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Approximation of Sine Series with Coefficient from Class of p-Supremum Bounded Variation Difference Sequences

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Abstract: Recently, the monotone decreasing coefficients of sine series has been generalized by class of p-supremum bounded variation sequences. Further, class of p-supremum bounded variation sequences can be generalized by class of p-supremum bounded variation difference sequences. In this study we compute error about approximation of sine series with coefficient from class p-supremum bounded variation difference sequences.

Key words: Difference sequence, error, p-supremum bounded variation, sine series, coefficients, approximation

INTRODUCTION

Chaundy and Jollife (1917) proved the following classical theorem:

Theorem 1: Suppose that $\{\alpha_k\}\subset[0, \infty)$ is decreasingly tending to zero. A necessary and sufficient conditions for the uniform convergence of the series is:

$$\sum_{k=1}^{\infty}\alpha_k \sin kx \quad \text{is } \lim_{k\to\infty} k\alpha_k = 0 \tag{1}$$

The decreasing monotone coefficients (Eq. 1) are said class of Monotone Sequences (MS) and has been generalized by many researchers such as Tikhonov (2008), Zhou *et al.* (2010) and Korus (2010). These classes are GMS (General Monotone Sequences), NBVS (Non-one sided Bounded Variation Sequences), MVBVS (Mean Value Bounded Variation Sequences) and SBVS (Supremum Bounded Variation Sequences). Zhou *et al.* (2010) proved that MS⊊GMS⊊NBVS and Korus showed that MVBVS⊊SBVS (Imron and Indrati, 2014).

Furthermore, Liflyand and Tikhonov (2011) generalized GMS to g^{MS}_p p-general monotone sequences), $1\leq p\leq\infty$. Let $\alpha = \{\alpha_n\}$ and β $\{\beta_n\}$ be two sequences of complex and positive numbers, respectively, a couple $(\alpha, \beta) \in g^{MS}_p$ if there exists C>0 such that:

$$\left(\sum_{k=n}^{2n-1}|\alpha_k-\alpha_{k+1}|^p\right)^{\frac{1}{p}} \leq C\beta_n$$

For p, $1\leq p<\infty$. Imron and Indrati (2013) generalized MVBVS and SBVS to $MVBVS_p$ (p-Mean Value Bounded Variation Sequences) and $SBVS_p$ (Supremum Bounded Variation Sequences). A couple $(\alpha, \beta)\in MVBVS_p$ if there exist C>0 and $\lambda\geq 2$ such that:

$$\left(\sum_{k=n}^{2n-1}|\alpha_k-\alpha_{k+1}|^p\right)^{\frac{1}{p}} \leq C\sum_{k=\left[\frac{\lambda}{n}\right]}^{[\lambda n]}\beta_k$$

$1\leq p<\infty$ and $(\alpha, \beta)\in SBVS_p$ if there exist C>0 and $\lambda\geq 1$ such that:

$$\left(\sum_{k=n}^{2n-1}|\alpha_k-\alpha_{k+1}|^p\right)^{\frac{1}{p}} \leq \frac{C}{n}\left(\sup_{m\geq\left[\frac{n}{\gamma}\right]}\sum_{k=m}^{2m}\beta_k\right)$$

$1\leq p<\infty$. A little modification of definition of $SBVS_p$ gives a class $SBVS2_p$. The couple (α, β) is p-supremum bounded variation sequences of second type, written $(\alpha, \beta)\in SBVS2_p$, if there exist C>0 and $\{b(k)\}\subset[0,\infty)$tending monotonically to infinity depending only on $\{\alpha_k\}$ such that:

$$\left(\sum_{k=n}^{2n-1}|\alpha_k-\alpha_{k+1}|^p\right)^{\frac{1}{p}} \leq \frac{C}{n}\left(\sup_{m\geq b(n)}\sum_{k=m}^{2m}\beta_k\right)$$

holds for p, $1\leq p<\infty$. Imron and Indrati (2013) have shown that $MVBVS_p⊊SBVS_p$. Imron and Indrati (2014) generalized of to in the following definition, we consider sequences $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ be two sequences of complex and positive numbers, respectively.

Definition 2: Let $n\in N$, a couple (a, β) is said to be p-Supremum Bounded Variation Sequences order n, written $(\alpha, \beta)\in SBVS_p(\Delta^n)$, if there exist positive constant C and $\gamma\geq 1$ such that:

$$\left(\sum_{k=n}^{2n-1}\left|\Delta^n\text{-}\alpha_k\right|^p\right)^{\frac{1}{p}} \le \frac{C}{n}\left(\sup_{i\ge\left[\frac{m}{\gamma}\right]}\sum_{k=i}^{2i}\beta_k\right) m\ge n,\ 1\le$$

$p \le \infty$, where $\Delta^n\text{-}\alpha_k = \Delta^{n-1}\text{-}\alpha_k\text{-}\Delta^{n-1}\text{-}\alpha_{k+1}$

Note that $(\alpha, \beta)\epsilon SBVS_p(\Delta^1)$ is exactly. This class more general than that one.

Definition 3: Let $n\epsilon N$, a couple (α, β) is said to be p-Supremum Bounded Variation of second type order n, written $(\alpha, \beta)\epsilon SBVS2_p(\Delta^n)$, if there exist $C>0$ and $\{b(k)\}\subset[0, \infty)$ tending monotonically to infinity depending only on $\{\alpha_k\}$ such that:

$$\left(\sum_{k=m}^{2m-1}\left|\Delta^n\text{-}\alpha_k\right|^p\right)^{\frac{1}{p}} \le \frac{C}{n}\left(\sup_{i\ge b(m)}\sum_{k=i}^{2i}\beta_k\right),\quad m\ge n$$

For $1\le p<\infty$. Note that $(\alpha, \beta)\epsilon SBVS2_p(\Delta^1)$ is exactly $(\alpha, \beta)\epsilon SBVS2_p(\Delta) = SBVS2_p$. In the present study by definition class of p-supremum bounded variation of difference sequences, like by Imron (2018) we shall compute the error sine series with coefficient from class p-supremum bounded variation difference sequences.

Definition and preliminaries: In the following definition, we consider sequences $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ and be two sequences of complex and positive numbers, respectively (Korus, 2010).

Definition 4: Let a class $SBVS_p(\beta, \Delta^n)$ be given. Class of $SBVS_p(\beta, \Delta^n)$ is defined as $\{\alpha: (\alpha, \beta)\epsilon SBVS_p(\Delta^n)\}$.

Definition 5: Let a class $SBVS2_p(\Delta^n)$ be given. Class of $SBVS2_p(\beta, \Delta^n)$ is defined as $\{\alpha: (\alpha, \beta)\epsilon SBVS2_p(\Delta^n)\}$. Furthermore, we discussed error computation of sine series with coefficients from class of supremum bounded p-variation difference sequences. Let:

$$f(x) = \sum_{v=1}^{\infty}\alpha_v \sin vx \qquad (2)$$

The function represent that the sum of series on these points converge so that the function $f\epsilon C([0, \pi])$. The notation $E_m(f)$ is the best approximation of f by trigonometric polynomials of order m (DeVor and Lorentz, 1991), where:

$$E_m(f) = \inf_{r\in Xm}\left\|f\text{-}p\right\| \qquad (3)$$

And p is a trigonometric polynomial order less or equal to n with:

$$p \in X_m = \left\{\begin{matrix} p\in C[0,2\pi]: p(x) = \alpha_0 + \\ \sum_{j=1}^{k}\left(\alpha_j \cos jx + cj \sin jx\right),\ k\le m \end{matrix}\right\}$$

And $\|.\|$ norm on L_1.

MATERIALS AND METHODS

This research was studied from literature and the supporting scientific journals to find a well understanding, then the results related to the research that has been published in the journal. In summary the method of the research is applying the new class of p-supremum bounded variation difference sequences to approximation of sine series.

RESULTS AND DISCUSSION

In this study, we compute the error about approximation of sine series with coefficient from class p-supremum bounded variation difference sequences.

Theorem 6: Let $\alpha\epsilon SBVS_p(\beta, \Delta^n)$, $1\le p<\infty$, with β real non-negative sequence, if $\{\alpha_n\}$ decreasing monotone:

$$\alpha_m = m^{-\frac{1}{p}}_{i\ge bm}\sup\sum_{k=i}^{2i}\beta_k$$

And:

$$\left(\sum_{k=m}^{\infty}\left|\Delta^t\text{-}\alpha_k\right|^p\right)^{\frac{1}{p}} < \left(\frac{C}{m}\sup_{i\ge bm}\sum_{v=i}^{2i}\beta_v\right)$$

m for $m1\le t\le n\text{-}1$, $m\ge n$, then:

$$\left|g(x)\text{-}S_{m-1}(g, x)\right| \le 6C(m+2M)m^{-\frac{1}{p}}\sup_{i\ge bm}\sum_{k=i}^{2i}\beta_k$$

where, C positive constant only depending on $SBVS_p(\beta, \Delta^n)$ with $m\ge n$, $x = \pi/M$ and $x\epsilon(0, \pi]$.

Proof: Calculate $g\ (x)\text{-}S_{m-1(g,\ x)}$ with:

$$S_m(g, x) = \sum_{j=1}^{m} cj \sin jx$$

We have:

$$g(x)\text{-}S_{m-1}(g, x) = \sum_{k=m}^{\infty} c_k \sin kx$$

Let x∈(0, π], for any we can find M∈N such that x∈(π/M+1, π/M]. Since:

$$\sum_{\nu=m}^{\infty} c_k \sin \nu x = \frac{\cos\frac{1}{2}x - \cos\left(m+\frac{1}{2}x\right)}{2\sin\frac{1}{2}x} \leq \frac{1}{\sin\frac{1}{2}x} \leq \frac{\pi}{x}$$

By Abel's transformation:

$$\sum_{k=m}^{\infty} c_k \sin \nu x = A = \sum_{k=m}^{\infty} \Delta c_k D_m(x) - c_m D_{m-1}(x) = S+T$$

With:

$$D_m(x) = \sum_{j=1}^{m} \sin jx$$

And:

$$S = \sum_{k=m}^{\infty} \Delta c_k D_m(x),\ T = -C_m D_m(x)$$

$$|S| \leq \frac{\pi}{x}|C_m| \text{ and } |T| \leq \frac{\pi}{x}|C_m|$$

So:

$$|A| \leq \frac{2\pi}{x}|C_m| \leq 2(M+1)|C_m|$$

$$\leq (m+2M)\left|\sum_{s=m}^{\infty} \Delta C_s\right|$$

$$\leq (m+2M)\sum_{s=m}^{\infty} |\Delta C_s|$$

Because of:

$$\Delta a_s = \Delta a_{s+1} + \Delta^2 a_{s+1} +, ..., +\Delta^{n-1} a_{s+1} + \Delta^n a_s \qquad (4)$$

We get:

$$|A| \leq (m+2M)\sum_{s=m}^{\infty} \left|\Delta\alpha_{s+1} +, ..., +\Delta^{n-1}\alpha_{s+1}\right| +$$

$$(m+2M)\sum_{s=m}^{\infty} \left|\Delta^n \alpha_s\right| = I_1 + I_2$$

By Holder inequality, we get:

$$I_1 \leq (m+2M)\sum_{s=0}^{\infty}\left[\left(\sum_{\nu=2^s m}^{2^{s+1}m-1} \left\|\Delta\alpha_{s+1} +, ..., \Delta^{n-1}\alpha_{s+1}\right\|^p\right)^{\frac{1}{p}} \left(\left(2^s m\right)\right)^{1-\frac{1}{p}}\right]$$

$$\leq (m+2M)\sum_{S=0}^{\infty}\left[\left(\frac{C}{\left(2^s m\right)^2} \sup_{i \geq b_{2^s m}} \sum_{\nu=i}^{2i} \beta_\nu\right)\left(\left(2^s m\right)\right)^{1-\frac{1}{p}}\right]$$

$$\leq 4\frac{(m+2M)(m)^{1-\frac{1}{p}}}{m} C \sup_{i \geq bm} \sum_{\nu=i}^{2i} \beta_\nu$$

$$\leq 4C(m+2M)m^{1-\frac{1}{p}} \sup_{i \geq bm} \sum_{k=i}^{2i} \beta_k$$

Further:

$$I_2 = (m+2M)\sum_{S=m}^{\infty} \left|\Delta^n \alpha_s\right| =$$

$$(m+2M)\sum_{S=0}^{\infty} \sum_{\nu=2^s m}^{2^{s+1}m-1} \left|\Delta^n \alpha_s\right| \leq$$

$$(m+2M)\sum_{S=0}^{\infty}\left[\left(\sum_{\nu=2^s m}^{2^{s+1}m-1} \left|\Delta^n \alpha_s\right|^p\right)^{\frac{1}{p}} \left(\left(2^s m\right)\right)^{1-\frac{1}{p}}\right]$$

We defined:

$$\alpha_m = m^{-\frac{1}{p}} \sup_{i \geq bm} \sum_{k=i}^{2i} \beta_k$$

And by Holder inequality we get:

$$I_2 \leq (m+2M)\sum_{S=0}^{\infty}\left[\left(\sum_{\nu=2^s m}^{2^{s+1}m-1} \left|\Delta^n \alpha_s\right|^p\right)^{\frac{1}{p}} \left(\left(2^s m\right)\right)^{1-\frac{1}{p}}\right]$$

$$\leq (m+2M)C\sum_{S=0}^{\infty} \frac{2^s m\, \alpha_{2^s m}}{2^s m}$$

$$\leq C\frac{(m+2M)}{m} m\alpha_m \sum_{s=0}^{t} \frac{1}{2^s}$$

$$\leq 2C(m+2M)m^{-\frac{1}{2}} \sup_{i \geq bm} \sum_{k=i}^{2i} \beta_k$$

So, we have:

$$\left|g(x) - S_{m-1}(g, x)\right| \leq 6C(m+2M)m^{-\frac{1}{p}} \sup_{i \geq bm} \sum_{k=i}^{2i} \beta_k$$

With:

$$m \geq n,\ x = \frac{\pi}{M} \text{ and } x \in (0, \pi]$$

Theorem 7: Let $\alpha\epsilon SBVS_p(\beta, \Delta^n)$ $1\leq p<\infty$ with β real non-negative sequence, if:

$$m\left(\sum_{k=m}^{\infty}\left|\Delta\alpha_{k+1}+,...,+\Delta^{n-1}\alpha_{k+1}\right|^p\right)^{\frac{1}{p}}\leq$$

$$\frac{C}{m}\left(\sup_{i\geq bm}\sum_{k=i}^{2i}\beta_k\right), m\geq n$$

And:

$$m^{1-\frac{1}{p}}\sup_{1\geq bm}\sum_{k=1}^{21}\beta_k$$

Decreasing monotone, then:

$$E_m(f)\leq 2\max_{v\in[1,m]} v\left|C_{v+m}\right|+6C\ m^{1-\frac{1}{p}}\sup_{i\geq bm}\sum_{k=i}^{2i}\beta_k+$$

$$6C(m+2M)(2m)^{-\frac{1}{p}}\sup_{i\geq bm}\sum_{k=i}^{2i}\beta_k$$

With C positive constant depending class of on $SBVS_p(\beta, \Delta^n)$ and $M = \pi/x$ for $x\epsilon(0, \pi]$, $m\geq n>1$.

Proof:

$$E_m(g) = \inf_{p\in Xm}\|g-p\|\leq\|g-p^{\square}\|$$

With:

$$p^{\square}(x) = \sum_{v=1}^{m}C_v\sin vx+\sum_{v=1}^{m}C_{m+v}\sin(m-v)x$$

We get:

$$\left|g(x)-\left(\sum_{v=1}^{m}C_v\sin vx+\sum_{v=1}^{m}C_{m+v}\sin(m-v)x\right)\right|\leq$$

$$\left|\sum_{v=m+1}^{\infty}C_v\sin vx-\sum_{v=1}^{m}C_{n+v}\sin(m-v)x\right|=$$

$$\left|\sum_{v=2m+1}^{\infty}C_v\sin vx+\sum_{v=n+1}^{m}C_v\sin vx-\sum_{v=1}^{m}C_{n+v}\sin(m-v)x\right|=$$

$$\left|\sum_{v=2m+1}^{\infty}C_v\sin vx+\sum_{v=1}^{m}C_{m+v}\left(\sin(m-v)x-\sin(m-v)x\right)\right|=$$

$$\left|\sum_{v=2+m1}^{\infty}C_v\sin vx-2\sum_{v=1}^{m}C_{m+v}\sin vx\cos mx\right|\leq$$

$$2\left|\sum_{v=1}^{m}C_{m+v}\sin vx\right|+\left|\sum_{v=2m+1}^{m}C_v\sin vx\right| = A+B$$

With:

$$A\ 2\left|\sum_{v=1}^{m}C_{m+v}\sin vx\right|$$

And:

$$B = \left|\sum_{v=2m+1}^{\infty}C_v\sin vx\right|$$

For any $x\epsilon(0, \pi)$ and $j = [\pi/2x]$ by [x] is integer part of x. If $j\geq m$, then:

$$A = \left|\sum_{v=1}^{m}C_{m+v}\sin vx\right|\leq x\sum_{v=1}^{m}v\left|\alpha_{m+v}\right|\leq$$
$$\pi\frac{m}{2j}\max_{1\leq v\leq m}v\left|\alpha_{m+v}\right|\leq\frac{\pi}{2}\max_{1\leq v\leq m}v\left|C_{m+v}\right| \quad (5)$$

If $j<m$, then sum:

$$\sum_{v=1}^{m}C_{m+v}\sin vx$$

Broken into:

$$\sum_{v=1}^{m}C_{m+v}\sin vx=\sum_{v=1}^{j}C_{m+v}\sin vx+\sum_{v=j+1}^{m}C_{m+v}\sin vx = I_1+I_2$$

By Eq. 5, we get:

$$\left|\sum_{v=j+1}^{m-1}\Delta C_{v+m}D_v(x)\right|\leq\frac{\pi}{2}\sum_{v=j+1}^{m-1}\left|\Delta C_{v+m}\right|\left|\ I_1\right|\leq$$
$$x\sum_{v=1}^{j}v\left|C_{v+m}\right|\leq\frac{\pi}{2}\max_{1\leq v\leq m}v\left|C_{m+v}\right| \quad (6)$$

By Abel's transformation:

$$\left|\ I_1\right|\leq\left|\sum_{v=j+1}^{m-1}\Delta C_{v+m}D_v(x)\right|+\left|\alpha_{2m}D_m^{\square}(x)\right|+\left|\alpha_{j+1+m}D_m^{\square}(x)\right|$$

With:

$$D_m^{\square}(x) = \sum_{v=1}^{s}\sin vx$$

For $x\epsilon(0, \pi]$ we find $|D_s^{\cdot}(x)|\leq\pi/x$, so that, from Eq. 4, we get:

$$\frac{\pi}{x}\sum_{v=j+1}^{m-1}\left|\Delta C_{v+m}\right|\leq\frac{\pi}{x}\sum_{v=j+m+1}^{2m-1}\left|\begin{matrix}\Delta\alpha_{v+1}+,...,+\\\Delta^{n-1}\alpha_{v+1}+\Delta^n\alpha_v\end{matrix}\right|\leq$$

$$\frac{\pi}{x}\sum_{v=j+m+1}^{2m-1}\left|\Delta\alpha_{v+1}+,...,+\Delta^{n-1}\alpha_{v+1}\right|+\left|\Delta^n\alpha_v\right|\leq$$

$$\frac{\pi}{x}\sum_{s=0}^{t}\sum_{v=j+m+1}^{\left[2(2)^{t+1}j+m\right)-1}\left|\Delta\alpha_{v+1}+,...,+\Delta^{n-1}\alpha_{v+1}\right|+\left|\Delta^n\alpha_v\right|$$

where, t non-negatif integer and $2^tj\leq m\leq 2^{t+1}j$. Further we defined:

$$\alpha_m = m^{-\frac{1}{p}}\sup_{i\geq bm}\sum_{k=i}^{2i}\beta_k$$

And we get:

$$\frac{\pi}{x}\sum_{s=0}^{t}\sum_{v=2^s j+m}^{\left[2(2)^{s+1} j+m\right)-1}\left|\Delta\alpha_{v+1}+, \ldots, +\Delta^{n-1}\alpha_{v+1}\right|+ \left|\Delta^n\alpha_v\right| = I_2+I_4$$

By Holder inequality, we get:

$$I_2 = \frac{\pi}{x}\sum_{s=0}^{t}\left[\left(\sum_{v=2^s j+m}^{\left[2(2)^{s+1} j+m\right)-1}\left\|\begin{matrix}\Delta\alpha_{v+1}+, \ldots, + \\ \Delta^{n-1}\alpha_{v+1}\end{matrix}\right\|^p\right)^{\frac{1}{p}}\left(\left(2^s j+m\right)\right)^{1-\frac{1}{p}}\right] \leq$$

$$\frac{\pi}{x}\sum_{s=0}^{t}\left[\left(\frac{C}{\left(2^s j+m\right)^2}\sup_{i\geq b_{2^s m}}\sum_{v=i}^{2i}\beta_v\right)\left(\left(2^s j+m\right)\right)^{1-\frac{1}{p}}\right] \leq$$

$$a(m)^{1-\frac{1}{p}} C \sup_{i\geq bm}\sum_{v=i}^{2i}\beta_v$$

$$I_4 \frac{\pi C}{x}\sum_{s=0}^{t}\frac{\left(2^s j+m\right)\alpha_{2^s j+m}}{2^s j+m} \leq \frac{\pi C}{xj}\sum_{s=0}^{t}\frac{1}{2^s} \leq 2C(1+m)^{1-\frac{1}{p}}\sup_{i\geq b1+m}\sum_{k=i}^{2i}\beta_k \leq 2C\, m^{1-\frac{1}{p}}\sup_{i\geq bm}\sum_{k=i}^{2i}\beta_k \quad (7)$$

Further:

$$\left|C_{2m}D_n(x)\right|+\left|C_{j+1+m}D_j(x)\right| \leq 2\operatorname{maks}_{1\leq v\leq n} v\left|C_{v+m}\right| \quad (8)$$

By Eq. 5-8, we have:

$$A \leq 2\max_{v\in[1,m]} v\left|C_{v+m}\right|+6C\, m^{-\frac{1}{p}}\sup_{i\geq bn}\sum_{k=i}^{2i}\beta_k \quad (9)$$

By Theorem 6, we have:

$$B = \left|\sum_{v=2m+1}^{\infty} C_v \sin vx\right| = \left|f(x)-S_{2m}(f, m)\right| \leq 6C(2m+2M)(2m)^{-\frac{1}{p}}\sup_{i\geq b2m}\sum_{k=i}^{2i}\beta_k \quad (10)$$

From Eq. 9 and 10, we have:

$$E_m(f) \leq 2\max_{v\in[1,m]} v\left|C_{v+m}\right|+6C\, m^{1-\frac{1}{p}}\sup_{i\geq bn}\sum_{k=i}^{2i}\beta_k + 6C(2m+2M)(2m)^{-\frac{1}{p}}\sup_{i\geq b2m}\sum_{k=i}^{2i}\beta_k$$

where, C positive constant depending class of $SBVS2_p$, $m\geq n$ and $M = \pi/x$ for $x\in(0, \pi]$.

CONCLUSION

In this study we have introduced the class $SBVS2_p(\Delta^n)$. We have investigated that error of sine series with coefficient from class p-supremum bounded variation difference sequences is:

$$E_m(f) \leq 2\max_{v\in[1,m]} v\left|C_{v+m}\right|+6C\, m^{1-\frac{1}{p}}\sup_{i\geq bn}\sum_{k=i}^{2i}\beta_k + 6C(2m+2M)(2m)^{-\frac{1}{p}}\sup_{i\geq b2m}\sum_{k=i}^{2i}\beta_k$$

where, C positive constant depending class of $SBVS2_p$, $m\geq n>1$ and $M = \pi/x$ for $x\in(0, \pi]$.

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