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Influence of Homogenous Part on the Total Solution to the Differential Equation of SDOF System Response under Harmonic Loading

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Abstract: The problem of SDOF response under harmonic load is usually faced the engineers in their design life. SDOF represents many structures in practice such as elevated liquid tower or tank. Any equipments or machines inside structure can exert a harmonic loading on the structure itself. The objective of this research is to find the complete solution to the differential equation of SDOF structure under sinusoidal loading. Further, state when the designer or mathematician can neglect the homogonous part of the total solution and when this part has a significant effect on the total solution. In this study, mathematics of the differential equation for particular solution is extended to include the homogenous solution. MATLAB was used to plot the homogenous, particular and the total solutions for the covering differential equation. Many solution variables (excitation frequency, loading frequency ratio and viscous damping ratio) were used to test the results obtained and generalized the solution. From the data collected, it was concluded when the homogenous solution part was neglected from the total solution, the results will have an error which depends on the value of the damping ratio and the time at which the total solution evaluated. If the value of the damping ratio is small, large window of time required to minimums the error. While a small windows of time required when the damping ratio has a larger values.

Key words: Homogenous solution, particular solution, steady state response, damping ratio, SDOF, harmonic loading

INTRODUCTION

In all the engineering dynamic of structures books, when they reached to the subject of the response of Single Degree of Freedom (SDOF) under harmonic loadings, they only stated that the homogonous part of the solution (transient component) vanished due to the presence of the exponential factor, leaving only the particular part (steady-state motion component) as illustrated in structural dynamic book like the one that authorized by Paz (2004). Clough and Penzien (1993) wrote in their book dynamic of structure that the homogonous part of the total solution of response of SDOF which represent the transient response which damps out in accordance with exponential factor while the second term represents the steady-state harmonic response which will continue indefinitely. As known, there are many applications on the SDOF system under sinusoidal loading in the literature but unfortunately most of the previous studies were neglecting directly the homogonous part from the total solution and their solutions are depending only on the particular one (Gattulli *et al*., 2004; Pandey and Benipal, 2006; Peng *et al*., 2007; Kavitha *et al*., 2016). The one important work in this field was done by Gil-Martin *et al*. (2012) which tried to include the transient part to the total solution especially for un-damped systems.

The well-known analytical solution for the steady state response of a SDOF system under harmonic loading is described in many structural dynamics textbooks. The reason of this study is the complete solutions that include the transient part and the only steady-state can differ significantly in some circumferences for both damped and un-damped systems. The objective of this study is to find an analytically expression for the SDOF system under sinusoidal loadings to the complete solution (steady state part plus transient part) for both damped systems. Further aims of this research are to determine when we can neglect the homogonous part without an appreciated error.

MATERIALS AND METHODS

The covering differential equation that represents the response of a damped Single Degree of Freedom (SDOF) system under sine harmonic loading is given as (Chopra, 2007):

$$M\ddot{u}(t)+C\dot{u}(t)+Ku(t) = P_o \sin \overline{w}t \tag{1}$$

Knowing that, the parameters of above equation can be defined physically as: M is the mass of the system, C is viscous damping coefficient, K is the stiffness of the system, P_O and $\overline{w}$ are the peak of the sinusoidal sine load and its loading frequency, respectively. Whereas the other parameters can be defined physically as follow:

$u(t)$ = Displacement time history of the SDOF system

$\dot{u}(t) = \frac{du}{dt}$ = Velocity time history of the system

$\ddot{u}(t) = \frac{d^2u}{dt^2}$ = Acceleration time history of the system

Equation 1 is a second ordinary linear differential equation with constant coefficient. The general total solution of this equation can be written as (Kreyszig, 2011):

$$u(t) = u_h(t) + u_p(t) \tag{2}$$

where, u(t) is the total solution of Eq. 1 while both $u_h(t)$ and $u_p(t)$ represent the homogonous solution part and particular solution part of Eq. 1, respectively. It is known that, the homogonous solution part of Eq. 1 means the solution of the following homogonous differential equation:

$$M\ddot{u}_h(t) + C\dot{u}_h(t) + Ku_h(t) = 0 \tag{3}$$

The solution of Eq. 3 is well known and can be written for under damped system (when the viscous damping ratio is less than one) which represents the actual situation of building engineering problem as (Villaverde, 2009):

$$u_h(t) = e^{-\xi wt}(c_1 \sin w_D t + c_2 \cos w_D t) \tag{4}$$

where, ξ is defined as a viscous damping ratio which is equal to C/C_{cr}, $C_{cr} = 2$ MW is defined as the critical viscous damping coefficient, w is the non-damped natural frequency of the system and $w_D = w\sqrt{1-\xi^2}$ is damped natural frequency of the system. The particular solution part of Eq. 1 means the solution of the following second order differential Eq. 5:

$$M\ddot{u}_p(t) + C\dot{u}_p(t) + Ku_p(t) = P_o \sin \overline{w}t \tag{5}$$

Mathematic says the general solution of Eq. 5 can be written as (Chopra, 2007):

$$u_p(t) = G_1 \sin \overline{w}t + G_2 \cos \overline{w}t \tag{6}$$

Differentiate Eq. 6 once and twice and by substituting the differentiable functions into Eq. 5 and after arrange and simplify the terms, G_1 and G_2 can be written as (Kreyszig, 2011):

$$G_1 = \frac{P_o}{K}\left[\frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2}\right] \tag{7}$$

where, $\beta = \overline{w}/w$ which is called the excitation or loading frequency ratio and:

$$G_2 = \frac{-P_o}{K}\left[\frac{2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}\right] \tag{8}$$

By substituting Eq. 7 and 8 into Eq. 6 with suitable arrangements, then the particular solution part representing into Eq. 6 becomes (Craig and Kurdila, 2006):

$$u_p(t) = \frac{P_o/K}{(1-\beta^2)^2 + (2\xi\beta)^2}\left[(1-\beta^2)\sin \overline{w}t - 2\xi\beta \cos \overline{w}t\right] \tag{9}$$

Adding Eq. 4 and 6 together to collect the total solution of the differential equation of SDOF system under harmonic loading as given in Eq. 2, the result yields:

$$u(t) = e^{-\xi wt}(c_1 \sin w_D t + c_2 \cos w_D t) + \frac{P_o/K}{(1-\beta^2)^2 + (2\xi\beta)^2}\left[(1-\beta^2)\sin \overline{w}t - 2\xi\beta \cos \overline{w}t\right] \tag{10}$$

Let us evaluate the constants c_1 and c_2 at specific initial conditions which are named physically at rest initial conditions. Mathematically at time equal zero; Both total initial displacement and velocity are equal zero. Simply from the first initial condition (zero displacement at time equal zero), c_2 will be as:

$$c_2 = \frac{2\xi\beta(P_o/K)}{(1-\beta^2)^2 + (2\xi\beta)^2} \tag{11}$$

To find c_1, differentiation of Eq. 10 is necessary to find the velocity time history which represents the total velocity response. The differentiation of Eq. 10 with respect to time will be as:

$$\dot{u}(t) = -\xi we^{-\xi wt}(c_1 \sin w_D t + c_2 \cos w_D t) + e^{-\xi wt}(c_1 w_D \cos w_D t - c_2 w_D \sin w_D t) + \frac{P_o/K}{\left[(1-\beta^2)^2 + (2\xi\beta)^2\right]}\left[(1-\beta^2)\overline{w}\cos \overline{w}t + 2\xi\beta\overline{w}\sin \overline{w}t\right] \tag{12}$$

Now, applying the second initial condition (zero velocity at time equal zero) with suitable simplification, c_1 can be written as:

$$c_1 = \frac{\beta(P_o / K)}{\sqrt{1-\xi^2}\left[(1-\beta^2)^2+(2\xi\beta)^2\right]}\left[2\xi^2-(1-\beta^2)\right] \quad (13)$$

So, after substituting Eq. 11 and 13 into Eq. 10, the total clear (without any constants) solution can be written as:

$$u(t) = e^{-\xi wt}\left(\begin{array}{l}\dfrac{\beta(P_o / K)}{\sqrt{1-\xi^2}\left[(1-\beta^2)^2+(2\xi\beta)^2\right]}\left[2\xi^2-(1-\beta^2)\right] \\ \sin w_D t+\dfrac{2\xi\beta(P_o / K)}{(1-\beta^2)^2+(2\xi\beta)^2}\cos w_D t\end{array}\right)+ \quad (14)$$
$$\frac{P_o / K}{(1-\beta^2)^2+(2\xi\beta)^2}\left[(1-\beta^2)\sin\bar{w}t-2\xi\beta\cos\bar{w}t\right]$$

Equation 14 represents the total solution to the differential equation of SDOF response under harmonic loading (Eq. 1) in term of both homogenous and particular parts. It is difficult to study the effect of the first part (homogenous part) of Eq. 14 analytically unless drawing this function by suitable and flexible function drawings software such as MATLAB as used in this work. For generality purposes, Eq. 14 can be written as:

$$u(t) = (P_o / K)R(t) \quad (15)$$

where, R (t) is called a resistance function which may be defined as the total displacement time response for unit (P_0/K). Really, the resistance function is depended on ξ, β and $\bar{w}$ which are classified as uncoupled variables. Knowing that w, W_D and $\bar{w}$ are each related with other by ξ or β.

RESULTS AND DISCUSSION

MATLAB was used in this research to draw Eq. 14 in term of the resistance function defined by Eq. 15. The drawings are divided mainly into three stages; The homogonous part of the total solution, the particular part of the total solution and the total solution itself. These stages are drawn based on the solution variables listed in Table 1. Three values of the excitation frequencies were used as in Table 1 which is simulating the real practice situation of low, medium and high loading frequencies.

Table 1: Solution variables

$\bar{w}$ (rad/sec)	β	ξ
20	0.7	0.02
		0.04
		0.06
		0.08
	1	0.01
		0.04
		0.06
		0.08
	$\sqrt{2}$	0.01
		0.04
		0.06
		0.08
200	0.7	0.01
		0.04
		0.06
		0.08
	1	0.01
		0.04
		0.06
		0.08
	$\sqrt{2}$	0.01
		0.04
		0.06
		0.08
2000	0.7	0.01
		0.04
		0.06
		0.08
	1	0.01
		0.04
		0.06
		0.08
	$\sqrt{2}$	0.01
		0.04
		0.06
		0.08

For each one of loading frequency, three values of excitation ratios were taken into account which represent that the natural frequency of the system is larger, equal and less than the excitation frequency. Finally four different values of viscous damping ratios were incorporated in this analysis. These values are descending from the low, intermediate to high value as shown in Table 1.

Figure 1-3 show the variation of the resistance function with time for homogenous, particular and total solutions, respectively. Fig. 1 represents the time history of homogenous part of the total solution of SDOF response under harmonic loads for four values of viscous damping (ξ = 0.02, 0.04, 0.06 and 0.08) at specific excitation frequency $\bar{w}$ = 20 rad/sec and at specific excitation frequency ratio β = 0.7. It was observed from this figure that all responses follow similar pattern for all values of ξ. The maximum response occurs at time approach zero and then the response decreases as the time is increasing. It is also shown as the viscous damping ratio increases, the time required to make the response approach zero is decreasing. For example at ξ = 0.02 the response approach zero at time equal 7 sec. while only 2 sec required to vanish the response at ξ = 0.08.

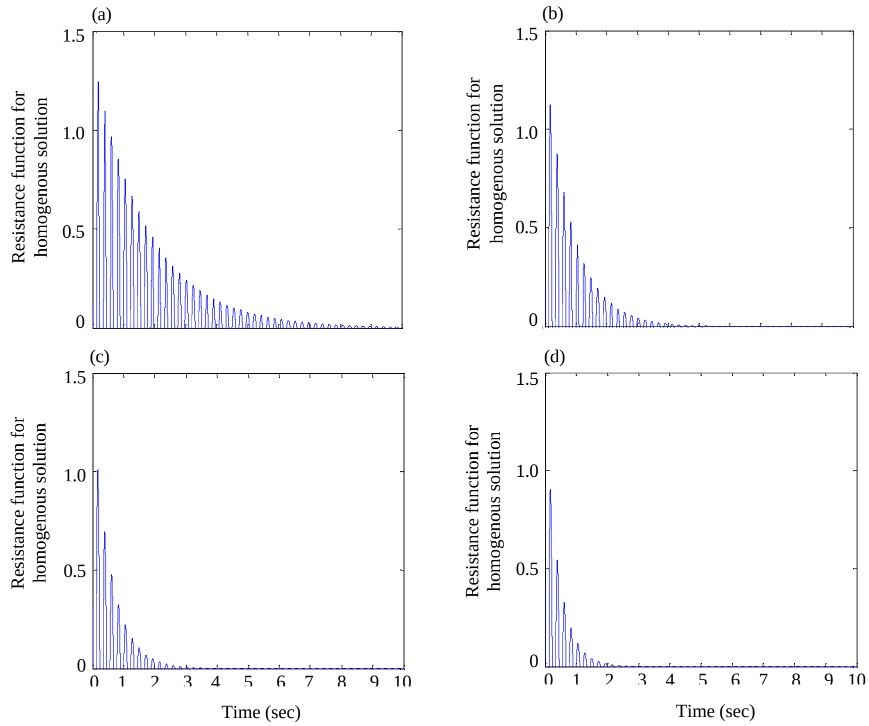

Fig. 1: Resistance function of homogenous solution for $\bar{w}$ = 20 rad/sec and β = 0.7: a) ξ = 0.02; b) ξ = 0.04; c) ξ = 0.06 and d) ξ = 0.08

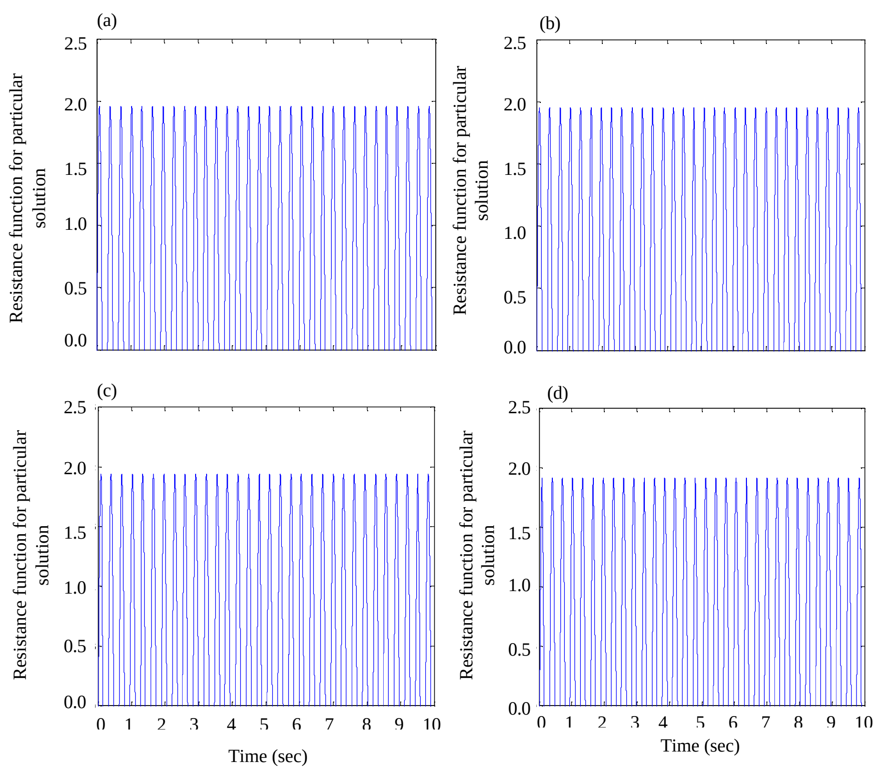

Fig. 2: Resistance function of particular solution for $\bar{w}$ = 20 rad/sec and β = 0.7: a) ξ = 0.02; b) ξ = 0.04; c) ξ = 0.06 and d) ξ = 0.08

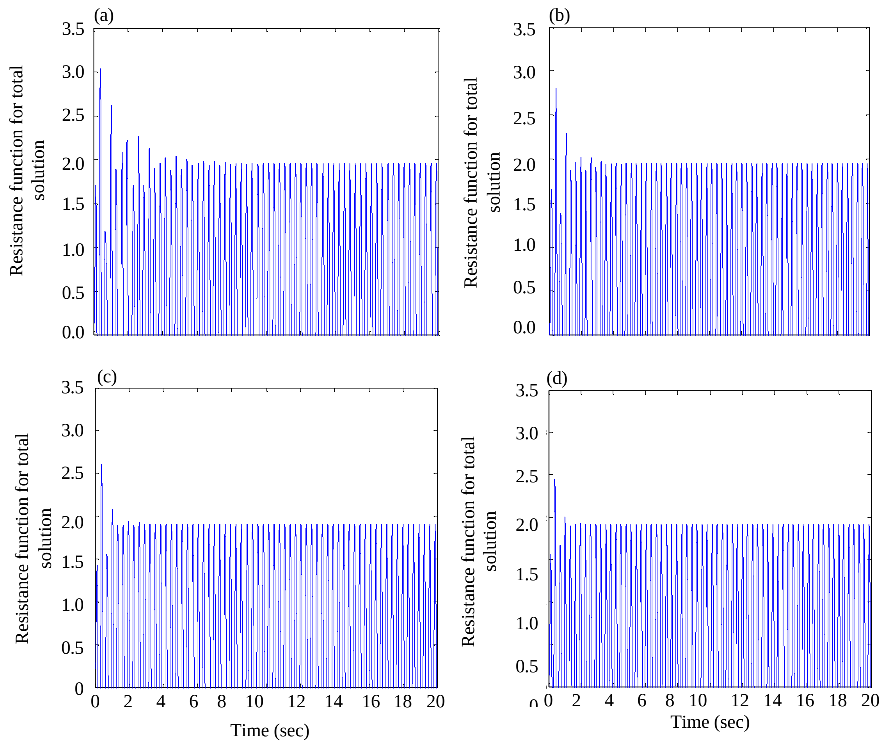

Fig. 3: Resistance function of total solution for $\bar{w}$ = 20 rad/sec and β = 0.7: a) ξ = 0.02; b) ξ = 0.04; c) ξ = 0.06 and d) ξ = 0.08

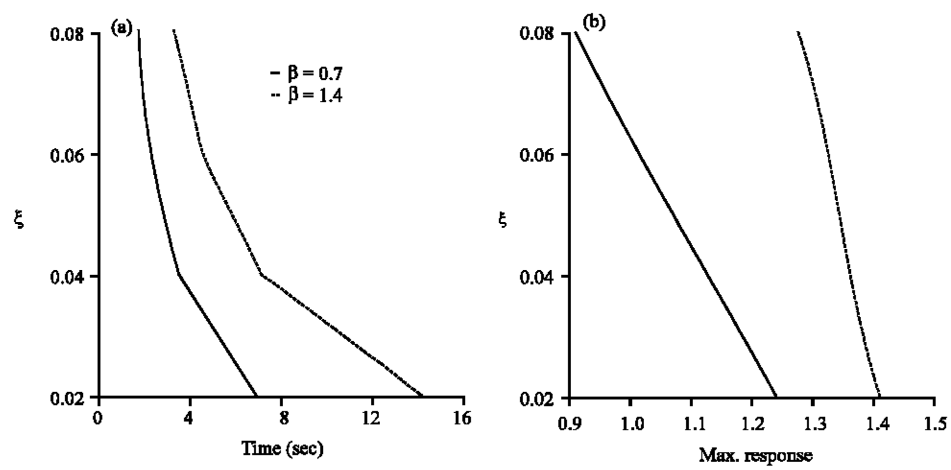

Fig. 4: Variation of: a) Zero response and b) Maximum response with ξ and β at $\bar{w}$ = 20 rad/sec

Figure 2 shows the time history of the particular solution part for four values of ξ at $\bar{w}$ = 20 rad/sec and at β = 0.7. This figure clearly indicates that this part of solution is in steady state. The total solution (both homogonous and particular parts) variation with time for the same above parameters (ξ = 0.02, 0.04, 0.06 and 0.08) ($\bar{w}$ = 20 rad/sec) and (β = 0.7) is illustrated in Fig. 3. It is obvious from this figure that the influence of the homogonous part on the total solution is marked at the beginning of time. Then, the homogonous part that called transient part is become negligible after a sufficient time window. The size of window depends on the value of viscous damping ratio. When ξ has a small value, large window of time required avoiding the effect of transient part of solution and small window required when has a large value.

Figure 4-7 show the homogenous part variation of zero response and maximum response with respect to four values of viscous damping ratios (ξ = 0.02, 0.04, 0.06 and 0.08) and for two values of excitation frequency ratios

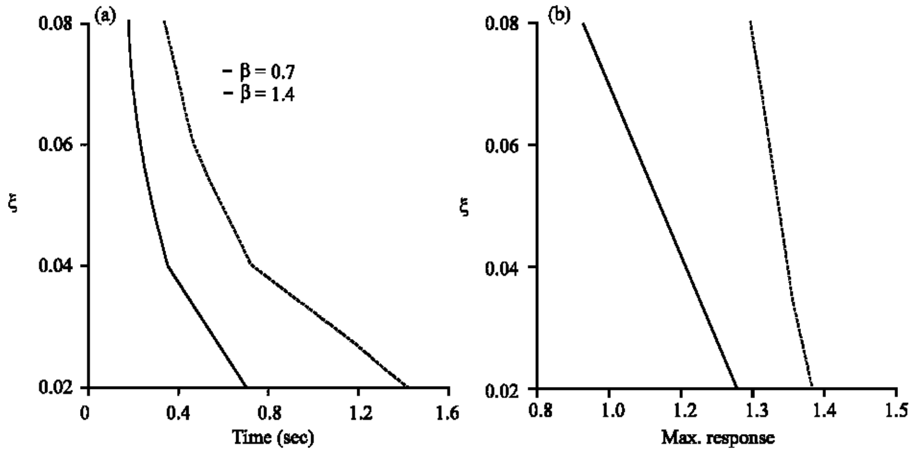

Fig. 5: Variation of: a) Zero response and b) Maximum response with ξ and β at $\bar{w}$ = 200 rad/sec

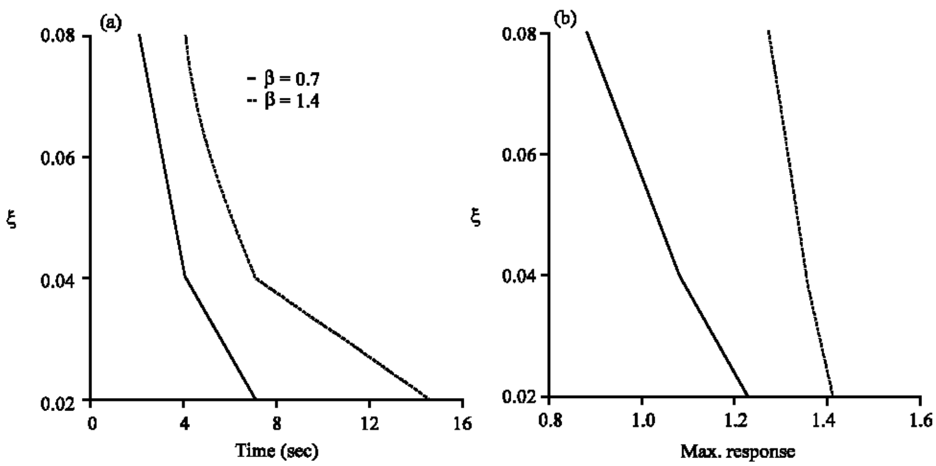

Fig. 6: Variation of: a) Zero response and b) Maximum response with ξ and β at $\bar{w}$ = 2000 rad/sec

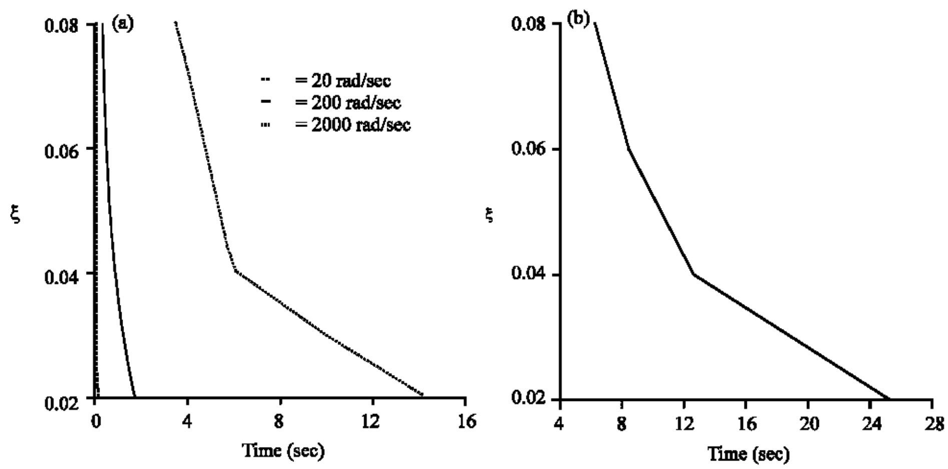

Fig. 7: Variation of: a) Zero response and b) Maximum response with ξ and β at $\bar{w}$ = 1

(β = 0.7 and 1.4) and at three values of excitation frequency ($\bar{w}$ = 20, 200 and 2000 rad/sec). It was observed from these figures, at small value of ξ, the zero response required a large time window for all low, medium and high loading frequencies. While the maximum response, at same values of β and for all values of $\bar{w}$, decreases as the viscous damping is decreasing. Finally, Fig. 7 illustrates the zero response and maximum response of homogenous part variation with ξ and for three values of excitation frequencies ($\bar{w}$ = 20, 200 and 2000 rad/sec) at β =1. The zero response is follow the same pattern for other value of β greater or less than one. While the maximum responses are identical for all the previous three values of $\bar{w}$.

CONCLUSION

According to above results and discussion, the following scientific points can be concluded: Canceling the homogenous solution part from the total solution of the response of SDOF system under harmonic loading resulted in small error which depends on the value of the damping ratio and the time at which the total solution found. If the value of the damping ratio is small, large window of time required to minimums the error. While small windows of time required when the damping ratio has a larger values.

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