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Existence of the Attractor of Recurrent Iterated Function System

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Abstract: In this research paper, we present a generalization of Iterated Function System (IFS) called the Recurrent Iterated Function System (RIFS). We explore RIFS, investigate the existence of the fractal attractor of RIFS and present its main properties. Finally, MATLAB program is presented to implement the algorithm for determining the attractor of the RIFS.

Key words: Fractals, iterated function system, fixed points, attractors, existence, algorithm

INTRODUCTION

Iterated Function System (IFS) theory is an important part of fractal theory; it is widely used in fields of image compression due to the pioneering works done by Bransley (1988a, b), Barnsley *et al*. (1989). Bransley (1988a, b) put forward another concept: Recurrent Iterated Function System (RIFS). It reflects the similarities among local regions of a graph. It can generate more complex graphics. That is recurrent modeling is the process of partitioning an object into components and representing each component as a collection of contraction copies of possibly itself and possibly other components. This representation is described by a RIFS which is based on the simpler case of the IFS which represents the entire object as a collection of contraction copies of itself. Barnsley gave a strict proof for the existence and uniqueness of the invariant measure (invariant set) of an RIFS. Based on researches of Barnsley, this study first extends the definition of IFS from the viewpoint of graph theory and product space, and then investigates the proof of the existence and uniqueness of the fractal attractor of the RIFS.

Recurrent modeling by Bransley (1988a, b) and Barnsley *et al*. (1989) is the process of partitioning an object into components and representing each component as a collection of contraction copies of possibly itself and possibly other components. This representation is described by a recurrent iterated function system, based on the simpler case of the iterated function system which ultimately represents the entire object as a collection of contraction copies of itself.

This study paper is organized as follows. The second section reviews the concepts that will help us to define several properties of digraphs. Section 3 presents the recurrent Hausdorff distance metric and recurrent Hutchinson operator whereas Section 4 develops the theory of RIFS and investigates the existence of the attractor of RIFS. Section 5 presents the implementation, results conducted and the discussions made. Finally, Section 6 concludes with directions for future research on the relationship between RIFS and recursive fractal interpolation function (RFIF) and provides theoretical basis for their applications.

MATERIALS AND METHODS

Diagraph: Graphs are described here as an ordered pair of sets. The first is the set of vertices, the second is the set of edges. Edges are denoted as ordered pairs of vertices. Now, let us fix the most important notions which furnish the more general setting called diagraph. All notions are well-known and may be found in the literature (Hart, 1996).

A digraph G is a set of N vertices $G_v = \{v_i\}^N_{i=1}$ and a set of edges G_e which are ordered pairs (v_i, v_j), $1 \leq i, j \leq N$ where $(v_i, v_j) \in G_e$ implies G contains a directed edge from v_i to v_j. Since, G_e is a set, the same edge cannot appear twice in G. Thus, the cardinality of G_e is at most the cardinality of G_v squared. The number of edges into a vertex is called the "in-degree" of the vertex. Similarly, the number of edges out of vertex is called "out-degree" of the vertex.

A directed (undirected) path of vertices $v_{i1}, v_{i2}, \ldots, v_{ik}$ connects vertex v_i to v_{i_k} if and only if for each pair of neighboring vertices $v_{i_j}, v_{i_{j+1}}$ the edge $(v_{i_j}, v_{i_{j+1}}) \in Ge$. We shall follow (Al-shameri, 2001; Hart, 1996) to review the concepts that will help us to define several properties of digraphs.

A cycle of edges is simply a path from a vertex to itself while diagraph G contains a (directed, undirected) cycle if and only if there exists a vertex $v_i \in G_v$ such that there exists a (directed, undirected) path of vertices in G_v connecting vertex v_i to itself. The term cycle will imply directed cycle. A cycle may be as simple as the edge (v_i, v_i). An "acyclic" digraph contains no cycles.

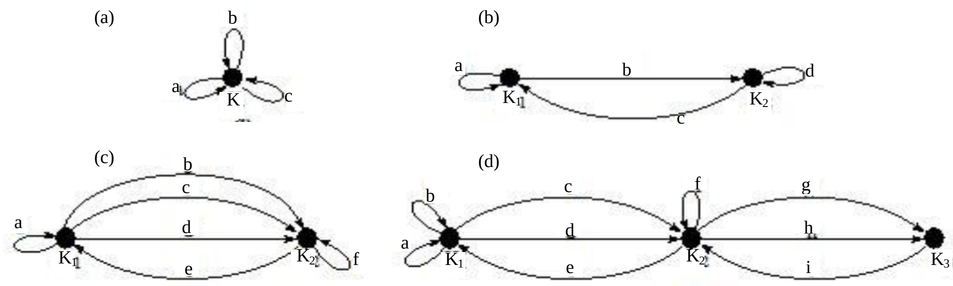

Fig. 1 a-d): Examples of diagraph

Figure 1a-d are examples of diagraph. The graph in Fig. 1a-c is a diagraph with the set of vertices $G_v = \{K_1, K_2\}$ and the set of edges $G_e = \{a, b, c, d, e, f\}$. We have, e.g., $i(c) = K_1$, $t(c) = K_2$ where i(c) and t(c) denoted to the initial and terminal vertex of the edge c, respectively where $I: G_e \rightarrow \mathbb{R}^+$, $t: G_e \rightarrow \mathbb{R}^+$ (Al-shameri, 2001; Edgar, 1990).

As with any geometric model, there are certain properties that recurrent models may satisfy. For example, the partitioning may satisfy the open set property and the directed graph that describes how components combine to form components may be weakly or strongly connected. The following concepts are needed for digraph connectedness. We shall follow (Deo, 1974).

A digraph G is (strongly, weakly)-connected if and only if every pair of vertices $v_i, v_j \in G_v$ is connected by a (directed, undirected) path of vertices (Al-shameri, 2001; Edgar, 1990). We follow the convention, implied from this definition, that strongly-connected implies weakly-connected which differs form (Deo, 1974). Hence, a weakly-connected digraph may or may not be strongly-connected too (Deo, 1974; Edgar, 1990; Hart, 1996). If digraph G is weakly connected, then the cardinality of G_e is at least one less than the cardinality of G_v.

A strongly-connected digraph necessarily contains a cycle. A strictly weakly-connected digraph necessarily contains no cycle. An Iterated Function System (IFS) is a couple (X, d) of a complete metric space together with a finite set of contraction mappings $f_n: X \rightarrow X$, n = 1, 2, ..., N where the metric d is a distance function between elements of X, i.e., $d: X \times X \rightarrow \mathbb{R}^+$. Note that, f_n transforms a subset of the complete metric space $A \subset X$ onto smaller subsets $f_n(S)$, i.e., an IFS models an object by constructing it out of smaller copies of itself (Barnsley and Demko, 1985). Symbolically, an IFS $\{X; f_1, f_2, ..., f_N\}$ models the set $A \subset X$ is called the attractor of the IFS as the union of attractorlets $A_n = f_n(A)$ as the solution A of :

$$A = \bigcup_{n=1}^{N} A_n$$

It is often convenient to write an IFS somewhat more briefly as IFS $\{X; f_{1\text{-}N}\}$ instead of IFS $\{X; f_1, f_2, ..., f_N\}$. Let, $\varsigma(X) = \{A \subset X: A \text{ is compact}\}$ be the collection or space of all compact subsets of the complete metric space X. Then, the metric d is now used to define a metric $\hbar(d)$ denotes the Hausdorff metric (distance) between elements A and B of $\varsigma(X)$ as follows. Let, $\hbar: \varsigma(X) \times \varsigma(X) \rightarrow \mathbb{R}^+$ be a mapping such that:

$$\hbar(A, B) = \max\{d(A, B), d(B, A)\}$$

is the Hausdorff metric between A and B of $\varsigma(X)$ (Al-Shameri, 2015; Barnsley and Demko, 1985; Falconer, 1999). The recurrent versions of the standard IFS analysis tools operate on N-tuples of sets. These will be denoted by:

$$S = (S_1, S_2, ..., S_N$$

A set N-tuple S of n-dimensional sets S_i belongs to the n×N dimensional space $(\mathbb{R}^n)^N$. Moreover, functions on these sets in this space are denoted with a superscripted N (i.e., $f \rightarrow f^N$).

One can think of $(\mathbb{R}^n)^N$ as N copies of $\mathbb{R}^n$ overlayed on top of each other. A set $S \subset \mathbb{R}^2$ is better understood by drawing each of its N parts A_i on a separate clear sheet. Then, the set S can be visualized in the space by overlaying all N transparent sheets (Barnsley *et al*., 1989). The subset relation in $(\mathbb{R}^2)^N$ is defined as the logical "and" of the subset relation of its components. That is, if $A = (A_1, A_2, ..., A_N)$ and $(B_1, B_2, ..., B_N)$. Then, $A \subset B$ if and only if, $A_i \subset B_i$, $\forall i = 1, ..., N$. Notice that, if $A \subset B$ then $U_i A_i \subset U_i B_i$.

The recurrent Hausdorff metric and recurrent Hutchinson operator: The extension of the Hausdorff distance (metric) as originally defined by Barnsley *et al*. (1989), combines the individual Hausdorff distances of the component sets. One possible space is $P(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$. A subspace of $P(\mathbb{R}^n)$ is ς the collection of non-empty compact subsets of $\mathbb{R}^n$. Then the Hausdorff distance between the points A and B in ς is defined by $\hbar(A, B) = d(A, B) \vee d(B, A)$ where $(A, B) = \max\{d(x, B): x \in A\}$

and $x \vee y$ means the maximum of x and y. We also call $\hbar$ the Hausdorff metric on ς. Now, $(\varsigma \hbar)$ is a complete metric space (Hart, 1996).

The recurrent version of Hausdorff metric denoted by $\hbar^N$ measures the distance between two subsets $A = (A_1, A_2, ..., A_N)$ and $B = (B_1, B_2, ..., B_N)$ of $(\mathbb{R}^n)^N (\text{or} \varsigma^N)$ as:

$$\hbar^N(A, B) = \max_{i=1,...,N} \hbar(A_i, B_i)$$

Now, $(\varsigma^N, \hbar^N)$ is a complete metric space (Hart, 1996). Here, a set $A \subset (\mathbb{R}^n)^N$ is compact if and only if every one of its parts A_i is compact in $\mathbb{R}^n$. A Recurrent Iterated Function System (RIFS) consists of a finite set of strictly contractive maps $\{f_i\}^N_{i=1}$ from $\mathbb{R}^n$ into itself, along with an N-vertex weakly-connected digraph $G = (G_v, G_e)$ containing some directed cycle from every vertex $v_i \in G_v$ back to itself. The digraph G is used to restrict map compositions. The iteration sequence $f_j \circ f_i$ is allowed if and only if a directed edge from vertex v_i to v_j exists in G. Symbolically, we write $((\{f_i\}^N_{i=1}, G$ for RIFS.

According to connectedness of the digraph many different names for IFS enhancement are given (Barnsley *et al*., 1989; Prusinkiewicz and Hammel, 1991; Prusinkiewicz *et al*., 1990; Reuter, 1987). In this study, we mention that (f^N, G) is a RIFS where $f^N = \{f_i\}^N_{i=1}$ and G is weakly-connected diagraph. If each partition is the union of images of every partition including itself, then the graph G is complete and the RIFS is simply an IFS. Hence, every IFS is an RIFS and we will focus in the upcoming section on existence of the attractor of RIFS representation.

The degree of overlap of the partitioning is dictated by the open set property. An RIFS $((\{f_i\}^N_{i=1}, G$ satisfies the open set property, if and only if there exists a set-vector $U = (U_1, U_2, U_3, ..., U_N)$ of open sets $U_i \subset \mathbb{R}^n$ such that $f(U) \subset U$ and $U_i \cap U_j = \phi \forall i = j$. An attractor is "just touching" (Bransley, 1988a, b), if and only if it is connected and its RIFS satisfies the open set property. The recurrent Hutchinson operator as first developed by Barnsley *et al*. (1989) is a generalization of the standard Hutchinson operator. Let $((\{f_i\}^N_{i=1}, G)$ be an RIFS. Then, the recurrent Hutchinson operator $f^N: (\mathbb{R}^n)^N \rightarrow (\mathbb{R}^n)^N$ is defined (Hutchinson, 1981):

$$f^N(A) = (f_1(A), f_2(A), ..., f_N(A))$$

where, $A = (A_1, A_2, ..., A_N) \in (\mathbb{R}^n)^N$ and:

$$f_j(A) = \bigcup_{\{i:(v_i, v_j) \in G_e\}} f_j(A_i)$$

The recurrent Hutchinson operator is a contraction. However, this proof will need the following and its proof based on the version that appears in Barnsley *et al*. (1989). Lemma (Hart, 1996) for collections $A = \{A_i\}^N_{i=1}$, $B = \{B_i\}^N_{i=1}$ where A_i, B_i are subset of metric space (X, d).

$$\hbar\left(\bigcup_{i=1,...,N} A_i, \bigcup_{i=1,...,N} B_j\right) \leq \sup_{i=1,...,N} \hbar(A_i, B_i)$$

Proof; Let:

$$\varepsilon = \hbar\left(\bigcup_{i=1,...,N} A_i, \bigcup_{i=1,...,N} B_j\right)$$

Then:

$$\bigcup_{i=1,...,N} A_i, \bigcup_{i=1,...,N} B_j + \varepsilon$$

For each $1 \leq i \leq N$ we have:

$$A_i \subset \bigcup_{j=1,...,N} B_j + \varepsilon$$

and specifically:

$$A_i \subset B_j + \varepsilon_{B_j}$$

where, ε_{Bi} is minimal still, $\varepsilon_{Bi} \geq \varepsilon$. Likewise, for each i there is a corresponding $\varepsilon_{Ai} \geq \varepsilon$. For each i the maximum thickening radius matches or exceeds the original thickening radius:

$$\max\left\{\varepsilon_{A_i}, \varepsilon_{B_j}\right\} \geq \varepsilon$$

as does its maximum over i. The proof is complete once the reader realizes that the left-hand side of this inequality is the Hausdorff distance between A_i and B_i and the right hand side is the Hausdorff distance between the union of A_i and the union of B_i. Now, we are ready to prove that the recurrent Hutchinson operator is a contraction.

Theorem (Hart, 1996): Let f^N be the recurrent version of Hutchinson operator of RIFS $(\{f_i\}^N_{i=1}, G)$. Then f^N is a contraction on the metric space $(\varsigma^N, \hbar^N)$.

Proof: Let, $A = (A_1, A_2, ..., A_N)$, $B = (B_1, B_2, ..., B_N) \in (\mathbb{R}^n)^N$. Consider the following chain of inequalities:

$$\begin{aligned}\hbar^N\left(f^N(A), f^N(B)\right) &= \max_{j=1,...,N} \hbar(f_j(A), f_j(B)) \\ &\leq \max_{i,j=1,...,N} \hbar(f_j(A_i), f_j(B_i)) \\ &\leq \max_{i,j=1,...,N} s_j \hbar(A_i, B_i) \\ &\leq s \max_{i=1,...,N} \hbar(f_j(A_i), f_j(B_i)) \\ &= s\hbar^N(f_j(A, B)\end{aligned}$$

where:

$$s = \max_{i=1,...,N} \text{Lip} f_i$$

Since, $s<1$, f^N is a contraction on the complete metric space $(\varsigma^N, \hbar^N)$.

Existness and uniquness of the attractor of recurrent iterated function system: Now, we consider fundamental result that a unique set be associated with an RIFS as a consequence of the following theorem and proof are based on (Barnsley *et al*., 1989).

Theorem 1 (Hart, 1996): For any RIFS $((\{f_i\}^N_{i=1}, G)$ there exists a unique N-tuple. $A\in(\mathbb{R}^n)^N$ of non-empty compact sets such that:

$$A = f^N(A)$$

Proof: Since, ς^N, h^N is complete metric space (the finite product of complete spaces (the finite product of complete: Theorem 76 of (Kaplansky, 1977) and f^N is a contraction on (ς^N, h^N) the Contraction Mapping Principle implies that f^N possesses a unique fixed point in (ς^N, h^N).

The attractor of a RIFS is not an N-tuple because we want to deal in an n-dimensional space not an n×N-dimensional space. Thus, we have from (Barnsley *et al*., 1989), the attractor of a RIFS.

Let, $A = (A_1, A_2, ..., A_N)$ be a set such that it is invariant under the recurrent Hutchinson operator of a RIFS (f^N, G):

$$f^N(A) = A$$

Then, the attractor A of the RIFS is given by:

$$A = \bigcup_{i=1,\ldots,N} Ai$$

We need the following result.

Corollary 1; (Hart, 1996): Let (X, d) be a complete metric space and let: $f:X\rightarrow X$ be a contraction is on X. Then all points in X converge to the same fixed point under iteration of f. Finally, we have the useful result that any initial set N-tuple will iterate to the attractor of an RIFS.

Corollary 2; (Hart, 1996): Consider the RIFS (f^N, G) $A\in(\mathbb{R}^n)N$ of non-empty compact sets such that $A = f^N(A)$. Let B be an N-tuple of nonempty bounded sets. Then:

$$\lim_{n\to\infty}(f^N)^{\circ n}(B) = A$$

Proof: Since, each component of B is bounded, there exists a compact set $B^+\supset B_i$ for all $B_i\in B$. Let:

$$B^+ = (B^+, B^+, ..., B^+)$$

Since, each component of B is non-empty, there exists a compact set $B^-\subset B^i$. Let:

$$B^- = (B^-, B^-, ..., B^-)$$

Then, by Corollary 1 and Theorem 1:

$$A\subset\lim_{i\to\infty}(f^N)^{\circ i}(B^-)\subset\lim_{i\to\infty}(f^N)^{\circ i}(B)\subset\lim_{i\to\infty}(f^N)^{\circ i}(B^+)\subset A$$

they are all equal.

RESULTS AND DISCUSSION

The attractor of RIFS is determined by the diagraph G or by the n×n transition matrix (adjacency matrix) of the diagraph representing the probabilities of the choice of the transformations (affine maps) w_i in the space $\mathbb{R}^2$ in which:

$$w_i\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a_i & b_i\\c_i & d_i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}e_i\\f_i\end{pmatrix}, i = 1,2,..., n$$

whereas n is the number of the transformations w_i defined in the space $\mathbb{R}^2$. The diagraph corresponding to the transition matrix for the RIFS:

$$P_1 = (P_{ij}) = \begin{pmatrix}0.3 & 0.6 & 0.1\\0.1 & 0.5 & 0.4\\0.4 & 0.4 & 0.2\end{pmatrix}, i, j = 1, 2,3$$

given in Fig. 2. The code for RIFS is presented in Table 1 and the attractor of RIFS is shown in Fig. 3 while the diagraph corresponding to the transition matrix::

$$P_2 = (P_{ij}) = \frac{1}{4}\times\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 0 & 1\\1 & 0 & 1 & 1\\0 & 1 & 1 & 1\end{pmatrix}, i, j = 1, 2,3, 4$$

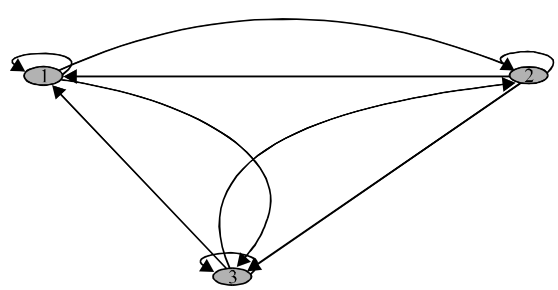

Fig. 2: Diagraph for RIFS code given in Table 1 whose RIFS attractor is a Version of Sierpiniski Gasket

given in Fig. 4. The code for RIFS is presented in Table 2 and the attractor of RIFS is shown in Fig. 5.

Both RIFS fractal attractors shown in Fig. 3 and 5 were constructed by using MATLAB program listed in the algorithm 1. This program is using the random iteration algorithm (Al-Shameri, 2015; Al-shameri, 2001; Barnsley, 1988a, b) for generation these fractal attractors of the RIFS in the space $\mathbb{R}^2$. The program inputs are p_0, w, p, n, k and r where p_0 is the initial point represented by a 1×2 matrix, p is the transition

Table 1: Recurrent IFS code for sierpiniski gasket

w_i	a_i	b_i	c_i	d_i	e_i	f_i
1	0.5	0	0	0.5	0	0
2	0.5	0	0	0.5	0	128
3	0.5	0	0	0.5	128	128

Table 2: Recurrent IFS code for quadtree fractal

w_i	a_i	b_i	c_i	d_i	e_i	f_i
1	0.5	0	0	0.5	0.0	0.0
2	0.5	0	0	0.5	0.5	0.0
3	0.5	0	0	0.5	0.0	0.5
4	0.5	0	0	0.5	0.5	0.5

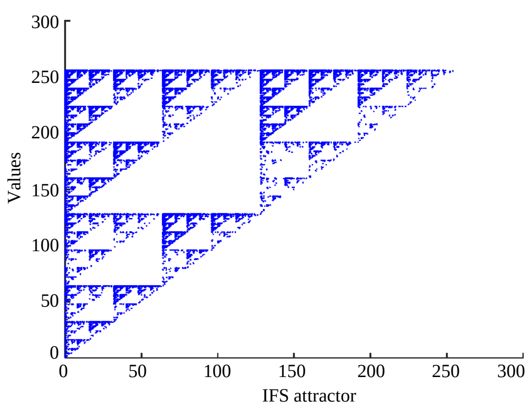

Fig. 3: Recurrent IFS attractor (Version of Sierpiniski Gasket) corresponding to the diagraph represented in Fig. 2 with RIFS code in Table 1

(adjacency) matrix n×n of the directed graph (diagraph) representing the probabilities (transition of the graph).

Algorithm 1:

```
%MATLAB program (Hahn and Valentine, 2007) plots the fractal attractor
of RIFS of
R2
clear all; clc
%The version of Sierpiniski Gasket
 w = [0.5 0.0 0.0 0.5 0 0
    0.5 0.0 0.0 0.5 0 128
    0.5 0.0 0.0 0.5 128 128]
p0 = [0 0]
P = [0.3 0.6 0.1
   0.1 0.5 0.4
   0.4 0.4 0.2]
%The quadtree fractal
%w = [0.5 0.0 0.0 0.5 0 0
    %0.5 0.0 0.0 0.5 0.5 0
    %0.5 0.0 0.0 0.5 0 0.5
    %0.5 0.0 0.0 0.5 0.5 0.5]
%p0 = [0 0]
%P = 1/4*[1 1 1 1
    %1 1 0 1
    %1 0 1 1
    %0 1 1 1]
Fig. 1
hold on
n = 4; k = 30000
for i = 1:n
  TP(i,1)=P(i,1)

for j = 2:n
   TP(i, j) = TP(i, j-1)+P(i, j)
  end
end
x = p0(1, 1); y = p0 (1, 2)
r = floor(n*rand-0.00001)+1
for i = 1:k
   newx = w (r, 1)*x+w (r, 2)*y+w(r, 5)
   newy = w(r, 3)*x+w(r, 4)*y+w(r, 6)
  x = newx; y = newy
  plot (x, y, 'b.' , 'MarkerSize', 2)
  s = rand (1)
  for j = 1:n
    if s<TP (r, j)
     r = j
     break
    end
  end
end
x = 0;
```

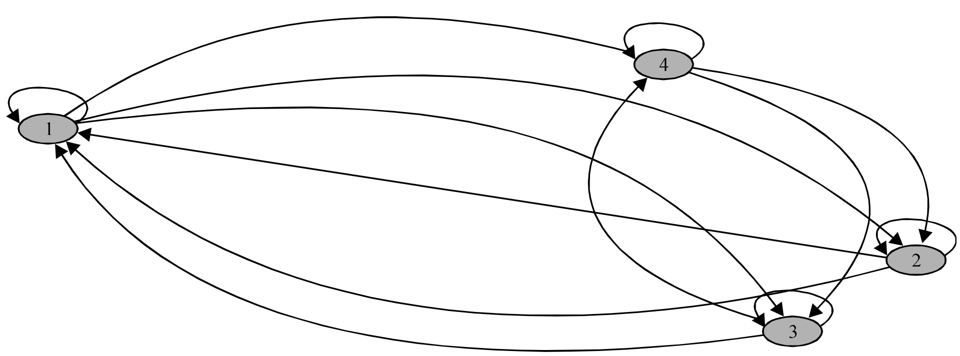

Fig. 4: Diagraph for RIFS code given in Table 2 whose RIFS attractor is a quadtree fractal

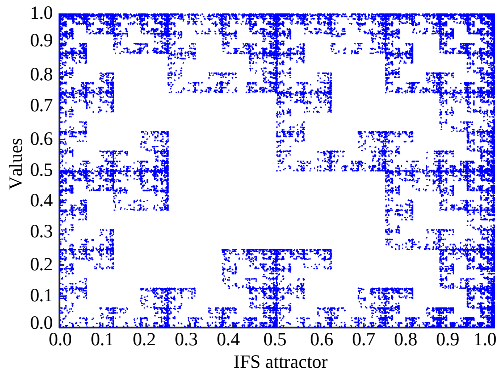

Fig. 5: Recurrent IFS attractor (quadtree fractal) corresponding to the diagraph represented in Fig. 4 with RIFS code in Table 2

The number of the transformations (affine maps) is n, k is the number of iterations, w is the n×6 matrix containing the parameters of the affine maps w_i and r is the random choice of the affine map w_i that it is applied to the initial point p_0 randomly. Each row i in Table 1 and 2 contains the 6 coefficients a_i, b_i, c_i, d_i, e_i and f_i of the affine maps w_i, i = 1, 2, ..., n representing the code of RIFS.

CONCLUSION

The mathematical principles behind RIFS have been introduced by Barnsley (1988a, b) and Barnsley *et al*., (1989). In this study we explored the RIFS and investigated the proof of the existence and uniqueness of its attractor. We observe that the RIFS from the viewpoint of graph theory reflects the similarities among local regions of a diagraph, it generates more complex fractal attractors using random iteration algorithm. Future research may focus on recursive fractal interpolation function which is an extension of fractal interpolation function. The latter is the attractor of IFS while recursive fractal interpolation function is the attractor of RIFS.

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