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# Several Norm Inequalities for Matrices Partitioned into a Small Number of Blocks

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**Abstract:** Positive matrices of the form $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ have a property that $\left\|\begin{pmatrix} A & X \\ X & B \end{pmatrix}\right\| \leq \|A+B\|$. In this study, we will give proof of the inequality $\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\| \leq \|A+B\|$ which proved before by Bourin in (2005) and use it to give the generalizations due to Ando and Zhanin (1999).

**Key words:** Symmetric norms, operator norm, normal matrices, unitarily congruent, matrix may, partitioned

## INTRODUCTION

Capital letters A, B, …, Z mean n-by-n complex matrices or operators on an n-dimensional Hilbert space H. If A is positive semidefinite, resp. positive definite, we write $A\geq 0$, resp. A>/0. Let $\|A\|$ denote a symmetric norm (or unitarily invariant norm) which satisfies $\|UAV\| = \|A\|$ for all A and for all unitaries U, V. Ando and Zhan (1999), Horn and Johnson (1991, 1985), Bhatia (1997), Bourin (2005, 2006a, b), Bhatia and Kittaneh (1990) and Bhatia and Kittaneh (2008). Throughout this study the symbol $\|.\|_\infty$ means the operator norm and for A, $B\geq 0$, we can write $A \prec_w B$ to mean that $\|A\|\leq\|B\|$ (Ando, 1994).

## MATERIALS AND METHDOS

### Preliminaries

**Definition 3.1:** For any matrix A , we define the absolute value |A| of A to be the positive semidefinite matrix square root of A*A. Then the singular values of A, $s_1(A)$, ..., $s_n(A)$ are defined to be the eigenvalues of |A| which ordered from largest to smallest $s_1(A) = s_2(A) = \geq, ..., \geq = s_n(A)$. In this study, we will apply Ky Fan's maximum principle to the Hermitian matrix A, we get for each k = 1, 2, ..., n that:

$$\sum_{j=1}^{k}\lambda(A) = \max \sum_{j=1}^{k}\langle Ax_j, y_j\rangle$$

where, the maximum is taken over all choices of orthonormal k-tuples $x_1, ..., x_k$ and $y_1, ..., y_k$. Given a natural number k such that $1\leq k\leq n$, the Ky Fan k-norm of a matrix A, denoted by is defined to be the sum of k largest singular values of A. That is:

$$\|A\|_{(k)} = \sum_{j=1}^{k}s_j(A)$$

Also, Fan dominance theorem saying that for A, B, then: $\|A\|\leq\|B\|$ for all symmetric norms if and only if $\|A\|_{(k)}\leq\|B\|_{(k)}$ for all k = 1, 2, ..., n.

## RESTULS AND DISCUSSION

### Several inequalities for block-matrices and the main result

**Lemma 4.1:** Let C, D, S, $T\geq 0$ such that $C\prec_w S$ and, $D\prec_w T$. Then:

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \prec_w \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$$

**Proof :** We have:

$$\sum_{j=1}^{k} \lambda_j(C\oplus D) = \max_{r+n=k}\left\{\sum_{j=1}^{k} \lambda_j(C) + \sum_{j=1}^{k} \lambda_j(D)\right\}$$

Also:

$$\sum_{j=1}^{r} \lambda_j(C) + \sum_{j=1}^{n} \lambda_j(D) \leq \sum_{j=1}^{r} \lambda_j(S) + \sum_{j=1}^{n} \lambda_j(T) \leq \sum_{j=1}^{k} \lambda_j(S\oplus T)$$

Ends the proof.

**Lemma 4.2:** Let C, $D\geq 0$. Then:

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \prec_w \begin{pmatrix} C+D & 0 \\ 0 & 0 \end{pmatrix}$$

**Proof:** We can write:

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$$\begin{pmatrix} C+D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C^{\frac{1}{2}} & D^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C^{\frac{1}{2}} & 0 \\ D^{\frac{1}{2}} & 0 \end{pmatrix}$$

Also:

$$\begin{pmatrix} C+D & 0 \\ 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} C & C^{\frac{1}{2}}D^{\frac{1}{2}} \\ D^{\frac{1}{2}}C^{\frac{1}{2}} & D \end{pmatrix} \simeq \begin{pmatrix} C & -C^{\frac{1}{2}}D^{\frac{1}{2}} \\ -D^{\frac{1}{2}}C^{\frac{1}{2}} & D \end{pmatrix}$$

where $\simeq$ means unitarily congruent. Also:

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \frac{1}{2} \begin{pmatrix} C & C^{\frac{1}{2}}D^{\frac{1}{2}} \\ D^{\frac{1}{2}}C^{\frac{1}{2}} & D \end{pmatrix} + \frac{1}{2} \begin{pmatrix} C & -C^{\frac{1}{2}}D^{\frac{1}{2}} \\ -D^{\frac{1}{2}}C^{\frac{1}{2}} & D \end{pmatrix}$$

Ends the proof. Now, Lemma 4.2 gives the proof of the inequality:

$$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| \leq \|A+B\|$$

**Theorem 4.1:** For all block-matrices whose entries are normal matrices of same size and for all symmetric norms:

$$\left\| \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} \right\| \leq \big\| |X|+|Y|+|W|+|Z| \big\|$$

**Theorem 4.2:** For all block-matrices whose entries are normal matrices of same size:

$$\left\| \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} \right\|_{\infty} \leq \max \left\{ \begin{matrix} \big\| |X|+|Y| \big\|_{\infty}, \big\| |W|+|Z| \big\|_{\infty} \\ \big\| |X|+|W| \big\|_{\infty}, \big\| |Y|+|Z| \big\|_{\infty} \end{matrix} \right\}$$

**Proof:** Let $A_1$, $A_2$, $B_1$, $B_2$ be positive and be contractions. We get:

$$A_1C_1B_1+A_2C_2B_2 = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Now we will use the Cauchy-Shwarz inequality $\|AB\| \leq \|A^*A\|^{1/2}\|BB^*\|^{1/2}$ and $\|ST\| \leq \|S\|_{\infty}\|T\|$ to get $\|A_1C_1B_1+A_2C_2B_2\| \leq \|A_1^2+A_2^2\|^{1/2}\|B_1^2+B_2^2\|^{1/2}$. But the polar decompositions $X = |X^*|^{1/2}U|X|^{1/2}$ and $Y = |Y^*|^{1/2}V|Y|^{1/2}$ show that:

$$\big\| |X+Y| \big\| \leq \big\| |X|+|Y| \big\|^{\frac{1}{2}} \big\| |X^*|+|Y| \big\|^{\frac{1}{2}} \quad (1)$$

For all X, Y. We will replace X and Y in Eq. 1 by:

$$\begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix} \text{and} \begin{pmatrix} 0 & Y \\ W & 0 \end{pmatrix}$$

And using normality for X, Y, W, Zwe get:

$$\left\| \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |X|+|W| & 0 \\ 0 & |Y|+|Z| \end{pmatrix} \right\|^{\frac{1}{2}} \left\| \begin{pmatrix} |X|+|Y| & 0 \\ 0 & |W|+|Z| \end{pmatrix} \right\|^{\frac{1}{2}}$$

This proves theorem 4.2. Also, we can prove theorem 4.1by using the inequality:

$$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| \leq \|A+B\|$$

for all A, $B \geq 0$:

## CONCLUSION

This study will gives the generlizations due to Ando and Zhan in (1999).


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