

On Classification of Fuzzy Set Theory

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Abstract: In this study three definitions about fuzzy sets have given and a new five families of fuzzy sets was introduced and added to the two families which introduced. After analytical study a classification was done for the families, also introduced four new types of neighborhoods. We redefined the first type of fuzzy points and introduced two new types of fuzzy points with some notes the investigation of basic operations on fuzzy sets are presented via illustrative examples.

Key words: Families, analytical study, classification, fuzzy points, fuzzy sets, illustrative

INTRODUCTION

Zadeh *et al.* (1975) put the underpinning of the fuzzy set theory his thoughts concentrated on the idea of degree or grade of membership which is the concept that became the backbone of fuzzy set theory. In present it was found that many studies noticed as engineers and statisticians that different problems in real life involved different types of uncertainty as vagueness and imprecision on data (Ma *et al.*, 2014). Through their published literature for either books or papers in many years ago and for different international publishment houses which are famous in the world. Zadeh *et al.* (1975), Dubois and Prade (1980), Klir *et al.* (1997) and Zimmermann (2001) they have talked about only two families of fuzzy sets. In the view of a deep analytical study for these literatures, we get a new five families, a classification for the families that involve the two families of Zadeh *et al.*, (1975), Dubois and Prade (1980), Klir *et al.* (1997) and Zimmermann (2001) which is named by us second family and third family, also our new five families are named first, fourth, fifth, sixth and seventh families, respectively, they are worth to care because they were played a necessary role via the several applications, for example, the engineering application in the case of studying the sections of a specified electrical waves, it is meant by a section is the partition of the domain which is studied in the partitions of the domain for the wave as an example, for the first family which is produced by us. And another example for the fourth family is the studying of different electrical waves on all the partitions of the domain or some domain partitions, here appear the importance of these new families and a third example, of applications is in fuzzy reliability safety to avoid roads risk is the design of the shassies of

vehicles, the metal used in the structure of shassies composed of a steel mixed with manganize have pneumatic holes as a qualitative property which make the shassies more flexiable to absorb the vibrated waves which happened especially in bad roads to a void the dangerous failure in vehicles by absorbing the impact in roads as a damped waves. An illustrative examples are presented about the families mentioned above.

The seven families have an influence on topological spaces, they open a wide area especially the seventh family in engineering, control systems, nano applications, neural networks applications, total quality management, GIS applications, economics, medical science, social science, environment, teaching evolution, computer science, etc., that is in a wide branches of science involves data which are not always all crisp, deterministic and precise in character. There are many theories as the theory of fuzzy sets which can be considered as mathematical tools for dealing with different types of uncertainty in these problems which represents the track of scientific competitive between the countries (Sabri and Ahmed, 2018).

In the current study the first type of fuzzy points is redefined and two new types of fuzzy points are introduced with examples and by virtue of the classifications of fuzzy sets. It can be characterized the fuzzy topology by means of neighborhood (in short nhd) systems, a four new types of nhd's which help the engineers and scientist for deep study to the graphs or a complete band of graphs on all the definite intervals assigned on the domain universe in order to study its properties and the range of variation happened in it between an interval and the another, it is the answer of the question why we classify the fuzzy sets?

MATERIALS AND METHODS

Preliminaries: In this study some definitions and notions about fuzzy sets and fuzzy topological spaces are given. These will be need it in later sections.

Definition 2.1; Zimmermann (2001): Let X be a universal set, then a fuzzy subset \tilde{A}_i of X_i is a set of ordered pairs:

$$\tilde{A} = \{x, \mu_A(x) : x \in X\}$$

$\mu_A(x)$ is called the membership function where $\mu_A : X \rightarrow [0, 1]$. Which assign a real number $\mu_A(x)$ in the interval $[0, 1]$, to each element $x \in X$. Where the value of $\mu_A(x)$ at x shows the grade of membership of x in A .

Definition 2.2; Zadeh et al. (1975): A fuzzy topology on a universe set X is a family of fuzzy sets in X which satisfies the following conditions:

$$\emptyset, X \in \tau$$

$$\text{If } A, B \in \tau, \text{ then } A \cap B \in \tau$$

$$\text{If } A_i \in \tau \text{ for each } i \in I \text{ then } \bigcup_{i \in I} A_i \in \tau$$

The, τ is called a fuzzy topology for X and the pair (X, τ) is a fuzzy topological space (simply fts). Any element of τ is called τ -open set or fuzzy open set, A fuzzy set is τ -fuzzy closed or fuzzy closed, if its complement is fuzzy open set (Rodabaugh and Klement, 2013). The indiscrete fuzzy topology on X consist of \emptyset and only. The discrete fuzzy topology on X contains all fuzzy subsets of X.

Definition 2.3; Zimmermann (2010) and Mendel (2017): Let $A = \{x, \mu_A(x) : x \in X\}$ and $B = \{x, \mu_B(x) : x \in X\}$ be two fuzzy sets in X. Then their union intersection and complement are also, sets with the membership functions defined as follows:

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X$$

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$$

$$\mu_{\lambda}(x) = 1 - \mu_A(x), \forall x \in X$$

Definition 2.4; Yen and Langari (1999) and Ma et al. (2014): A fuzzy set A w.r.t. the universe set U is called convex iff for all $u_1, u_2 \in U, \mu_A(\lambda u_1 + (1-\lambda) u_2) \geq \min(\mu_A(u_1), \mu_A(u_2))$ where $\lambda \in [0, 1]$. The support of A which is

denoted by $\text{sup}(A)$ is the set of the element that have non zero degrees of membership in A that is:

$$\text{sup } p(A) = \{u : u \in U \text{ and } \mu_A > 0\} \text{ and}$$

$$A \text{ is called normal, if } \exists u \in U, \mu_A(u) = 1$$

Definition 2.5; Mercer et al. (2002): A fuzzy point is a circular region representing the uncertain location of a normal Euclidean point. It is in the plane is considered to be a closed disc (a circle and its interior)

Definition 2.6; Chang (1968): A fuzzy set a in a fuzzy topological space (X, τ) is a fuzzy nhd of a fuzzy set B in X iff \exists an open set $O \in \tau \ni B \subseteq O \subseteq A$.

Definition 2.7; Tripathy and Ray (2014): A fuzzy point x_λ is said to be quasi-coincident (briefly q-coincident) with a fuzzy set A in X which is denoted by $x_\lambda qA \leftrightarrow \lambda + A_x > 1$ or $\lambda > A^c(x)$.

Example 2.8: Let $X = \{a, b, c\}$ be a set and $a_{0.5}$ be a fuzzy point in X and A-C are fuzzy sets in X defined as below:

$$A(a) = 0.1, B(a) = 0.9, C(a) = 0.8$$

$$A(b) = 0.6, B(b) = 0.3, C(b) = 0.4$$

$$A(c) = 0.5, B(c) = 0.4, C(c) = 0.6$$

$a_{0.5}$ is q-coincident with a fuzzy set B, since, $0.5 + B(a) = 0.5 + 0.9 = 1.4$.

Definition 2.9; Tripathy and Ray (2014): A fuzzy set A is said to be quasi coincident with a fuzzy set B and is denoted by AqB iff $\exists x \in X \ni A(x) + B(x) > 1$. Return to example (2.8): $A(c) + C(c) = 0.5 + 0.6 = 1.1 > 1$, so, AqC but $A(c), B(c)$ are not q-coincident, since, $A(c) + B(c) = 0.5 + 0.4 = 0.9$.

Families of fuzzy sets: In this study we will give some definitions and obtain a new families of fuzzy sets which are worth to care. According to, the current analytical study for fuzzy sets a classification of fuzzy sets is presented for the families and we defined and denoted these families. Basic operations on fuzzy sets are investigated via. an illustrative examples. Notes are listed about the relation between the families.

We introduce the following definitions

Definition 3.1.1: Let $\mu(X, I) = \{f : f : X \rightarrow I\}$ be the set of all functions f on the universe set X to the unit interval $I = [0, 1]$.

Definition 3.1.2: Let $\Phi_A(x) = \{f_i A: \forall f_i \in \mu(X, 1), f_{iA}(x) = \{f_i(x), \forall x \in A\}$ be the set of all restriction functions in $\mu(X, I)$ with respect to the subset A of X.

$$\tilde{A} \cap \tilde{B} = \min_{x \in X} \{f_A(x), f_B(x), \forall x \in X\}$$

Definition 3.1.3: Let $\Psi_{A_i}(X) = \{f_{iA_i}: \forall f_i \in (\mu, I)\} \forall A_i \subseteq X$, where $f_{iA_i}, \forall x \in A_i$ be the set of all restriction functions in $\mu(X, I)$ with respect to the subsets A_i of X.

$$\tilde{A} \cap \tilde{B} = \min_{x \in A \cap B} (f_A(x), f_B(x))$$

First family: We introduce a new family which it means that it is containing all fuzzy sets that depends on all sets contained in the power set of $X(A \in 2^X)$ for fixed function, f such that $f: X \rightarrow I$, we call it the first family and denoted it I_1^x that is:

and in the case $x \in X$ it is not valid. From above we conclude that the intersection and union of two fuzzy sets valid within the common points between the domains of the fuzzy sets i.e. the first coordinate in which their functions in this points are defined on both sets.

$$I_1^x = \left\{ \begin{array}{l} \tilde{A}_1; \tilde{A}_1 = \{(x, f_{A_1}(x)) : \forall x \in A_1 \\ \forall A_1 \in 2^X, f_{A_1}(x) = \{f(x), \forall x \in A_1\} \end{array} \right.$$

In example 3.2.1 in I_1^x we illustrate that the set \tilde{A} contains $\{1\}$ which means that its function defined only in the point 1 and undefined in other points $\{2, 3, \dots, 100\}$ whereas the function of set \tilde{B} is defined in $\{1, 2, 99\}$ and undefined in other points on X, so, the two operations intersection and union valid only in the common points between the domains of both fuzzy sets and these compel us to extend the fuzzy sets (i.e., universal set X in order that all the functions of fuzzy sets become defined).

Example 3.2.1: Let $X = \{1, 2, \dots, 100\}$ and

$$f(x) = \begin{cases} \frac{1}{x} & \forall x \in X/\{100\} \\ 0 & x = 100 \end{cases}$$

For $A = \forall A \in P(X), \tilde{A} = \{(x, f_A(x)) : f_A(x) = f(x), \forall x \in A\}$. For $A = \{1\} \forall x \in X/\{1\}$ undefined $\tilde{A} = \{(1, 1)\}, B = \{1, 2, 99\}, \tilde{B} = \{(1, 1), (2, 1/2), (99, 1/99)\}$.

Second family: The second family which we denoted it by I_2^x at the subset A of X. In other words, we fixed the subset A and compute the restriction for all functions f in $\mu(X, I)$, that is:

Note; By Ma et al. (2014), Klir and Yaun (2002): The fuzzy subset A of a fuzzy set B is defined as follows:

$$I_1^x = \left\{ \tilde{A} : \tilde{A} = \left\{ \begin{array}{l} (x, f_{iA}(x)), \forall f_i \in \Phi_A, (x) f_{iA}(x) = \\ f_i(x), \forall x \in A \end{array} \right. \right\}$$

$$\text{if } \tilde{A} \subseteq \tilde{B} \text{ then } \mu(A) \leq \mu(B) \forall x \in X$$

Example 3.3.1: Let:

$$X = \{0, 1, 2, 3, 4, 5\} \text{ and } f(x) = \frac{1}{1+x} \forall x \in A$$

In our study, $\tilde{A} \subseteq \tilde{B} \leftrightarrow (\mu(A) \leq \mu(B) \forall x \in X)$

$$f(0) = 1, f(1) = \frac{1}{2}, f(2) = \frac{1}{3}, f(3) = \frac{1}{4}, f(4) = \frac{1}{5}, f(5) = \frac{1}{6}$$

The subset for fuzzy sets in I_1^x is defined as follows: $\forall \tilde{A}, \tilde{B} \in I_1^x$ the fuzzy subset \tilde{A} to the fuzzy subset \tilde{B} is as follows $\tilde{A} \subseteq \tilde{B}$ iff : $1-A \subseteq B$ $2-f_A(x) \leq f_B(x), \forall x \in A$ where

And let:

$$g(x) = \frac{1}{3+2x}, \forall x \in A$$

$$\tilde{A} = \{(x, f_A(x)) | x \in A\}$$

Let:

$$A = \{0, 1, 2\} \subseteq X$$

$$\tilde{B} = \{(x, f_B(x)) | x \in A\}$$

$$\tilde{B} = \{(x, f_A(x)) ; x \in A\} = \left\{ (0, 1), \left(1, \frac{1}{2}\right), \left(2, \frac{1}{3}\right) \right\}$$

Intersection and union of two fuzzy sets:

Let $\tilde{A}, \tilde{B} \in I_1^x$, then

$$\tilde{C} = \{x, g_A(x) | x \in A\} = \left\{ \left(0, \frac{1}{3}\right), \left(1, \frac{1}{5}\right), \left(2, \frac{1}{7}\right) \right\}$$

$$\tilde{A} \cap \tilde{B} = \min_{x \in A \cap B} \{f_A(x), f_B(x), \forall x \in A \cap B\}$$

The subset for a fuzzy set is defined as follows:

$$\forall \tilde{A}, \tilde{B} \in I_2^X$$

$$\tilde{A} \subseteq \tilde{B} \leftrightarrow f(x) \leq g(x) \quad \forall x \in X$$

$\tilde{A} \cap \tilde{B} \leftrightarrow f_A(x) \leq g_A(x), \forall x \in A$, where

Intersection and union of two fuzzy sets; Let:

$$\tilde{A} = \{(x, f_A(x), \forall x \in A), \tilde{B} = \{(x, g_A(x), \forall x \in A)\}$$

$$\tilde{A}, \tilde{B} \in I_3^X, \text{ then}$$

Intersection and union of two fuzzy sets; Let:

$$(A)^0 \cap B^0 = \min_T(x \in X) \{(f(x), (g(x)))\}$$

$$\tilde{A}, \tilde{B} \in I_2^X, \text{ then}$$

$$(A)^0 \cup B^0 = \max_T(x \in X) \{(f(x), (g(x)))\}$$

$$\tilde{A} \cap \tilde{B} = \min_{x \in A} \{(f_A(x), (g_A(x))), \forall x \in A \cap B\}$$

Fourth family: It means that is containing all fuzzy sets depends on all functions $f_{i_{A_i}} \in \Psi_{A_i}(x)$ and we donate this family by I_4^X that is:

$$\tilde{A} \cup \tilde{B} = \min_{x \in A} \{(f_A(x), (g_A(x)))\}$$

$$I_4^X = \{\tilde{A}_i; \tilde{A}_i = \{x_i, f_{i_{A_i}}(x); f_{i_{A_i}} \in \Psi_{A_i}(x)\}$$

Third family: The third family which it means that it is containing all fuzzy sets which depends on all functions $f_i \in \mu(X, I)$, we denoted it by I_3^X , that is:

Example 3.5.1; Let:

$$I_3^X = \{\tilde{A}, \tilde{A} = (x, f(x), \forall f_i \in \mu(X, I))\}$$

$$X = \{x_1, x_2, \dots, x_9\}, \quad i = 1, 2, \dots, 9$$

And:

$$A = \{x_2, x_4, x_6, x_8\} \quad B = \{x_1, x_3, x_5, x_7, x_9\}$$

Example 3.4.1: Let $X = \{1, 2, 3, 4, 5\}$ and $f(x) = 1/x, \forall x \in X$:

$$g(x) = \frac{1}{1+x}, \quad \forall x \in X$$

$$f_{(x_i)} = \frac{1}{(1+i)^2 - 1}, \quad g_{(x_i)} = \frac{1}{2i^2 - 1}$$

$$h(x) = \frac{1}{|2-x|}, \quad \forall x \in X$$

$$\tilde{A} = \{(x, f(x)), \quad \forall x \in A\} =$$

$$\left\{ \left(x_2, \frac{1}{8}\right), \left(x_4, \frac{1}{24}\right), \left(x_6, \frac{1}{48}\right), \left(x_8, \frac{1}{80}\right) \right\}$$

And let:

$$\tilde{A} = \{(x, f(x), \forall x \in X), \tilde{B} = \{(x, g(x), \forall x \in X)\}$$

$$\tilde{B} = \{(x, g_B(x)), \quad \forall x \in B\} =$$

$$\left\{ (x_1, 1), \left(x_3, \frac{1}{17}\right), \left(x_5, \frac{1}{49}\right), \left(x_7, \frac{1}{97}\right), \left(x_9, \frac{1}{161}\right) \right\}$$

$$\tilde{C} = \{(x, h(x), \forall x \in X), \text{ then}$$

$$\tilde{A} = \left\{ (1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right) \right\}$$

$$\tilde{B} = \left\{ \left(1, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{4}\right), \left(4, \frac{1}{5}\right), \left(5, \frac{1}{6}\right) \right\}$$

$$\tilde{C} = \left\{ (1, 1), (3, 1), \left(4, \frac{1}{2}\right), \left(5, \frac{1}{3}\right) \right\}$$

The subset for a fuzzy set is defined as follows:

$$\forall \tilde{A}, \tilde{B} \in I_4^X$$

Where:

$$\tilde{A} = \{(x, f_A(x)) | x \in A\}, \quad \tilde{B} = \{(x, g_B(x)) | x \in B\}$$

The subset for a fuzzy set is defined as follows:

$$\forall \tilde{A}, \tilde{B} \in I_3^X$$

$$\tilde{A} \subseteq \tilde{B} \leftrightarrow A \subseteq B$$

Where:

$$f_A(x) \leq g_B(x) \quad \forall x \in A$$

$$\tilde{A} = \{(x, f(x), x \in X)\}$$

Intersection and union of two fuzzy sets; Let:

$$(B)^0 = \{(x, g(x), x \in X)\}$$

$$\tilde{A}, \tilde{B} \in I_4^X, \text{ then}$$

$$\tilde{A} \cap \tilde{B} = \min_{x \in A \cap B} \{f_A(x), g_B(x)\}$$

$$\tilde{A} \cap \tilde{B} = \min_{x \in A \cap B} \{f_A(x), g_B(x)\}$$

Note: About the relation between 4th family and other families: in case the sets which belong to 2^X are fixed this implies to the second family, so, the fourth family is considered more general than the third family. In case the functions are fixed in other words all the functions is a one function and sets which belong to 2^X are vary, we get the first family, so, it is considered more general from the first family. In case the sets which belongs to 2^X are fixed i.e., $\{\forall A \in 2^X, A = X\}$. And the functions are vary we obtain the second family. In other words the fourth family is the most general than the other families and the study of its properties and conclusion are more complicated. From all above appears that it is necessary to define the fuzzy sets on all the points of the space in the first, second and fourth families, respectively and it becomes as follows:

$$f_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ f(x) & \text{if } x \in A \end{cases}$$

Note: As an extension for the first, second and fourth families, we are introduced a new added families named 5th-7th, respectively as follows.

Fifth family: We introduce the fifth family and denoted it by I_5^X and defined as follows:

$$I_5^X = \{\tilde{A}, \forall A \subset X | f: X \rightarrow I, \forall A \in 2^X\}$$

$$\tilde{A} = \begin{cases} (x, f_{A_i}(x)), f_{A_i}(x) = 0 & \forall x \notin A_i \\ f_{A_i}(x) = f(x) & \forall x \in A_i \end{cases}$$

Example 3.6.1; Return to example (3.2.1): We can rewrite \tilde{A}, \tilde{B} as follows:

$$\tilde{A} = \{(1, 1), (2, 0), (3, 0), \dots, (100, 0)\}$$

$$\tilde{B} = \{(1, 1), (2, \frac{1}{2}), (3, 0), (4, 0), \dots, (98, 0), (99, \frac{1}{99}), (100, 0)\}$$

Sixth family: We introduce the sixth family and denoted it by $I_6^X, \tilde{A}, \tilde{B}$ and defined it as follows:

$$I_6^X = \{\tilde{A}, f_i \in \Phi_A(x)\}$$

Where:

$$\tilde{A} = \{(x, f_{iA}(x)); f_{iA}(x) = 0 \quad \forall x \notin A; f_{iA}(x) = f_i(x), \forall x \in A$$

Example 3.7.1: Return to example (3.3.1) in $I_2^X, \tilde{B}, \tilde{C}$ are rewritten as follows:

$$\tilde{B} = \{(0, 1), (1, \frac{1}{2}), (2, \frac{1}{3}), (3, 0), (4, 0), (5, 0)\}$$

$$\tilde{C} = \{(0, \frac{1}{3}), (1, \frac{1}{5}), (2, \frac{1}{7}), (3, 0), (4, 0), (5, 0)\}$$

Seventh family: We can introduce a seventh family and denoted it by I_7^X and defined it as follows:

$$I_7^X = \begin{cases} \tilde{A}_i, A_i = \{(x, f_{iA_i}(x)), f_{iA_i} \in \Psi_{A_i}(x); \\ f_{iA_i}(x) = 0 \quad \forall x \notin A_i, f_{iA_i} = f_i(x) \quad \forall x \in A_i \end{cases}$$

It is considered the more general with respect to other families.

Example 3.8.1: Return to example (3.5.1), \tilde{A} and \tilde{B} are rewritten as follows:

$$\tilde{A} = \left\{ \begin{aligned} &(x_1, 1), (x_2, \frac{1}{8}), (x_3, 0), (x_4, \frac{1}{24}), (x_5, 0) \\ &(x_6, \frac{1}{48}), (x_7, 0), (x_8, \frac{1}{8}), (x_9, 0) \end{aligned} \right\}$$

$$\tilde{B} = \left\{ \begin{aligned} &(x_1, 1), (x_2, 0), (x_3, \frac{1}{17}), (x_4, 0), (x_5, \frac{1}{49}) \\ &(x_6, 0), (x_7, \frac{1}{97}), (x_8, 0), (x_9, \frac{1}{161}) \end{aligned} \right\}$$

Illustrative examples: We appended the following illustrative examples about families:

Example 3.9.1: Let the universe of discourse X is the positive real numbers, $0 \leq x \leq 1$:

$$\tilde{A} = \{(x, \mu_A(x)) | x \in X\}$$

$$\tilde{B} = \{(x, V_B(x)) | x \in X\}$$

where, $\mu_A(x), V_B(x)$ are defined as follows:

$$\mu_A(x) = \begin{cases} 0 & 0 \leq x \leq 0.5 \\ \frac{5(x-0.5)^2}{1+(x-0.5)^2} & 0.5 < x \leq 1 \end{cases}$$

$$V_B(x) = \frac{1}{1+(x-0.7)^4} \quad 0 \leq x \leq 1$$

Where:

- \bar{A} = Damping ratio
- x = Considerably large than 0.5
- \bar{B} = Damping ratio
- x = Approximately 0.7

In many branches of engineering the damping ratio is dimensionless measure describing two oscillations in a system after a disturbance.

$$\mu_A(x) \cap V_B(x) = \begin{cases} \min \left\{ \left[\frac{5(x-0.5)^2}{1+(x-0.5)^2} \right], \left[\frac{1}{1+(x-0.7)^4} \right] \right\} & x \leq 0.5 \\ 0 & x > 0.5 \end{cases}$$

And:

$$\mu_A(x) \cup V_B(x) = \max \left\{ \left[\frac{5(x-0.5)^2}{1+(x-0.5)^2} \right], \left[\frac{1}{1+(x-0.7)^4} \right] \right\}$$

Note that:

$$A \cap A^c = \emptyset, \quad A \cup A^c = X$$

where, A^c represents the complements of A and also for the set B.

Example 3.9.2: The example is a modification for an example which is taken from (Chen and Pham, 2000), let:

$$A_1(x) = \begin{cases} (x-2)+1 & x \in [1, 2] \\ (2-x)+1 & x \in [2, 3] \\ 0 & \text{else where} \end{cases}$$

$$A_4(x) = \frac{1}{1+10(x-2)^2}, \quad A_3(x) = e^{-|5(x-2)|}$$

$$A_1(x) = \begin{cases} \left[\frac{1+\cos(2\pi(x-2))}{2} \right] & \text{when } x \in \left[\frac{1}{2}, \frac{3}{2} \right] \\ 0 & \text{elsewhere} \end{cases}$$

Represents the membership functions for fuzzy sets A_1 - A_4 of real numbers close Fig. 1 and 2. Each of these fuzzy sets refer in particular ship for the general concept of a class of real numbers that are close (Fig. 2 and 3).

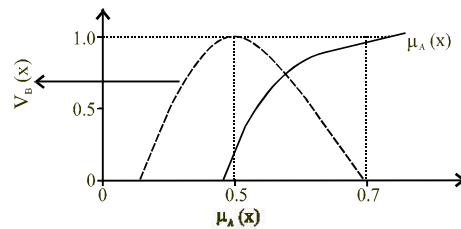


Fig. 1: The graph of example 3.9.1

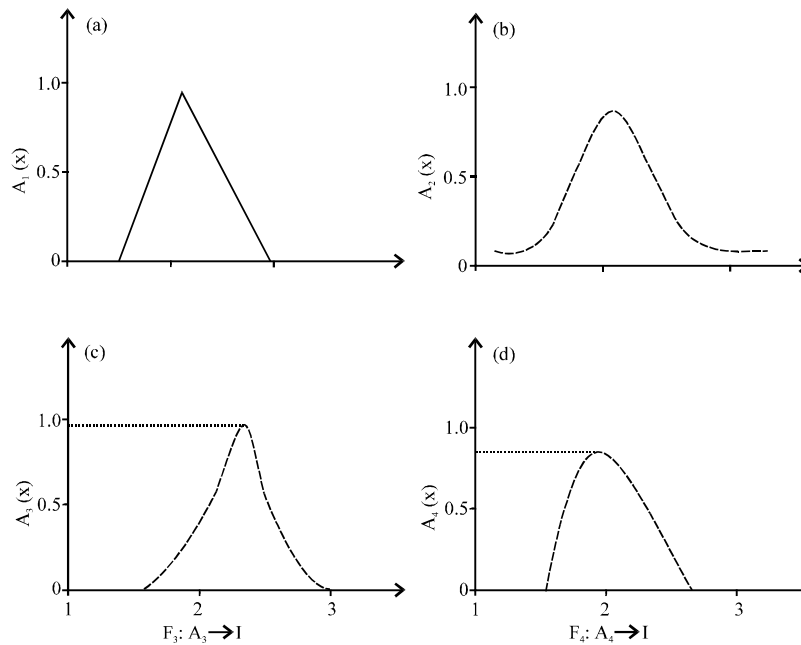


Fig. 2: The graphs of example 3.9.2: a) $F_1: A_1 \rightarrow 1$; b) $F_2: A_2 \rightarrow 1$; c) $F_3: A_3 \rightarrow 1$ and d) $F_4: A_4 \rightarrow 1$

Example 3.9.3: Let us assume that:

$$\begin{aligned} \tilde{A} &= \text{"x considerably } > 20\text{"} \\ \tilde{B} &= \text{"x approximately } 21\text{"} \end{aligned}$$

Characterized by:

$$\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x) \rangle \mid x \in A \}$$

Where:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq 20 \\ \left(1 + (x-20)^{-2}\right)^{-1} & x > 20 \end{cases}$$

And:

$$\tilde{B} = \{ \langle x, \mu_{\tilde{B}}(x) \rangle \mid x \in B \}$$

Where:

$$\begin{aligned} \mu_{\tilde{B}}(x) &= \left(1 + (x-21)^4\right)^{-1} \\ \mu_{\tilde{A} \cap \tilde{B}}(x) &= \begin{cases} \min \left[\left(1 + (x-20)^{-2}\right)^{-1}, \left(1 + (x-21)^4\right)^{-1} \right] & x > 20 \\ 0, & x \leq 20 \end{cases} \\ \mu_{\tilde{A} \cup \tilde{B}}(x) &= \begin{cases} \min \left[\left(1 + (x-20)^{-2}\right)^{-1}, \left(1 + (x-21)^4\right)^{-1} \right], & x \in X \end{cases} \end{aligned}$$

The distance between two fuzzy sets: Several definitions are given for distance between two fuzzy sets, (Rodabaugh and Klement, 2013; Wierman, 2010). In our study the distance between two fuzzy sets \tilde{A} and \tilde{B} which denoted by $d(\tilde{A}, \tilde{B})$ is:

$$d(\tilde{A}, \tilde{B}) = |f_{\tilde{A}}(x) - f_{\tilde{B}}(x)|$$

It means the difference between the two images of the functions for the two fuzzy sets, respectively, it must to be defined on the two functions on the same point, it is not possible that one of the functions defined at a point and the another not define in the same point. The distance

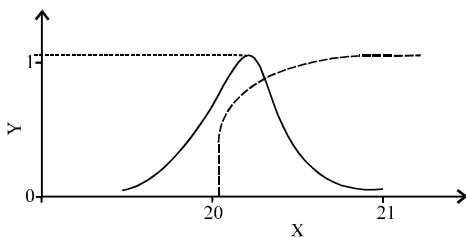


Fig. 3: Depict the example 3.9.3

between two fuzzy sets it must to be in the common points and their functions are also defined at these points. In the case of the generalization of the fuzzy sets according to the domain X, it meant by this the families (5-7) in addition to the 3rd family, so, the different definitions is valid at these families.

RESULTS AND DISCUSSION

Fuzzy points and its neighborhood (nhd) structure: A study of fuzzy points is presented, we redefined the first type of fuzzy points and introduced two new types of fuzzy points with illustrative examples. The fuzzy topology can be characterized by means of nhd systems, so, we introduce four types of nhd's which aid the engineers and scientist mathematically to solve different problems of uncertainty.

In view of the definition of fuzzy topological spaces by Chang (1968) and the revision of the definition of the fuzzy topological space by Lowen (1976) and according to the presented families of fuzzy sets in our research, the general definition with respect to the seventh family is as follows:

Definition 4.1: A fuzzy topology on a universe X is a class τ of fuzzy sets in X satisfying the following conditions: $\tau \subset \mathcal{F}_X$ satisfy the following conditions:

$$\begin{aligned} \tau_1 &: (\forall k \in [0, 1])(k \in \tau) \\ \tau_2 &: O_1 \in \tau \text{ and } O_2 \in \tau \rightarrow O_1 \cap O_2 \in \tau \\ \tau_3 &: (\forall j \in J)(O_{j \in \tau}) \rightarrow \bigcup_{j \in J} O_{j \in \tau} \end{aligned}$$

where, k resembles the constant X- $\{k\}$ mapping on X. According to, the definition above as in classical topology one can defined the class τ all closed fuzzy sets in (X, τ) and interior, closure and boundary of any fuzzy set in X. For more detail (Ruan, 1995) (Fig. 4).

On the different notions of nhd in fuzzy topological spaces, the concept of a nhd in fuzzy topological space relies on the fuzzification of a point and the corresponding membership relation. The concept of fuzzy point in a universe X is as follows:

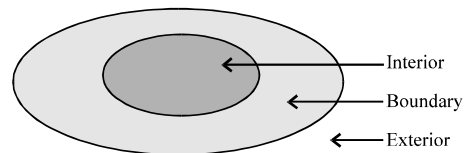


Fig. 4: Depicts the concept of interior, exterior and of a fuzzy set

Types of fuzzy points

Definition 4.2.1; Tripathy and Ray (2014): A fuzzy set A in fuzzy topological space (X, τ) is called quasi-nhd of x_λ iff A₁ ∈ τ ⇒ A₁ ⊆ A and x_λ ∈ A₁ the family of all Q-nbd of x_λ called the system of Q-nbd of x_λ.

Definition 4.2.2; Palaniappan (2002): A fuzzy point x_λ is called an adherence point of a fuzzy set A if and only if every Q-nhd of x_λ is quasi coincident with A. Now, we redefined the first type of fuzzy points and introduced two new fuzzy points as in below:

Type 1: For some x ∈ X and 0 < α ≤ 1, we say that p_α^x is classical fuzzy point on the fuzzy point of type I if:

$$P_{\alpha}^x(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Example 4.2.3: Let X = {x₁, x₂, x₃} and α = 0.131:

$$P_{\alpha}^{x_2} = \{(x_1, 0), (x_2, 0.131), (x_3, 0)\}$$

Type 2: Let A ⊆ X and for some 0 < α ≤ 1. The formula p_α^A is called the fuzzy point of type II where:

$$P_{\alpha}^A(y) = \begin{cases} P_{\alpha}^y & \text{if } y \in A \\ 0 & \text{if } y \notin A \end{cases}$$

Example 4.2.4: Let X = {x₁, x₂, x₃}, A = {x₂, x₃}, α = 1:

$$P_1^A = \{(x_1, 0), (x_2, 1), (x_3, 1)\}$$

Note that:

$$P_1^A = P_{\alpha}^x \bigcup_{x \in A}$$

Type 3: For any 0 < α < 1, the formula p^x is called the fuzzy point of type III such that:

$$P_{\alpha}(y) = \alpha \forall y \in X$$

Example 4.2.5: Let X = {x₁, x₂, x₃}, α = 0.23:

$$P_{\alpha} = P_{0.23} = \{(x_1, 0.23), (x_2, 0.23), (x_3, 0.23)\}$$

Noted that:

$$P_{\alpha} = P_{\alpha}^x \bigcup_{x \in A}$$

also, we can denote the fuzzy point of type III by p_α^x. We introduce a new two nhds denoted by (L-A) as: a fuzzy

set \tilde{A} is called fuzzy nhd. of the fuzzy point P_α iff a fuzzy set \tilde{A} is a nhd, of the fuzzy point p_α^x ∀ x ∈ X. A fuzzy set \tilde{A} is called quasi fuzzy nhd. of the fuzzy point p_α iff a fuzzy set \tilde{A} is quasi fuzzy nhd of p_α^x ∀ x ∈ X.

Note: A fuzzy point P_α is a fuzzy (limit, adherent interior, exterior and boundary) point iff p_α^x ∀ x ∈ X is a fuzzy (limit, adherent interior, exterior and boundary) point. A fuzzy point P_α is a quasi (limit, adherent interior, exterior and boundary) point iff p_α^x ∀ x ∈ X is quasi fuzzy (limit, adherent interior, exterior and boundary) point. A fuzzy point P_α is a fuzzy (limit, adherent interior, exterior and boundary) of a fuzzy set \tilde{A} , iff ∀ x ∈ X, p_α^x is a fuzzy (limit, adherent interior, exterior and boundary) point of a fuzzy set \tilde{A} .

Different nhd classes that have been introduced by several researcher like Ludescher, Kerre, Warren, Pu, Mashhour are stated by Ruan (1995). Also, other two (L-A) are introduced as:

- A is a (L-A) nhd ⇔ ∃ $\tilde{O} \in \tau \ni P_{\alpha} \in \tilde{O} \subseteq \tilde{A}$, where \tilde{O} is an open fuzzy set
- A is a (L-A) nhd ⇔ (∃ $\tilde{O} \in \tau$) (P_α ∈ \tilde{O} and $\tilde{O} \subseteq \tilde{A}$)

CONCLUSION

In this study, we have given definitions for three functions of fuzzy sets and obtain five new families of fuzzy sets which added to two families of Zadeh *et al.* (1975), Dubois and Prade (1980), Klir *et al.* (1997) and Zimmermann (2001). After an analytical study for fuzzy sets, the families are classified into seven families which have an important role in solution of the uncertainty problems in real life to help the engineers and scientists. The seventh family is considered the more general family with respect to the other families. Also, four new types of neighborhoods are introduced through we redefined the first type of fuzzy points and introduced two new types of fuzzy points with illustrative examples.

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