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On the Metro Domination Number of Cartesian Product of $P_m\_P_n$ and $C_m\_C_n$

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Abstract: Let G = (V, E) be a graph. A set S⊆V is called resolving set if for every u, v∈V there exist w∈V such that d(u, w) ≠ = d(v, w). The resolving set with minimum cardinality is called metric basis and its cardinality is called metric dimention and it is denoted by β(G). A set D⊆V is called dominating set if every vertex not in D is adjacent to at least one vertex in D. The dominating set with minimum cardinality is called domination number of G and it is denoted by γ(G). A set which is both resolving set as well as dominating set is called metro dominating set. The minimum cardinality of a metro dominating set is called metro domination number of G and it is denoted by γβ(G). In this study we determine on the metro domination number of cartesian product of P_m P_n and C_mC_n.

Key words: Metric dimension, landmark, dominating set, metro dominating set, cardinality, product

INTRODUCTION

All the graphs considered are simple, finite and connected. Given a graph G = (V, E) and u, v∈V, d_G (u, v) (or simply d(u, v)) denotes as distance between u and v in G, i.e., the length of a shortest u-v path.

Harary and Melter (1976) introduce the metric dimension graph G. The vertex and edge sets of a graph G are represented as V(G) and E(G). The distance between vertices u, w∈V(G) is represented as dG(v, w) or d(v, w). A vertex a∈V(G) resolves a pair of vertices v, w∈V(G) if d(v.a) ≠ d(w, a). A set of vertices S⊆V(G) resolves G and S is a resolving set of G, if every pair of distinct vertices of G are resolved by some vertex in S. A resolving set S of G with minimum cardinality is a metric basis of G and its cardinality is the metric dimension of G, denoted by β(G).

Let D be a dominating set, if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number of the graph G and denoted by γ(G). The cartesian product of two graphs G, H is a graph with vertex set V(G) V(H) and ((g_1, h_1), (g_2, h_2)) ∈E(GH) if and only if either g_1 = g_2 and (h_1, h_2) ∈E(H) or (g_1, g_2) ∈E(G) and h_1 = h_2 (Dirac, 1952).

Vizing (1963) was initiated by the domination number of the cartesian products of graphs and also were intensively investigated in the past (Vizing, 1963; El-Zahar and Pareek, 1991; Jacobson and Kinch, 1983). We define a metro dominating set and which can be served as a better alternating for the locating dominating set as A dom inating set D of V(G) having the property that for each pair of vertices u, v there exists a vertex a in D such that d(u, a)≠d(v, a) is called the metro dominating set of G. The minimum cardinality of a metro dominating set of G is called metro domination number of G, denoted by γβ(G) (Buckley and Harary, 1990).

Remark 1.1: For any connected graph G,γβ(G)≥max f{γG), β(G)}.

MATERIALS AND METHODS

On the metro domination number of Cartesian product of P_mP_n: In this study, we determine some of known result on the metro domination number of cartesian product of P_m, P_n.

Theorem 2.1: For all m, n,γβ (P_2, P_n) = [n+1/ 2] n≥4.

Proof: Consider P_2, P_n as two canonical copies of P_n with vertices lebeled u_1, u_2, ... , u_n and v_1, v_2, ... , v_n with for each i, u_i v_i the only edges between the two paths. Domination number γ(P_2, P_n) = [n+1/2] (Jacobson and Kinch, 1983) and metric dimension β(P_m, P_n) = 2 (Khuller *et al.*, 1996). The dominating set is also serves as metric set, thus, (Eq. 1):

$$\gamma\beta(P_2, P_n) \geq \left\lceil \frac{n+1}{2} \right\rceil \quad (1)$$

To prove the reverse inequality we find a metro dominating set of cardinality n+1/2. We define a set D as follows:

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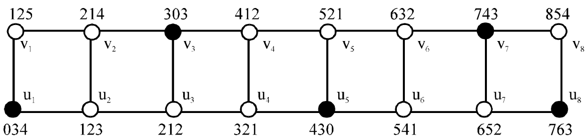

Fig. 1: $\gamma\beta(P_2, P_8) = 5$

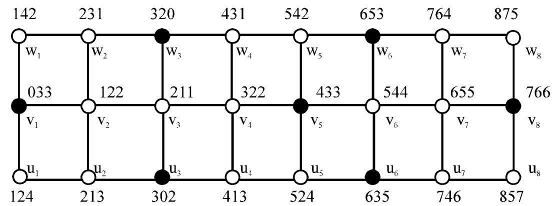

Fig. 2: $\gamma\beta(P_3, P_8) = 7$

- $D_1 = \{u_{4k-3}: k\geq 1\}$, n = 1(mod 4)
- $D_2 = \{v_{4k-1}: k\geq 1\}$, n = 3 (mod 4)

Choose D in above cases and then $|D| = [n+1/2]$ proved and D is a dominating set, in fact for every $v_j \in V\text{-}D$ by the choice of D, at least one of v_{j-1} or v_{j+1} must be in D and which dominates v_j, by using Khuller *et al*. (1996), D is resolving set. Hence, (Eq. 2):

$$\gamma\beta(P_2, P_n) \leq \left\lceil \frac{n+1}{2} \right\rceil \tag{2}$$

Therefore, from Eq. 1 and 2:

$$\gamma\beta(p_2, P_n) = \left\lceil \frac{n+1}{2} \right\rceil \tag{3}$$

Ex: The minimal metro dominating set of a graph G of Fig. 2 is 5:

Theorem 2.2; For all m, n:

$$(P_3, P_n) = \begin{cases} n - \left\lceil \frac{n-1}{4} \right\rceil & \text{if } n \geq 3,\ n \neq 4k+6,\ k \geq \text{ifn} \\ (n+1) - [n-1] & \text{if } n = 4k + 6k \geq 1 \end{cases}$$

Proof: Consider P_3, P_n as three canonical copies of P_n with vertices lebeled $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ with for each I, u_i, v_i, w_i the only edges between the three paths. Dominatio number $\gamma(P_3, P_n)$ = n- [n-1/4] (Jacobson and Kinch, 1983) and metric dimension $\beta(P_m, P_n) = 2$ (Khuller *et al*., 1996). The dominating set is also serves as metric set, thus, (Eq. 4):

$$\gamma\beta(P_3, P_n) \geq n - \left\lceil \frac{n\text{-}1}{4} \right\rceil \tag{4}$$

To prove the reverse inequality we find a metro dominating set of cardinality n [n-1/4]. We define a set D as follows:

- $D_1 = \{u_{4k-1}: k\geq 1\}$, n = 3(mod4)
- $D_2 = \{v_{4k-3}: k\geq 1\}$, n = 1(mod4)
- $D_3 = \{w_{4k-1}: k\geq 1\}$, n = 3(mod4)

Choose D in above cases and then $|D|$ = n-[n-1/4] proved and D is a dominating set, in fact for every $v_j \in V\text{-}D$, by the choice of D, at least one of v_j-1 or v_j+1 must be in D and which dominates v_j, by using Khuller *et al*. (1996), D is resolving set. Hence, (Eq. 5):

$$\gamma\beta(P_3, P_n) \leq n - \left\lceil \frac{n\text{-}1}{4} \right\rceil \tag{5}$$

Therefore, from Eq. 4 and 5:

$$\gamma\beta(P_3, P_n) = n - \left\lceil \frac{n\text{-}1}{4} \right\rceil \tag{6}$$

Ex: The minimal metro dominating set of a graph G, of Fig. 3 is 7.

Theorem 2.3: For all m, n, $\gamma\beta(P_4, P_n)$ = n, n = 4, 7, 8 and n $\geq$10.

Proof: Consider P_4, P_n as four canonical copies of P_n with vertices labeled $u_1, u_2,: u_n, v_1, v_2, \ldots, v_n\ w_1, w_2, \ldots, w_n$ and x_1, x_2, x_n with each for i, u_i, v_i, w_i, x_i, the only edges between the four paths. Domi-nation number $\gamma(P_4, P_n)$ = n (Jacobson and Kinch, 1983) metric dimention $\beta(P_m, P_n) = 2$ (Khuller *et al*., 1996). The dominating set is also serves as metric set, thus, (Eq. 7):

$$\gamma(P_4, P_n) \geq n \tag{7}$$

To prove the reverse inequality we find a metro dominating set of cardinality n. We define a set D as follows:

Case; 1: Suppose n = 4 + 3k for some integer k$\geq$0. Let. $D_1 = \{u_3, v_1, w_4, x_2\}$ $D_2 = \{x_{6t+1}, u_{6t}: t = 0, 1, \ldots, [k/2]\}$ and $D_3 = \{u_{6t+3}, w_{6t+4}, x_{6t+2}: t = 0, 1, \ldots, [k/2]\}$. Set $D = D_1 \cup D_2 \cup D_3$. It is easy to show that D is a dominating set of P_4, P_n of order n (Chartrand *et al*., 2000; Elumalai and Karthikeyan, 2014).

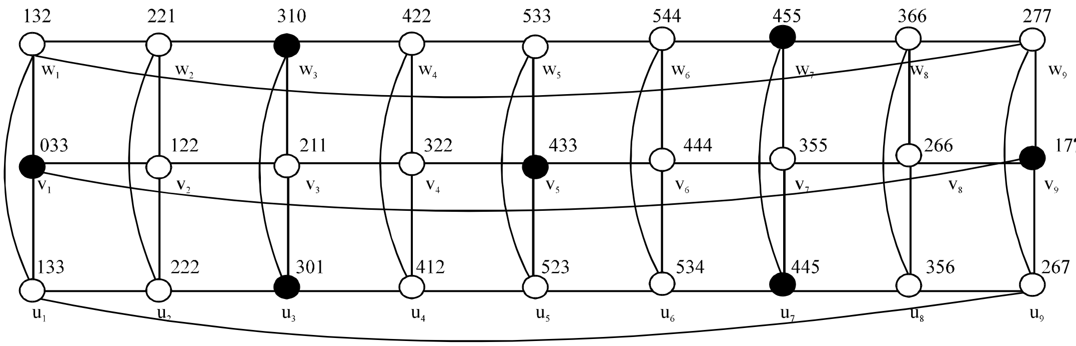

Fig. 3: $\gamma\beta(C_2, C_8) = 7$

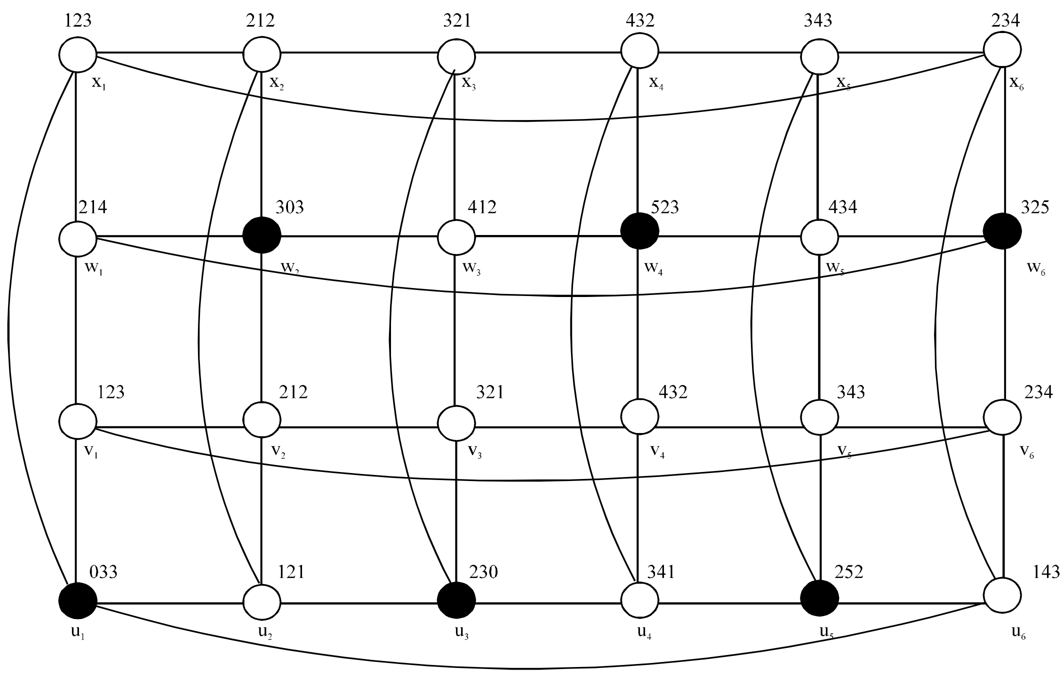

Fig. 4: $\gamma\beta(C_2, C_6) = 6$

Case 2: Suppose $n = 8+3k$ for some integer $k\geq 0$. Let. $D_1 = \{u_3, v_1, w_4, x_2, x_7, u_5, w_8, x_6\}$, $D_2 = x_{6t+4}, v_{6t+5}, u_{6t+3}$: $t = 0, 1, \ldots, [k/2]\}$ and $D_3 = \{u_{6t+11}, w_{6t+12}, x_{6t+10}$: $t = 0, 1, \ldots, [k/2]\}$. Set $D = D_1 \cup D_2 \cup D_3$. Again, it can shown that D is a dominating set of P_4, P_n of order n.

Case 3: Suppose $n = 12+3k$ for some integer $k\geq 0$. Let. $D_1 = \{u_3, v_7, u_{11}, v_1, u_5, v_9, w_4, w_8, w_{12}, x_2, x_6, x_{10}\}$, $D_2 = x_{6t+8}, v_{6t+9}, u_{6t+7}$: $t = 0, 1, \ldots, [k/2]\}$ and $D_3 = \{u_{6t+11}, w_{6t+12}, x_{6t+10}$: $t = 0, 1, \ldots, [k/2]\}$. Set $D = D_1 \cup D_2 \cup D_3$. As above, D is dominating set of P_4, P_n of order n.

Choose D in above cases and $|D| = n$ proved and D is a dominating set, in fact for every $v_j \in V\text{-}D$, by the choice of D, at least one of v_{j-1} or v_{j+1} must be in D and which dominates v_j by using Khuller *et al*. (1996), D is resolving set. Hence, $\gamma\beta(P_4, P_n) \leq n$. Therefore, from Eq. 1 and 2 $\gamma\beta(P_4, P_n) = n$:

RESULTS AND DISCUSSION

Generalisation result

Theorem 3.1: For all m, n, $\gamma\beta(P_m, P_n) \leq mn/5$.

Proof: Label the vertices of P_m, P_n as y_{ij} for $1 \leq i \leq m$, $1 \leq j \leq n$. Consider the vertices y_{ij}, where, $j = 2_i \pmod 5$. There are not more than mn/5 of these vertices and they dominate of all P_m, P_n except for approximately 1/5 th those vertices of the form y_{lj}, y_{mj}, y_{il}, y_{in}. It is sufficient to show that not more than $2/5\,(m+n)+2$ vertices of P_m, P_n are not dominated by this set. Hence, by using Jacobson and Kinch (1983), for lower bound, $1/5\ \gamma\beta(P_m, P_n) \leq 1/mn)\ (mn/5+2/5\ (m+n)+2)$. By using Khuller *et al*. (1996), we note that the dominating set which satisfies the above condition also serves as metric basis. Thus, $\gamma\beta(P_m, P_n) \leq mn/5$.

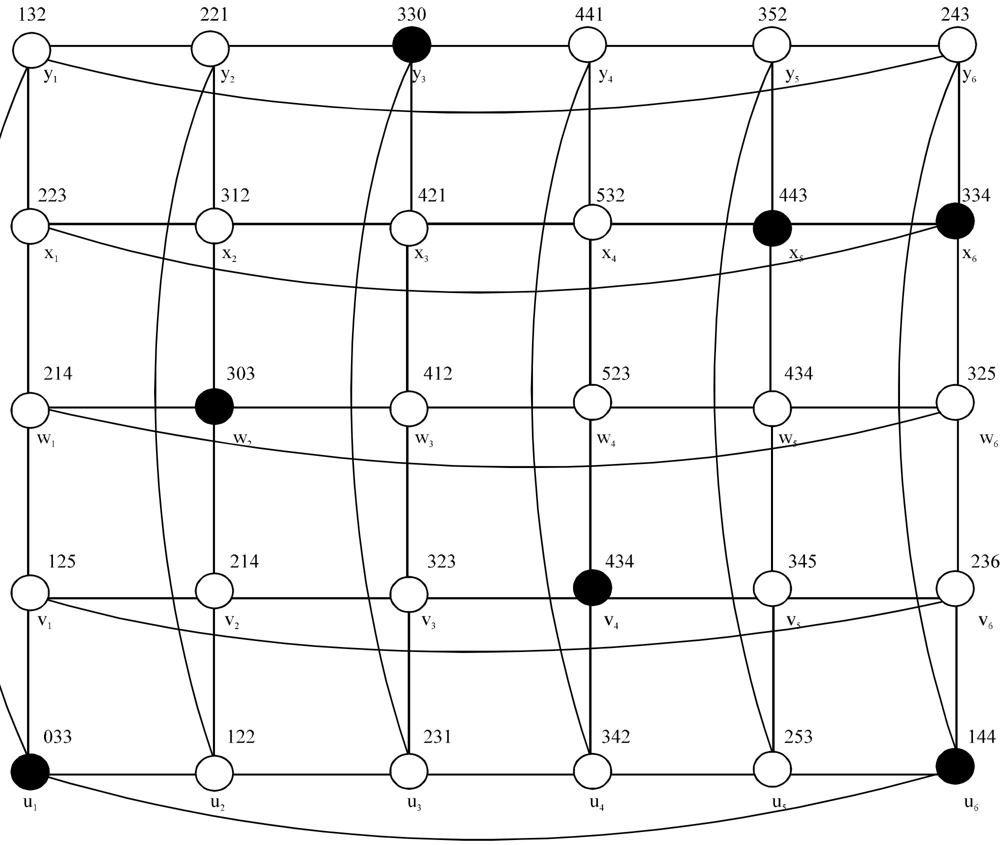

Fig. 5: $\gamma\beta(C_5, C_6) = 7$

On the metro domination number of cartesian product of C_m, C_n: In this study we determine some of known result on the metro domination number of cartesian product of C_m, C_n, where m = 3, 4, 5 and 6.

Theorem 4.1: For all m, n, $\gamma\beta(C_3, C_n) = n - [n/4]$, $n \geq 4$.

Proof: Let D be a dominating set of C_3, C_n. Let $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ are the vertices of C_1-C_3, respectively such that for each i, i = 1, 2, ..., n-1. Domination number $\gamma(C_3\text{-}C_n) = n\,[n/4]$ (Klavzar and Seifter, 1995) and metric dimension $\beta(C_m, C_n)$ is 3 if m or n is odd and 4 otherwise (Caceres *et al*., 2007). The dominating set is also serves as metric set, thus, (Eq. 8):

$$\gamma\beta(C_3, C_n) \geq n - \left[\frac{n}{4}\right] \tag{8}$$

To prove the reverse inequality we find a metro dominating set of cardinality $n[n/4]$. We define a set D as follows:

- $D_1 = u_{4l-1}$: $l \geq 1$, $n = 3(\text{mod } 4)$
- $D_2 = v_{4l-3}$: $l \geq 1$ g, $n = 1(\text{mod } 4)$
- $D_3 = w_{4l-1}$: $l \geq 1$ g, $n = 3(\text{mod } 4)$

Choose D in the above cases and then $|D| = n-[n/4]$ proved and D is a dominating set, in fact for every $v_j \in V\text{-}D$, by the choice of D, at least one of v_{j-1} or v_{j+1} must be in D and which dominates v_j, by using Caceres *et al*. (2007), D is resolving set. Hence, (Eq. 8):

$$\gamma\beta(C_3, C_n) \leq n - \left[\frac{n\text{-}1}{4}\right] \tag{9}$$

Therefore, from 11 and 12 $\gamma\beta(C_3, C_n) \geq n/4$.

Ex: The minimal metro dominating set of a graph G, of Fig. 3 is 7.

Theorem 4.2: For all m, n: $\gamma\beta(C_4, C_n) = n$, $n \geq 4$.

Proof: Let D be a dominating set of C_4, C_n. let $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$, $w_1, w_2, \ldots, w_n$ and $x_1, x_2, \ldots, x_n$ are the vertices of C_1-C_4 respectively such that for each i, i = 1, 2, ..., n-1. Domination $\gamma(C_4, C_n) = n$ (Klavzar and Seifter, 1995) and metric dimension $\beta(C_m, C_n)$ is 3 if m or n is odd and 4 otherwise (Caceres *et al*., 2007). The dominating set is also serves as metric set, thus, (Eq. 9):

$$\gamma\beta(C_3, C_n) \geq n - \left\lceil \frac{n}{4} \right\rceil \quad (10)$$

To prove the reverse inequality we find a metro dominating set of cardinality n. We define a set D as follows:

- $D_1 = \{u_{2l-1}: l \geq 1\}$, n = 1(mod2)
- $D_2 = \{w_{2l}: l \geq 1\}$ g, n = 2(mod2)

Choose D in the above cases and then |D| = n proved and D is a dominating set, in fact for every $v_j \in V$- D by the choice of D, at least one of v_{j-1} or v_{j+1} must be in D and which dominates v_j by using Caceres *et al.* (2007), D is resolving set. Hence, (Eq. 10):

$$\gamma\beta(C_4, C_n) \geq n \quad (11)$$

Therefore, from Eq. 10 and 11:

$$\gamma\beta(C_4, C_n) = n \quad (12)$$

Ex: The minimal metro dominating set of a graph G, of Fig. 4 is 6.

Theorem 4.3: For all m, n:

$$(C_5, C_n) = \begin{cases} n+1 & \text{if } n \geq 3,\ n \neq 5k, k \geq 1 \\ n & \text{if } n = 5k,\ k \geq 1 \end{cases}$$

Proof: Let D be a dominating set of C_5, C_n. Let $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$, $w_1, w_2, \ldots, w_n$, $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are the vertices of C_1-C_5 respectively such that for each i, i = 1, 2, ... , n-1. Dominatio number $\gamma(C_5\text{-}C_n) = n+1$ (Klavzar and Seifter, 1995) and metric dimension $\beta(C_m, C_n)$ is 3 if m or n is odd and 4 otherwise (Caceres *et al*., 2007). The dominating set is also serves as metric set, thus, $\gamma\beta(C_5, C_n) \geq n+1$. To prove the reverse inequality we find a metro dominating set of cardinality n+1. We define a set D as follows:

- $D_1 = \{u_{5l-4}: l \geq 1\}$, n = 1(mod 5)
- $D_2 = v_{5l-1}: l \geq 1\}$, n = 4(mod 5)
- $D_3 = w_{5l-3}: l \geq 1\}$, n = 2(mod 5)
- $D_4 = x_{5l}: l \geq 1\}$, n = 5(mod 5)
- $D_5 = y_{5l-2}: l \geq 1\}$, n = 3(mod 5)

Choose D in the above cases and then |D| = n+1 proved and D is a dominating set in fact for every $v_j \in V$-D by the choice of D, at least one of v_{j-1} or v_{j+1} must be in D. and which dominates v_j, by using Caceres *et al.* (2007), D is resolving set. Hence, (Eq. 12):

$$\gamma\beta(C_5, C_n) \geq n+1 \quad (13)$$

Therefore, from Eq. 11 and 12:

$$\gamma\beta(C_5, C_n) \leq n+1 \quad (14)$$

Ex: The minimal metro dominating set of a graph G, of Fig. 5 is 7.

Theorem 4.2: For all m, n: $\gamma\beta(C_6, C_n) = 2n$, $n \geq 4$.

Proof: Let D be a dominating set of C_6, C_n. Let $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n$, $w_1, w_2, \ldots, w_n$, ... , $x_1, x_2, \ldots, x_n$, y_1, y_2, y_n, and $z_1, z_2, \ldots, z_n$ are the vertices of C_1-C_6 and C_n, respectively such that for each i, i = 1, 2, ... , n-1. By using Klavzar and Seifter (1995) and metric dimension $\beta(C_m, C_n)$ is 3 if m or n is odd and 4 otherwise (Caceres *et al*., 2007). The dominating set is also serves as metric set, thus, $\gamma\beta(C_6, C_n) \geq 2n$. To prove the reverse inequality we find a metro dominating set of cardinality 2n. We define a set D as follows:

- $D_1 = v_{2l-1}: l \geq 1\}$, n = 1(mod 2)
- $D_2 = w_{2l}: l \geq 1\}$, n = 2(mod 2)
- $D_3 = y_{2l-1}: l \geq 1\}$, n = 1(mod 2)
- $D_4 = z_{2l}: l \geq 1\}$, n = 2(mod 2)

Choose D in the above cases and then |D| = 2n proved and D is a dominating set, in fact for every $v_j \in V$-D by the choice of D, at least one of v_{j-1} or v_{j+1} must be in D and which dominates v_j by using Caceres *et al.* (2007), D is resolving set. Hence, Hence $\gamma\beta(C_6, C_n) \leq 2n$. Therefore, from 1 and 2 $\gamma\beta(C_6, C_n) = 2n$.

CONCLUSION

Determination of the locating domination number of a graph is equivalent to finding the least number of monitors that can do the contain task in a given graph or network. Metro dominating set served as better alternating set for the locating dominating set. Using this concepts we are finding the results for Cartesian product of some graphs which is absolutely finer than the locating domination number.

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