, z \in X$
- $x*y = 0 \Rightarrow x = y$, for all $x, y \in X$
- $(x*(x*y))*y = 0$, for all $x, y \in X$
- $(x*y)*z = (x*z)*y$, for all $x, y, z \in X$
- $(x*z)*(y*t) = (x*y)*(z*t)$, for all $x, y, z, t \in X$

Definition 7; Abbass and Mohammed (2013): A subset R of a BH-algebra X is said to be regular if it satisfies: $(\forall x \in R)(\forall Y \in X)(x*y \in R \Rightarrow y \in R)$.

Definition 8; Jun *et al.* (1998): Let, X be a BH-algebra and $I(\neq \emptyset) \subseteq X$. Then, I is called an ideal of X if it satisfies:

- $0 \in I$
- If $x*y \in I$ and $y \in I \Rightarrow x \in I$, for all $x \in X$

Definition 9; Saeid *et al.* (2009): An ideal I of a BCH-algebra X is called a closed ideal of X if for every $x \in I$, we have $0*x \in I$. We generalize the concept of an ideal to a BH-algebra.

Definition 10: An ideal I of a BH-algebra X is called a closed ideal of X if: $0*x \in I$, for all $x \in I$.

Definition 11; Abass and Dahham (2012a): An ideal I of a BH-algebra X is called a completely closed ideal of X if: $x*y \in I$, for all $x, y \in I$.

Definition 12; Abbass and Mahdi (2016): Let, X be a BH-agebra and I be a subset of X. Then, I is called a BH-ideal of X if it satisfies the following conditions:

- $0 \in I$
- $x*y \in I$ and $y \in I$ imply $x \in I$
- $x \in I$ and $y \in X$ imply $x*y \in I$, $I*X \subseteq I$

Definition 13; Abbass and Mhadi (2014): A nonempty subset I of a U-BG-BH-algebra X is called a U-BG-ideal of X related to U if it satisfies:

- $0 \in I$
- $x*y \in I \Rightarrow x \in I$, for all $x \in U$, $y \in I$

Definition 14; Abbass and Dahham (2012b): A filter of a BH-algebra X is a non-empty F of X such that:

- F_1: if $x \in F$ and $y \in F$, then $y*(y*x) \in F$ and $x*(x*y) \in F$
- F_2: if $x \in F$ and $x*y = 0$ then $y \in F$. Further F is a closed filter if $0*x \in F$, for all $x \in F$. In sequel we shall denote $y*(y*x)$ by x^y

Definition 15; Abbass and Dahham (2012b): Let, X be a BH-algebra and F is a filter. Then, F is completely closed filter if $x*y \in F$, for all $x, y \in F$.

Definition 16; Abbass and Hamza (2017): A nonempty subset F of a U-BG-BH-algebra X is called a U-BG- filter of X, if it satisfies (F_1) and:

- F_3: if $x \in F$ and $x*y = 0$ then $y \in F$. for all $y \in U$

Theorem 3; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra and F be a U-BG-filter of X such that $x*y \neq 0$, for all $y \notin F$ and $x \in F$. Then F is a filter of X.

Proposition 1; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra. Then, every filter of X is a U-BG-filter of X.

Proposition 2; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra and let $\{F_i, i \in \lambda\}$ be a family of U-BG-filters of X. Then, $\bigcap_{i \in \lambda} F_i$ is a U-BG-filter of X.

Proposition 3; Abbass and Hamza (2017): Let, X be a U-BG-BH-algebra and let $\{F_i, i \in \lambda\}$ be a chain of U-BG-filters of X. Then, $\bigcup_{i \in \lambda} F_i$ is a U-BG-filter of X.

Remark 1: Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be BH-algebra. A mapping $f: X \rightarrow Y$ is called a homomorphism if $f(x *_X y) = f(x) *_Y f(y)$ for any $x, y \in X$. A homomorphism f is called a monomorphism (resp., epimorphism) if it injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebra X and Y are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism f:

$X \to Y$. For any homomorphism f: $X \to Y$, the set {x∈X: f (x) = 0_Y} is called the kernel of f, denoted by ker (f), the set {f(x):x∈X} is called image of f, denoted by Im (f). Notice that $f(0_X) = 0_Y$. By Jun *et al.* (1998) and the set {x∈X: f(x) = y, for some y∈Y} is preimage of f, denoted by $f^{-1}(Y)$ by Abbass and Mhadi (2014).

Theorem 4; Abbass and Hamza (2017): Let, f: $(X,*,0) \to (Y,*',0')$ be a U-BG-BH- monomorphism and let F be a U-BG-filter of X. Then, f(F) is f(U)-BG-filter of Y.

Theorem 5; Abbass and Hamza (2017): Let, $f:(X,*,0) \to (Y,*', 0')$ be a U-BG-BH-isomorphism. If F is a U-BG-filter of Y. Then, $f^{-1}(F)$ is $f^{-1}(U)$-BG-filter of X.

Proposition 4; Abbass and Hamza (2017): Let, X and Y be two U-BG-BH-algebras and $f:(X,*,0) \to (Y, *', 0')$ be a BH-homomorphism. Then, ker(f) is a U-B G-filter of X.

Remark 2; Kim and Kim (2008): Let (X, *, 0) be a BG-algebra and let, N be a normal subalgebra of X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if x*y∈N where x, y∈X. Then, it is easy to show $\sim_N$ is an equivealence relation on X. Denote the equivalence class containing x by $[x]_N$, i.e., $[x]_N$ ={y∈X:$x \sim_N y$} and let X/N = {$[x]_N$: x∈X}. If * denoted on X/N by $[x]_N [Y]_N = [xy]_N$. Then (X/N,*, $[0]_N$) is a BG-algebra and it is called quotient BG-algebra of X by N, the researchers by Abbass and Dahham (2016), generalized this concept to BH-algebra to obtain (X/N, *, $[0]_N$) quotient BH-algebra of X by N.

Theorem 6; Abbass and Hamza (2017): Let (X, *, 0) be a U-BG- BH-algebra and N be a normal subalgebra, if F is a U-BG-filter in X, then, F/N N is U/N-BG- filter of (X/N, *', $[0]_N$).

Proposition 5; Abbass and Mhadi (2016): Let, X be a U-BG-BH-algebra. Then, every BH-ideal is a completely closed U-BG-ideal of X.

Theorem 7; Abbass and Dahham (2012): Let, N be a normal subalgebra of a BH-algebra X. Then, X/N is a BH-algebra.

Remark3; Abbass and Hamza (2017): Let {(X_i,*, 0_i): i∈λ} be a family of U_i-BG-BH-algebra. Define the Cartesian product of all X_i, i∈λ to be the structure $\prod_{i\in\lambda} X_i = (\prod_{i\in\lambda} X_i, \circledast, (0_i))$ where, $\prod_{i\in\lambda} X_i$ is the set of tuples {(x_i): for all i∈λ and x_i∈X} and whose binary operation ⊛ is given by $(x_i)\circledast(y_i) = (x_i *_i y_i)$, for all i∈λ and x_i, y_i∈X_i, Note that the binary operation ⊛ is componentwise.

Theorem 8; Abbass and Hamza (2017): Let ($\prod_{i\in\lambda} X_i$, ⊛, (0_i)) be a $\prod_{i\in\lambda} U_i$-BG-BH-algebra. If {(F_i, $*_i$, 0_i): i∈λ} be a family of U-BG-filter of X_i. Then $\prod_{i\in\lambda} F_i$ is a $\prod_{i\in\lambda} U_i$-BG-filter of the product algebra $\prod_{i\in\lambda} X_i$.

Definition 17; Zhang *et al.* (2001): A BH-algebra X is said to be normal BH-algebra if it satisfies the following conditions:

- 0*(x*y) = (0*x)*(0*y) for all x, y∈X
- (x*y)*x = 0*y, for all x, y∈X
- (x*(x*y))*y = 0 for all x, y∈X

RESULTS AND DISCUSSION

In this study, the notion of closed and completely closed U-BG-filter of U-BG-BH-algebra are introduced for our discussion, we shall link these notions with the notions which mentioned in preliminaries.

Definition 18: A U-BG-filter F of a U-BG-BH-algebra X is called a closed U-BG-filter of X if: 0*x∈F for all x∈F.

Example 1: Consider the U-BG-BH-algebra X = {0, 1, 2, 3, 4} with binary operation "*" defined as follows Table 1: where, U = {0, 1, 2}, the U-BG-filter F = {1, 3} is a closed U-BG-filter of X. But the U-BG-filter F = {1, 4} is not a closed U-BG-filter, since, 4∈F and 0*4 = 3∉F.

Definition 19: A U-BG-filter F of a U-BG-BH-algebra X is called a completely closed U-BG-filter of X if: x*y∈F for all x, y∈F.

Example 2: Consider the U-BG-BH-algebra X in example 1, the U-BG-filter F = {0, 1, 3} is a completely closed U-BG-filter of X but the U-BG-filter F = {0, 1, 4} is not a completely closed U-BG-filter of X, since, 1, 4∈F but 1*4 = 2∈F.

Remark 4: Let, X be a U-BG-BH-algebra. The filters F = {0} and F = X are completely closed U-BG-filters of X which are called a trivial completely closed U-BG-filters of X.

Proposition 6: Let, X be a U-BG-BH-algebra. Then, every closed filter of X is a closed U-BG-filter of X.

Proof: Directly by Proposition 1 and Definition 10.

Remark 5: The converse of Proposition 7 is not correct in general as in the following example.

Table 1: Closed U-BG-filter

*	0	1	2	3	4
0	0	1	2	3	3
1	1	0	1	3	2
2	2	2	0	1	2
3	3	3	1	0	0
4	4	3	1	4	0

Example 3: Consider the U-BG-BH-algebra X in Example (1), the U-BG-filter F = {0, 1, 3} of X is a closed U-BG-filter of X but it is not a closed filter of X because F is not a filter of X, since, 3∈F, 3*4 = 0 but 4∉F.

Proposition 7: Let X be a U-BG-BH-algebra. Then every completely closed filter of X is a completely closed U-BG-filter of X.

Proof: Its directly by Definition (19) and Proposition (1).

Remark 6: The converse of Proposition (7) is not correct in general as in the following example.

Example 4: Consider the U-BG-BH-algebra X in example 1, the U-BG-filter F = {0, 1, 3} of X is a completely closed U-BG-filter of X but it is not a completely closed filter of X because F is not a filter, since, 1∈F, 1*4 = 0 but 4∉F.

Proposition 8: Let, X be a U-BG-BH-algebra and F be a completely closed U-BG-filter of X. Then, 0∈F.

Proof : Let, F be a completely closed U-BG-filter of X and let x∈F. then x*x∈F (since, F is a completely closed U-BG-filter of X). So, by using Definition (2) (I), 0∈F (since, x*x = 0).

Proposition 9: Let, X be a U-BG-BH-algebra. Then, every completely closed U-BG-filter of X is a closed U-BG- filter of X.

Proof: Its directly from Proposition (8) and by applying Definition (19), we get 0*x∈I, so, I is a closed U-BG-filter of X.

Remark 7: The converse of Proposition (9) is not correct in general as in the following example.

Example 5: Consider a U-BG-BH-algebra, X = {0, 1, 2, 3, 4} with binary operation "*" defined as follows Table 2: where, U = {0, 1, 2}. The U-BG-filter F = {0, 3, 4} a closed U-BG-filter of X but it is not a completely closed U-BG-filter, since, 3, 4∈F and 3*4 = 2∉F.

Proposition 10: Let, X be a U-BG-BH-algebra and F be a completely closed U-BG-filter of X. Then, F is BH-algebra with the same binary operation on X and the constant 0.

Table 2: Closed U-BG-filter is not a completely closed U-BG-filter

*	0	1	2	3	4
0	0	1	2	0	0
1	1	0	2	2	1
2	2	1	0	1	2
3	3	2	3	0	2
4	4	1	2	0	0

Proof: straightforward.

Theorem 9: Let, X be a U-BG-BH-algebra and let F be a U-BG-filter of X. Then, F is a completely closed U-BG-filter of X if and only if F is a subalgebra of X contain in U.

Proof: Let, F be a completely closed U-BG-filter of X and let x, y∈F, then, x*y∈F (since, F is a completely closed U-BG-filter of X.), so, F is a subalgebra of X. Conversely, let F is a subalgebra. Let x, y∈F, so we have x*(x*y)∈F and y*(y*x)∈F [by Definition (4)],

Let x∈F, x*y = 0, y∈U, we get x∈U (Since F⊆U) and applying Lemma (1) (iii), we get x = y, hence, y∈F, therefore, F is a U-BG-filter of X. Now, let x, y∈F, so, x*y∈F for all x, y∈F (since, F is a subalgebra). Then F is completely closed U-BG-filter of X.

Lemma 2: Let, X be a U-BG-BH-algebra and let, N be a normal subset of X contain in U. Then, N is a completely closed U-BG-filter of X.

Proof: Directly from Theorem 1 and 9.

Proposition 11: Let, X be a U-BG-BH-algebra and F be a completely closed U-BG-filter of X. Then, F is a completely closed U-BG-ideal of X.

Proof: Let, F be a completely closed U-BG-filter of X.

- 0∈F (by Proposition 8)
- Let, x*y∈F, y∈F, x∈U. Since, F is a completely closed U-BG-filter of X. Then (x*y)*y∈F. Since, F is a U-BG-filter of X, so, we have y∈U. Now, take y = 0

Since, U is a BG-algebra, then (x*0)*0∈F, so, x∈F[by Definition (1) (ii).

- Let x, y∈F, so, x*y∈F (Since, F is a completely closed U-BG-filter of X), therefore, F is a completely closed U-BG-ideal of X

Proposition 12: Let, X be U-BG-BH-algebra and F be a closed U-BG-filter such that x*y≠0, for all y∉F and x∈F. Then, F is a closed filter of X.

Proof: Let, F be a closed U-BG-filter of X. Then, F is a U-BG-filter of X by applying Theorem 3, we get F is a filter of X. Now, let, x∈F. Then, 0*x∈F (since, F is a closed U-BG-filter). Therefore, F is a closed filter of X.

Proposition 13: Let, X be U-BG-BH algebra and F be a completely closed U-BG-filter of X such that x*y≠0, for all y∉F and x∈F. Then, F is a completely closed filter of X.

Proof: Let, F be a completely closed U-BG-filter of X. Then, F is a U-BG-filter of X. By applying Theorem 3, we get, F be a filter of X. Now, let x, y∈F, so, x*y∈F (since, F is a completely closed U-BG-filter). Therefore, F is a completely closed filter of X.

Theorem 10: Let, X be a normal U-BG-BH algebra and let, R be regular subset of X such that R⊆U. If R is a U-BG-ideal, then, R is a U-BG-filter of X.

Proof: Let, R be a U-BG-ideal of X.

- Let, x, y∈R, since, R is a U-BG-ideal. Then 0∈R. Now, (x*(x*y))*y∈R (By Definition (17) (iii)], so, x*(x*y)∈R [by Definition (13) (ii)]. Similarly, y*(y*x)∈R
- Let x∈R, x*y = 0, y∈U, then, x*y∈R and x∈R. By Definition 7, we get, y∈R, therefore, R is a U-BG-filter of X

Corollary 1: Let, X be a normal U-BG-BH algebra and let, R be regular subset of X which is contain in U. If R is a completely closed U-BG-ideal of X, then, R is a completely closed U-B-filter of X.

Proof: Let, R be a completely closed U-BG-ideal. Then, R is a U-BG-ideal of X by using theorem 10, we get R is a U-BG-filter of X. Now, let x, y∈R, hence, x*y∈R (since, R is a completely closed U-BG-ideal). Therefore, R is a completely closed U-B-filter of X.

Theorem 11: Let, X be an associative U-BG-BH-algebra and F be a U-BG-filter contain in U. Then, F is a completely closed U-BG-filter if and only if F is a completely closed U-BG-ideal of X.

Proof: Let, F be a completely closed U-BG- filter: 0∈F (by Proposition 8). Let, x*y∈F, y∈F, x∈U, then (x*y)*y∈F (since, F is a completely closed U-BG-filter). So, x* (y*y) ∈F (by Definition (6) by applying Definition (2) (i), we get, x*0∈F, hence, x∈F (By using Definition (2)(iii)). Therefore, F is a U-BG- ideal of X.

Let, x, y∈F. Then, x*y∈F (since, F is a completely closed U-BG-filter), so, F is a completely closed U-BG-ideal. Conversely, let F be a completely closed U-BG-ideal of X.

Table 3: Closed U-BG-ideal is not a closed U-BG- filter

*	0	1	2	3	4
0	0	1	2	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Let, x, y∈F, then x*y∈F, y*x∈F (since, F is a completely closed U-BG- ideal), then, we have, x*(x*y)∈F and y*(y*x) ∈F. Let, x∈F, x*y = 0,y∈U, since, F⊆U, we have x⊆U, so, x = y (since, U is a BG-algebra), then, y⊆F, hence, F is a U-BG-filter of X. Let, x, y∈F, then, x*y∈F (since, F is a completely closed U-BG-ideal), so, F is a completely closed U-BG-filter of X.

Theorem 12: Let, X be an associative U-BG- BH-algebra. Then, every closed U-BG-ideal of X is a closed U-BG-filter of X.

Proof: Let, X be an associative BH-algebra and let I be a closed U-BG-ideal of X.

- Let, x, y∈I, since, X is an associative, we obtain, y*(y*x) = (y*y)*x=0*x∈I (Since I is a closed U-BG-ideal) and x*(x*y) = (x*x)*y = 0*y∈I
- Let, x∈I and y∈U such that x*y = 0. Thus, x = y (by Theorem (2) (v)), so, y∈I, then we get, I is a U-BG-filter of X
- Let, x∈I. By Definition (8) (i), we obtain 0∈I, so, 0*x∈I (since, I is a closed U-BG- ideal), therefore, I is a closed U-BG- filter of X.

Remark 8: If X is not associative U-BG-BH-algebra then the Theorem 12 is not correct in general as in the following example:

Example 6: Let, X be a U-BG-BH-algebra, X = {0, 1, 2, 3, 4} with binary operation "*" defined as follows Table 3 and U = {0, 2} the closed U-BG-ideal I = {0, 1} is not a closed U-BG-filter, since, I is not a filter. Since, 1∈I, 1*2 = 0, 2∈U but 2∉I.

Proposition 14: Let, X be a U-BG-BH-algebra and the right cancellation low holds in X. Then, every BH-ideal of X is a completely closed U-BH-filter of X.

Proof: Let, I be a BH-ideal of X. By Proposition 5, we get I is a completely closed U-BG-ideal of X:

- Let, x, y∈I, so, we have x*(x*y)∈F, y*(y*x)∈F (since, I is a completely closed U-BG-ideal of X
- Let, x∈I, x*y = 0, y∈U, then, x*y = y*y, so, by Lemma (1) (i), we obtain x = y, imply that y∈F. Therefore, I is a U-BG-filter of X

- Let, x, y∈I. By Definition (12) (iii), we get x*y∈I. Then, I is a completely closed U-BG-filter of X

Remark 9: The converse of Proposition 14 is not correct in general as in the following example:

Example 7: Consider U-BG-BH-algebra X in Example 1, F = {0, 3} is a completely closed U-BG-filter of X but it is not a BH-ideal of X, since, 3∈F, 2∈X but 3*2 = 1 ∉F.

Proposition 15: Let $\{F_i, I\in\lambda\}$ be a family of closed U-BG-filters of a U-BG-BH-algebra X. Then, $\bigcap_{i\in\lambda} F_i$ is a closed U-BG-filters of X.

Proof: Since, F_i is a closed U-BG-filters, ∀i∈λ. then, F_i is a U-BG-filter, ∀i∈λ (by Definition 18), so, $\bigcap_{i\in\lambda} F_i$ is a U-BG-filter (by Proposition 2).

Now, let $x\in\bigcap_{i\in\lambda} F_i$, hence, xλ F_i∀i∈λ, so, $0*x\in F_i$∀i∈λ (Since, F is a closed U-BG-filter of X. By Definition 18), then $0*x\in\bigcap_{i\in\lambda} F_i$. So, we get, $\bigcap_{i\in\lambda} F_i$ is a closed U-BG-filter of X.

Proposition 16: Let $\{F_i, i\in\lambda\}$ be a family of completely closed U-BG-filters of a U-BG-BH-algebra X. Then, $\bigcap_{i\in\lambda} F_i$ is a completely closed U-BG-filter of X.

Proof: Since, F_i is a completely closed U-BG-filter of X,$\forall_i\in\lambda$, then, F_i is a U-BG-filter of X, $\forall_i\in\lambda$ (by Definition 19). By proposition 2, we obtain $\bigcap_{i\in\lambda} F_i$ is a U-BG-filter of X. Now, let x, y $\in\bigcap_{i\in\lambda} F_i$, so, x, y∈$F_i$ $\forall_i\in\lambda$, then, x*y$\forall_i\in\lambda F_i$, $\forall_i\in\lambda$ (since, F is a completely closed U-BG- filter), hence, x*y $\in\bigcap_{i\in\lambda} F_i$. Therefore, $\in\bigcap_{i\in\lambda} F_i$ is a completely closed U-BG-filter of X.

Proposition 17: Let $\{F_i, i\in\lambda\}$ be a chain of closed U-BG-filters of a U-BG-BH-algebra X. Then, $\bigcup_{i\in\lambda} F_i$ is a closed U-BG-filter of X.

Proof: Since, F_i is a closed U-BG-filter of X, $\forall_i\in\lambda$, then, F_i is a U-BG-filter of X, $\forall_i\in\lambda$ (by Definition 18), so, $x\in\bigcup_{i\in\lambda} F_i$ is a U-BG-filter of X (by Proposition 3). Now, let $x\in\bigcup_{i\in\lambda} F_i$, Then, there exist F_i, $F_k\in\{F_i\}$i∈λ such that x∈F_i, y∈F_k and either $F_i\subseteq F_k$ or $F_k\subseteq F_i$ (since, $\{F_i\}$, i∈λ is a chain]. If $F_i\subseteq F_k$, then, x∈F_k, hence, 0*x∈F_k (since, F_k is a closed U-BG-filter) Similarly, if $F_k\subseteq F_i\Rightarrow 0*y\in\bigcup_{i\in\lambda} F_i$, therefore, $\bigcup_{i\in\lambda} F_i$ is a closed U-BG-filter of X.

Proposition 18: Let, $\{F_i, i\in\lambda\}$ be a chain of completely closed U-BG-filters of a U-BG-BH-algebra X. Then, $\bigcup_{i\in\lambda} F_i$ is a completely closed U-BG-filter of X.

Proof: Since, F_i is a completely closed U-BG-filter of X, $\forall_i\in\lambda$, we get F_i is a U-BG-filter of X, $\forall_i\in\lambda$. By definition (19), Therefore, $\bigcup_{i\in\lambda} F_i$ is a U-BG-filter of X (by Proposition 3). Now, let x, y∈ $\bigcup_{i\in\lambda} F_i$, so, there exist F_i, $F_k\in\{F_i\}$i∈λ such that x∈F_i, y∈F_k, then, either $F_i\subseteq F_k$ or $F_k\subseteq F_i$ (since, $\{F_i\}$, i∈λ is a chain). If $F_i\subseteq F_k$, hence, x, y∈F_k, so, x*y∈F_k (since, F_k is a completely closed U-BG filter]. Similarly, if $F_k\subseteq F_i\Rightarrow x*y\in\bigcup_{i\in\lambda} F_i$. Therefore, $\bigcup_{i\in\lambda} F_i$ is a completely closed U-BG-filter of X.

Proposition 19: Let, f:(X,*,0)→(Y,*$'$, 0$'$) be a U-BG-BH-monomorphism and let F be a closed U-BG-filter of X. Then, f(F) is a closed f(U)-BG-filter of Y.

Proof: Let, F be a closed U-BG-filter of X, then, F is a U-BG-filter of X (by Definition (18)), so, by Theorem (4), we obtain f(F) is a f(U)-BG-filter of Y. Now, let y∈f(F). Then, there exist a∈F such that y = f(a), hence, $0'*'y = 0'*'f(a) = f(0)*'f(a) = f(0*a)$.

Since, 0*a∈F (by Definition 18), we get f(0* a)∈f (F), so, $0'*'y\in f(F)$. Therefore, f(F) is a closed f(U)-BG-filter of Y.

Proposition 20: Let, f:(X,*,0)→(Y,*$'$, 0$'$) be a U-BG-BH-monomorphism and let F be a completely closed U-BG-filter of X. Then, f(F) is a completely closed f(U)-BG-filter of Y.

Proof: Let, F be a completely closed U-BG-filter of X, so, F is a U-BG-filter of X (by Definition 19), then, f(F) is a f U-BG-filter of Y (Theorem 4).

Now, let, x, y∈f (F), so, there exist a, b∈F such that x = f(a), y = f (b), then $x*'y = f(a)*'f(b) = f(a*b)$. Since, F is a completely closed U-BG-filter of X, then, we obtain a*b∈F, so, f(a* b)∈f(F), hence, $x*'y\in f(F)$. Therefore, f(F) is a completely closed f(U)-BG-filter of X.

Theorem 13: Let, f:(X, *, 0)→(Y,*$'$, 0$'$) be a U-BG-BH-isomorphism. If F is a closed U-BG-filter of Y. Then, $f^{-1}(F)$ is a closed $f^{-1}(U)$-BG-filter of X.

Proof: Let, F be a closed U-BG-filter of Y. Then, F be a U-BG-filter of Y (by Definition 18), so, $f^{-1}(F)$ is a $f^{-1}(U)$-BG-filter of X (by Theorem 5). Now, let, $y\in f^{-1}(F)$, hence, we have, f(y)∈F and $0'*'f(y)\in F$ (since, F is a closed U-BG-filter of Y), then, f(0*y)∈F, so, we get, $0*y\in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a closed $f^{-1}(U)$-BG-filter of X.

Theorem 14: Let, f:(X, *, 0)→(Y, $*^{/}$, $0^{/}$) be a U-BG-BH-isomorphism. If F is a completely closed U-BG-filter of Y. Then, f^{-1}(F) is a completely closed f^{-1}(U)-BG-filter of X.

Proof: Let, F be completely a closed U-BG-filter of Y. So, F be a U-BG-filter of Y (by Definition 19), then, f^{-1}(F) is a f^{-1}(U)-BG-filter of X (by Theorem 5). Now, let, x, y∈f^{-1} (F), hence, f(x)∈F, f(y)∈F. Since, F is a completely closed U-BG-filter of Y, we obtain f(x)$*^{/}$ f(y)∈F, then, f (x*y)∈F, so, we get x*y∈f $^{-1}$(F), therefore, f^{-1} (F) is a completely closed f^{-1} (U)-BG-filter of X.

Proposition 21: Let, f:(X, *, 0)→(Y, $*^{/}$, $0^{/}$) be a U-BG-BH-homomorphism. Then ker (f) is a completely closed filter of X.

Proof: Let, f:(X, *, 0)→(Y, $*^{/}$, $0^{/}$) be a U-BG-BH-homomorphism. Then ker(f) is a U-BG-filter (by Proposition 4). Now, let, x, y∈ker (f), we have f(x) = f (y) = $0^{/}$ and then, f(x*y) = f(x) $*^{/}$ f(y) = $0^{/}$ by Remark (1), we get x*y∈ker (f), therefore, ker (f) is a completely closed U-BG-filter of X.

Proposition 22: Let, f:(X, *, 0)→(Y, $*^{/}$, $0^{/}$) be a U-BG-BH-homomorphism. Then, ker(f) is a closed filter of X.

Proof: Directly by Proposition 9 and 21.

Proposition 23: Let, X be a U-BG-BH-algebra, N be a normal subalgebra of X and F be a closed U-BG-filter of X. Then, F/N is a closed U/N-BG-filter of X/N.

Proof: Let, F be a closed U-BG-filter of X, so, F is a U-BG-filter of X, (by Definition 18), then, F/N is a U/N-BG-filter of X/N (by Theorem 6). Now, let $(0)_N$, $(y)_N$ ∈F/N, since, 0*y∈F (by F is closed U-BG-filter], so $[0]_{N*}$ $[y]_{N=}$ $[0*y]_N$ ∈F/N, therefore, F/N is a closed U/N-BG-filter in X/N.

Proposition 24: Let, X be a U-BG-BH-algebra, N be a normal subalgebra of X and F is a completely closed U-BG-filter of X. Then, F/N is a completely closed U/N-BG-filter of X/N.

Proof: Let, F be a completely closed U-BG-filter of X by Definition (19), we obtain F is a U-BG-filter of X, then, F/N is a U/N-BG-filter of X/N [by Theorem (6).

Now, let $[x]_N$, $[y]_N$∈F/N, so, $[x]_{N*}$ $[y]_{N=}$ $[x*y]_N$∈F/N [since x*y∈F by F is completely closed U-BG-filter], then, we get F/N is a completely closed U/N-BG-filter in X/N.

Corollary 2: Let, X be a U-BG-BH-algebra, N be a normal subalgebra of X and F is a completely closed U-BG-filter in X. Then, F/N is a closed U/N-BG-filter of X/N.

Proof: Let, F be a completely closed U-BG- filter of X. By Proposition (9), we get F is a closed U-BG- filter of X, then, F/N is a closed U/N-BG-filter in X/N. By Proposition (24).

Theorem 15: Let ($\prod_{i\in\lambda}$ X_i, ⊛,(0_i)) be a $\prod_{i\in\lambda}$ U_i-BG-BH-algebra. If {F_i: i∈λ} be a family of a closed U_i-BG-filter of X_i. Then $\prod_{i\in\lambda}$ F_i is a closed $\prod_{i\in\lambda}$ U_i-BG-filter of the product algebra $\prod_{i\in\lambda}$ X_i.

Proof: Let {F_i: i∈λ} be a family of a closed U_i-BG-filter of X_i. By Definition (18), we obtain {F_i: i∈λ} be a family of a U_i-BG-filter of X_i, then, $\prod_{i\in\lambda}F_i$ is a $\prod_{i\in\lambda}U_i$-BG-filter of $\prod_{i\in\lambda}$ X_i (by Theorem 8). Now, let y = (yi)∈$\prod_{i\in\lambda}F_i$ for all y_i∈F_i and i∈λ, so, (0_i)⊛(y_i) = $(0_i{*_i}y_i)$, since, F_i is a closed U_i-BG-filter of X_i, then $0{*_i}y_i$∈F_i (By definition 18), hence, (0)⊛ $_i(y_i)$∈$\prod_{i\in\lambda}F_i$, then, $\prod_{i\in\lambda}F_i$ is a closed $\prod_{i\in\lambda}U_i$-BG-filter of $\prod_{i\in\lambda}X_i$.

Theorem 16: Let ($\prod_{i\in\lambda}X_i$, ⊛, (0_i)) be a $\prod_{i\in\lambda}U_i$-BG-BH-algebra. If {(F_i, $*_i$, 0_i): i∈λ} be a family of a completely closed U_i-BG-filter of X_i. Then, $\prod_{i\in\lambda}F_i$ is a completely closed $\prod_{i\in\lambda}U_i$-BG-filter of the product algebra $\prod_{i\in\lambda}X_i$.

Proof: Let {F_i: i∈λ} be a family of a completely closed U_i-BG-filter of X_i. Then {F_i: i∈λ} be a family of a U_i-BG-filter of X_i (by Definition 19), so,$\prod_{i\in\lambda}F_i$ is a $\prod_{i\in\lambda}U_i$-BG-filter of $\prod_{i\in\lambda}X_i$ (by Theorem 8). Now, let x = (x_i), y = (y_i)∈ $\prod_{i\in\lambda}F_i$ for all x_i, y_i,∈F_i,x_i, y_i∈F_i and i∈λ, then x⊛y = (x_i)⊛(y_i) = $(x_i{*_i}y_i)$∈ $\prod_{i\in\lambda}F_i$, since, F_i is a completely closed U_i-BG-filter of X_i, then $x_i{*_i}y_i$∈F_i, hence, (x_i)⊛(y_i)∈ $\prod_{i\in\lambda}F_i$. Therefore, $\prod_{i\in\lambda}F_i$ is a completely closed $\prod_{i\in\lambda}U_i$-BG-filter of $\prod_{i\in\lambda}X_i$.

Proposition 25; (Extension property for closed U-BG-filter in U-BG-BH-algebra): Let X be a normal U-BG-BH-algebra F is a completely closed U-BG-filter of X, G is a closed ideal of X such that G⊆U. Then G is a completely closed U-BG-filter of X.

Proof: Let x, y∈G, since, F be a completely closed U-BG-filter of X, then, 0∈F (by Proposition 8), so, 0∈G (since, F⊆G) by Definition (17), we get (x*(x*y))*y∈G. So, by Definition (8), we obtain x*(x*y)∈G. Similarly, y*(y*x)∈G.

Let, x∈G, x*y = 0, y∈U, so, x∈U (since, G⊆U). Then, x*y = y*y, imply that x = y (By Lemma (1)), so, we obtain y∈G. Therefore, G is U-BG-filter of X. Now, since, G is a closed ideal of X, thus, 0*y∈G (by Definition 10). By Definition (17), we obtain (x*y)*x∈G. So, we have (x*y)*x∈G, x∈G, then x*y∈G (Since, G is an ideal of X), therefore, G is a completely closed U-BG-filter of X.

CONCLUSION

In this study, the notions of closed and completely closed U-BG-filter of U-BG-BH-algebra are introduced. Furthermore, the results are examined in terms of the relationship between closed and completely closed U-BG-filters. In addition, the relationship between the closed and completely closed U-BG-filters with the other filters as well as some special ideals are also presented the important characteristics of closed and completely closed U-BG-filters are analyzed.

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