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The Reliability of Small World Network

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Abstract: The asymptotic spanning tree entropy is a natural measure related to topological and dynamic properties of networks, namely their reliability. However, computing the number of spanning trees of networks using classical algebraic methods is very demanding tasks on term computational resources and time, especially for large networks such as complex networks. In this study, we give an analytic formula for the number of spanning trees and the asymptotic entropy of two type of small-world networks G_k and C_k by using two methods of decomposition based on geometrical transformation which are Bipartite method and Reduction method. Then, we estimate and compare the robustness level of the networks G_k and C_k that have the same average degree of nodes.

Key words: Number of spanning trees, asymptotic entropy, network reliability, complex network, small world network, bipartite method, reduction method

INTRODUCTION

In real life, often complex systems can be modeled by networks, the nodes represent the components of the system and links symbolize their interaction. For example, social networks, airline networks, biological networks and so on (Zhang, 2015; Reggiani *et al*., 2010; Javari *et al*., 2016). The complex systems characterized by their structural properties such as degree correlation, average distance, clustering coefficient,network synchronization, number of spanning trees and other characteristics (Newman *et al*., 2006; Cancho *et al*., 2001; Newman, 2003; Nishikawa and Motter, 2006). Many researches have proven rigorously relations between topological and dynamical properties of network. Then, knowing the topological properties of network, we expect accurately its behavior and its dynamical characteristics. In graph theory, the asymptotic spanning tree entropy present a powerful tools to interpret and analyze the relationship between structural proprieties and reliability of networks. The robustness and reliability is the ability of a network to continue performing well when it is subject to failures (Wang *et al*., 2006; Myrvold *et al*., 1991; Colbourn, 1987). The asymptotic spanning tree entropy is very useful to measure the level of reliability and robustness of networks.

In this study, we use the spanning trees entropy to estimate and compare the robustness of two kind of small-world networks G_k and C_k. The entropy of G_n^k denoted by $\rho(G_k)$ defined (Lyons, 2005) as the limiting value:

$$\rho(G_k) = \lim_{|V(G_k)| \to +\infty} \frac{\log \tau(G_k)}{|V_{(G_k)}|}$$

First, we compute the number of spanning trees or what called the complexity $\rho(G_k)$. The best known method that computes the number of spanning trees of a graph G is the matrix-tree theorem (Kirkoff, 1847). However, computing the number of spanning trees using this method having high complexity $\Theta(n^3)$ is a demanding and difficult task namely for large graphs such as complex network “Small world network”. For this reason, there has been much interest to find alternative methods to avoid the tedious calculations by giving explicit expressions for the number of spanning trees for some networks families such as Sierpinski gaskets grids and lattices (Wu, 1977; Nikolopoulos and Papadopoulos, 2004; Shrock and Wu, 2000; Chang *et al*., 2007; Teufl and Wagner, 2011; Daouad, 2013; Liang *et al*., 2014). But most methods proposed require a lot of algebraic calculation. In this research, we propose an efficient combinatorial method based on principle of geometrical transformation (Bipartite and reduction) to compute the number of spanning trees

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in the small world network G_k and C_k. Then, we compare the robustness level of G_k and C_k which have the same average degree of nodes.

MATERIALS AND METHODS

Methods of decomposition (reduction and bipartite): In this study, we introduce two approaches used in the construction of the small world networks G_k and C_k, the reduction method and the bipartite method. Reduction and bipartite are tow methods of decomposition based on principle geometrical transformation consist to reduce the number of vertex and edges in a multiple graph G.

Let G be a planar graph. The reduced graph Red_2 (G) is the graph obtained when we add a new edge that connects each two adjacent vertex of G. If we add y-1 edges the obtained graph denoted by Red_y (G), called the y-reduced graph of G (Fig. 1).

Propriety: Let G be a planar graph and Red_y (G) its y-reduced graph, some structural parameters of Red_y (G) depending on the parameters of G are given by:

- The number of vertex in Red_y (G) is: $|V(Red_y(G))| = |V(G)|$
- The number of edges is: $|E(Red_y(G))| = y\times|E(G)|$
- The number of faces is: $|F(Red_y(G))| = |F(G)|+(y-1)\times|E(G)|$

Theorem 1: Let G be a planar graph and Red_y (G) the y-reduced graph of G. The number of spanning trees in Red_y (G) is given by Lotfi *et al*. (2015):

$$\tau(Red_y(G)) = y^{|V_G|-1}\times\tau(G) \quad (1)$$

Let G be a planar graph. If we add a new vertex in each edge of G, the graph obtained Bip_2 (G) is The Bipartite graph of G. When we add x-1 new vertex, the obtained graph denoted by Bip_x (G), called the x-Bipartite graph of G (Fig. 2). Now, we give the same structural proprieties for the Bip_x (G).

Propriety 2: Let G be a planar graph and Bip_x (G) its y-reduced graph

- The number of vertex in Bip_x (G) is: $|V(Bip_x(G))| = |V(G)|+(x-1)\times|E(G)|$
- The number of edges is: $|E(Bip_x(G))| = x\times|E(G)|$
- The number of faces is: $|F(Bip_x(G))| = |F(G)|$

Theorem 2: Let G be a planar graph and Bip_x (G) the x-Bipartite graph of G. The number of spanning trees in Bip_x (G) is given by Lotfi *et al*. (2015):

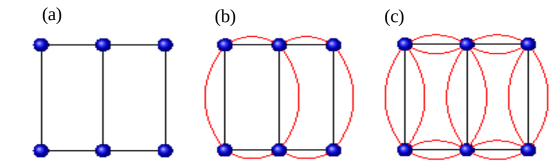

Fig. 1: A graph G, 2-reduced graph and 3-reduced graph: a) G; b) Red_2 (G) and c) Red_3 (G)

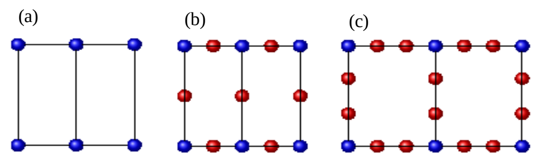

Fig. 2: A graph G, 2-bipartite and 3-bipartite graphs: a) G; b) Bip_2 (G) and c) Bip_3 (G)

$$\tau(Bip_x(G)) = x^{|F_G|-1}\times\tau(G) \quad (2)$$

The entropy of the small world networks G_k and C_k: In this study, reduction method and bipartite method are combined in a recursive way to construct the small world networks G_k and C_k where G_k and C_k are the network formed from G_{k-1} and C_{k-1} the networks of previous iteration k-1, after the reduction transformation and the bipartite transformation. In this research, we study the both cases:

- Reduction transformation followed by bipartite transformation:

$$G_n^k = Bip_x \circ Red_y(G_{k-1}) = Bip_x(Red_y(G_{k-1}))$$

- Bipartite transformation followed by reduction transformation:

$$C_n^k = Red_y \circ Bip_x(C_{k-1}) = Red_y(Bip_x(C_{k-1}))$$

The entropy of G_k: The construction of the small world networks G_k is iterative as follow: $G_k = Bip_x{}^\circ Red_y (G_{k-1}) = Bip_x (Red_y (G_{k-1}))$, the process start with G_0 which is simple path contains two vertex, at the next step, the reduced method is applied upon on G_0 adding y-1 new edges, followed by the bipartite method adding x-1 vertex in each edge of the network Red_y (G_0). So, the network G_1 is formed after these geometrical transformations as is illustrated in Fig. 3 where x = 4 and y = 3 $G_1 = Bip_4{}^\circ Red_2 (G_0) = Bip_4 (Red_2 (G_0))$. Then, the process go to the next iteration in the same way until the kth iteration. We give the topological proprieties of the network G_k.

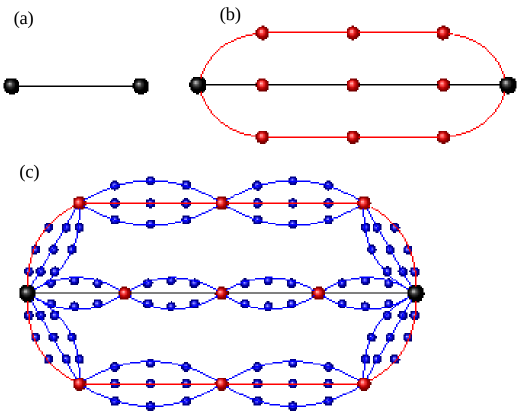

Fig. 3: Networks of the first three iterations of G_k: a) G_0; b) G_1 and c) G_2

Lemma 3.1: Let G_k a small world network. The number of vertex, edges and faces are given by:

- $|E(G_k)| = (xy)^k$
- $V(G_k)| = 2+y(x-1)\frac{(xy)^k-1}{xy-1}$
- $|F(G_k)| = 1+(y-1)\frac{(xy)^k-1}{xy-1}$

Theorem 3.2: The number of spanning trees in the the small world network G_k is given by:

$$\tau(G_k) = x^{\alpha}y^{\beta} \quad (3)$$

$$\alpha = \left(\frac{(y-1)[xy((xy)^k-1)-k(xy-1)]}{(xy-1)^2}\right);$$

$$\beta = \left(\frac{(x-1)(y((xy)^k-1)+k(y-1)(xy-1)}{(xy-1)^2}\right)$$

Proof: According to the construction of G_k:

$$\tau(G_k) = \tau(Bip_x \circ Red_y(G_{k-1})) = \tau(Bip_x(Red_y(G_{k-1})))$$

by using theorem 1 and 2, we get:

$$\tau(G_k) = x^{(|F(Red_y(G_{k-1})|-1)}\tau(Red_y(G_{k-1}))$$

$$\tau(G_k) = x^{(|F(Red_y(G_{k-1})|-1)}y^{(|V((G_{k-1})|-1)}\tau(G_{k-1})$$

by using property 1, we substitute $|F(Red_y(G_{k-1}))|$ by its value we obtain:

$$\tau(G_k) = x^{(|F(G_{k-1})|+(y-1)|E(G_{k-1})|-1)}y^{(|V((G_{k-1})|-1)}\tau(G_{k-1})$$

$$\tau(G_{k-1}) = x^{(|F(G_{k-2})|+(y-1)|E(G_{k-2})|-1)}y^{(|V((G_{k-2})|-1)}\tau(G_{k-2})$$

$$\tau(G_1) = x^{(|F(G_0)|+(y-1)|E(G_0)|-1)}y^{(|V((G_0)|-1)}\tau(G_0)$$

where, $\tau(G_0) = 1$. We multiply these equations, then, we get:

$$\tau(G_k) = x^{\sum_{i=0}^{k-1}(|F(G_i)|+(y-1)|E(G_i)|-1)}y^{\sum_{i=0}^{k-1}(|V((G_i)|-1)}$$

using lemma 3.1 to substitute $|V(G)|$, $|F(G_i)|$ and $|E(G_i)|$:

$$\tau(G_k) = x^{\sum_{i=0}^{k-1}(\frac{(y-1)((xy)^i-1)}{(xy-1)}+(y-1)(xy)^i)(}y^{\sum_{i=0}^{k-1}\frac{y(x-1)((xy)^i-1)}{(xy-1)}+1)}$$

we calculate the both summation, then, we get the result.

Corollary 3.3: Let G_k be a small world network, the asymptotic spanning tree entropy of G_k given by:

$$\rho(G_k) = \frac{(y-1)x\log(x)+(x-1)\log(y)}{(x-1)(xy-1)} \quad (4)$$

Proof: As is already defined, the asymptotic entropy of G_k is a the limiting value:

$$\rho(G_k) = \lim_{|G_k|\to+\infty}\frac{\log\tau(G_k)}{|G_k|}$$

Using the above theorem and substituting $\tau(G_k)$ by its expression where:

$$|V(G_k)| = 2+y(x-1)\frac{(xy)^k-1}{xy-1}$$

We get:

$$\rho(G_k) = \lim_{|G_k|\to+\infty}\frac{\log(\tau(G_k))}{|G_k|}$$

$$\lim_{|G_k|\to+\infty}\frac{\log(x^{(\frac{(y-1)[xy((xy)^k-1)-k(xy-1)]}{(xy-1)^2})}y^{(\frac{(x-1)(y((xy)^k-1)+k(y-1)(xy-1)}{(xy-1)^2})})}{2+y(x-1)\frac{(xy)^k-1}{xy-1}}$$

$$= \frac{(y-1)x\log(x)+(x-1)\log(y)}{(xy-1)(x-1)}$$

Hence, the result.

The entropy of C_k: The construction of the small world networks C_k is similar to the construction of G_k it is as follow:

$$C_k = Red_y \circ Bip_x(C_{k-1}) = Red_y(Bip_x(C_{k-1}))$$

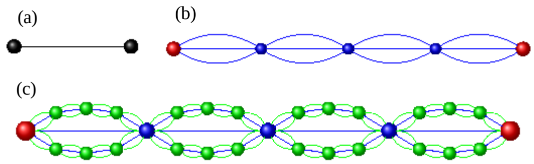

Fig. 4: Networks of the first three iteration of C_k; a) C_0; b) C_1 and c) C_2

At the first iteration, we apply the bipartite method by adding new x-1 vertex to C_0, then, we apply the reduced method, adding y-1 new edges between each two adjacent vertex in the network $Bib_x(C_0)$. So, the network C_1 is built after these geometrical transformations (Fig. 4) with x = 4 and y = 2 ($C_1 = Red_2 \circ Bip_4(C_0)$). Then with the same manner, the process is repeated in the next iteration until the kth iteration.

Lemma 4.1: Let C_k a small world network. The structural proprieties of the network C_k are given by:

- The number of vertex: $|E(C_k)| = (xy)^k$
- The number of edges $|V(C_k)| = 2+(x-1)(xy)^k{}_{-1}/xy-1$
- The number of faces $|F(C_k)| = 1+x(y-1)(xy)^k{}_{-1}/xy-1$

Theorem 4.2: The number of spanning trees in the the small world network C_k is given by:

$$\tau(C_k) = y^{\lambda}x^{\mu} \quad (5)$$

$$\lambda = \frac{(x-1)xy((xy)^k-1)+kx(y-1)(xy-1)}{(xy-1)^2}$$

$$\mu = \frac{x(y-1)\left[((xy)^k-1)-k(xy-1)\right]}{(xy-1)^2}$$

Proof: According to the construction of C_k:

$$\tau(C_k) = \tau(Red_y \circ Bip_x(C_{k-1})) = \tau(Red_y(Bip_x(C_{k-1})))$$

By using theorem 1 and 2, we get:

$$\tau(C_k) = y^{(|VBip_x(C_{k-1})|-1)}\tau(Bip_x(C_{k-1}))$$

$$\tau(C_k) = y^{(|VBip_x(C_{k-1})|-1)}x^{(|F((C_{k-1})|-1)}\tau(C_{k-1})$$

By using property 2, we substitute $|V(Bip_x(C_{k-1}))|$ by its value we obtain:

$$\tau(C_k) = y^{(|V(C_{k-1})|+(x-1)|E(C_{k-1})|-1)}x^{(|F((C_{k-1})|-1)}\tau(C_{k-1})$$

$$\tau(C_{k-1}) = y^{(|V(C_{k-2})|+(x-1)|E(C_{k-2})|-1)}x^{(|F((C_{k-2})|-1)}\tau(C_{k-2})$$

$$\vdots$$

$$t(C_1) = y^{(|V(C_0)|+(x-1)|E(C_0)|-1)}x^{(|F((C_0)|-1)}\tau(C_0)$$

where $\tau(C_0) = 1$. We multiply these equations, then we get:

$$\tau(C_k) = y^{\sum_{i=0}^{k-1}(|V(C_i)|+(x-1)|E(C_i)|-1)}x^{\sum_{i=0}^{k-1}(|F((C_i)|-1)}$$

using lemma 4.1 to substitute $|V(G_i)|$, $|F(G_i)|$ and $|E(G_i)|$:

$$\tau(C_k) = y^{\sum_{i=0}^{k-1}\left(1+\frac{(x-1)((xy)^i-1)}{(xy-1)}+(x-1)(xy)^i\right)}x^{\sum_{i=0}^{k-1}\left(\frac{x(y-1)((xy)^i-1)}{(xy-1)}\right)}$$

we calculate these summation, then, we get the result.

Corollary 4.3: Let C_k be a small world network, the asymptotic spanning tree entropy of G_k given by:

$$\rho(C_k) = \frac{xy(x-1)\log(x)+x(y-1)\log(y)}{(x-1)(xy-1)} \quad (6)$$

Proof: As is already defined, the asymptotic entropy of C_k is:

$$\rho(C_K) = \lim_{|C_k| \to +\infty} \frac{\log\tau(C_k)}{|C_k|}$$

Using the above theorem and substituting (C_k) by its expression where:

$$V(C_k)| = 2+(x-1)\frac{(xy)^k-1}{xy-1}$$

we get:

$$\rho(C_k) = \lim_{|C_k| \to +\infty} \frac{\log\tau(C_k)}{|C_k|}$$

$$\lim_{|C_k| \to +\infty} \frac{\log\left(y^{\left(\frac{(x-1)xy((xy)^k-1)+kx(y-1)(xy-1)}{(xy-1)^2}\right)} x^{\left(\frac{x(y-1)[((xy)^k-1)-k(xy-1)]}{(xy-1)^2}\right)}\right)}{2+(x-1)\frac{(xy)^k-1}{xy-1}}$$

$$= \frac{xy(x-1)\log x+x(y-1)\log y}{(x-1)(xy-1)}$$

RESULTS AND DISCUSSION

Results interpretation: Now we compare the asymptotic entropy of the two complex networks G_k and C_k which have the same average degree. First we give some particular values for x and y, for example, we choose x = 4, we get curves of $\rho(C_k)$ and $\rho(G_k)$ depending on y. And we give y = 3, we get curves of $\rho(C_k)$ and $\rho(G_k)$ depending on x. Next, we plot $\rho(C_k)$ and $\rho(G_k)$ as multiple variables

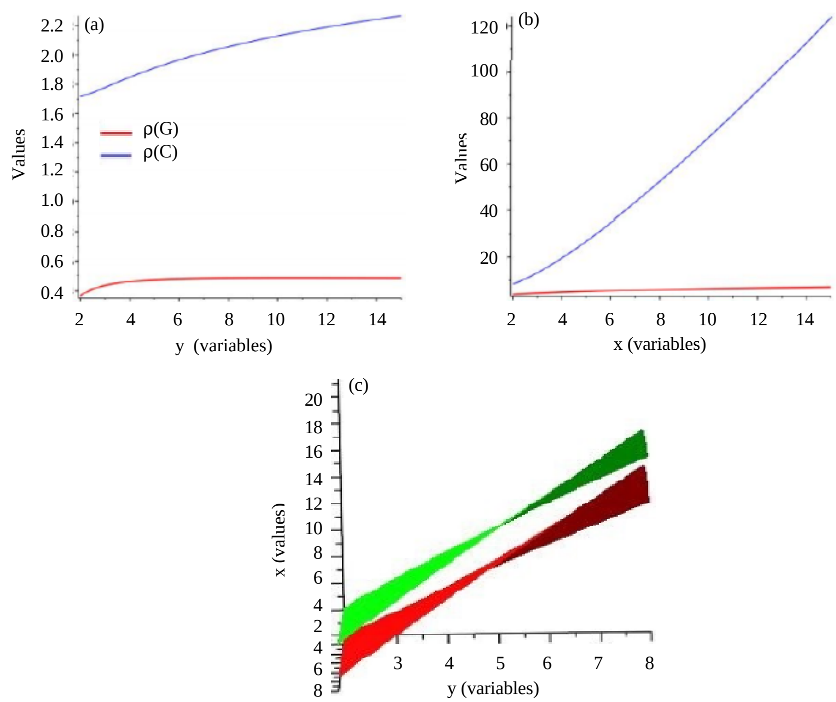

Fig. 5: Multiple variables functions: a) The curves are ploted with x = 4; b) The curves are ploted with x = 3 and c) The curve pf ρ(G) is red and ρ(C) is green

functions that depend on x and y. All obtained Fig. 5 show clearly $\rho(C_k)$ is higher than $\rho(G_k)$. Indeed, algebraically we prove $\rho(C_k)$ is higher than $\rho(G_k)$:

$$\rho(C_k)-\rho(G_k)=\frac{xy(x-1)\log x+x(y-1)\log y}{(x-1)(xy-1)}-\frac{(y-1)x\log x+(x-1)\log y}{(x-1)(xy-1)}$$

$$\upsilon(C_k)-\rho(G_k)=\frac{x(yx-2y+1)\log(x)+(xy-2x+1)\log(y)}{(x-1)(xy-1)}$$

$(x\geq 2)$ and $(y\geq 2)\Rightarrow(C_k)-\rho(G_k)\geq 0$. It means the network C_k has more spanning trees than the other network G_k. Thus, This result proves the network C_k is more robust and more reliable than the network G_k. Then, owing to the growth of the number of spanning trees in C_k which provides more connection between nodes related by eventual interrupted links that ensures more reliability and robustness and avoid having dysfunction of this network.

CONCLUSION

The asymptotic spanning tree entropy of a network is used to measure reliability and robustness of network. In this study, We proposed two combinatorial approaches: the bipartite and the reduction to construct two examples of small world network by giving their topological properties, computing their number of spanning trees. Finally, we evaluated their spanning tree entropy in order to estimate and compare the level of robustness of these two type of complex network that have the same average degree.

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