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Lie Group of Solving System for Partial Differential Equations

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Abstract: In this procedure, we established some examples for system 1st order partial differential equations to find the general solution using Lie group.

Key words: PDEs, lie group, invariance, transform, order partial, solution

INTRODUCTION

Lie group theory of deterministic differential equations is well comprehension in literature (Hydon, 2000; Ibragimov, 1999; Bluman *et al*., 2010; Bluman and Anco, 2008; Nass, 2014) and can applied for many substantial applications in the context of differential equations. For instance, for determination of group-invariant solutions, solving the 1st order DE, reducing order for higher ODE, reducing the number of variables of partial differential equations and finding conservation laws.The powerful and by now rather standard tool in the study of deterministic nonlinear problems is symmetry analysis of differential equations (Olver, 1986; Stephani, 1989; Cicogna and Gaeta, 1999). The theory of infinitesimal symmetries of Ordinary and Partial Differential Equations (ODEs and PDEs, respectively) is a classical research topic in applied mathematics, providing powerful tools both for investigating the qualitative behavior of differential equations and for obtaining some explicit expression for their solutions (Stephani, 1989; Olver, 2012).

MATERIALS AND METHODS

Example (1): solve the system of PDE by symmetry method (of one-dimension shallow water):

$$\begin{aligned} &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x} = 0 \\ &\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x} = 0 \\ &\frac{\partial z}{\partial t}+v\frac{\partial z}{\partial x}+z\frac{\partial v}{\partial x} = 0 \end{aligned} \tag{1}$$

Write the vector field as:

$$X = \zeta\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial x}+\sigma\frac{\partial}{\partial v}\psi\frac{\partial}{\partial z} \tag{2}$$

Note that the operator X depend on the variables t, x, u, v, z as follows:

$$\begin{aligned} &\zeta = \zeta(t, x) \\ &\eta = \eta(t, x) \\ &\sigma = \sigma(t, x, u) \\ &\tau = \tau(t, x, v) \\ &\psi = \psi(t, x, z) \end{aligned} \tag{3}$$

Now, we must find the 1st prolongation of the vector field write as in style:

$$\begin{aligned} X^{[1]} = X+\zeta_{u_t}\frac{\partial}{\partial u_t}+\zeta_{u_x}\frac{\partial}{\partial u_x}+\zeta_{v_t}\frac{\partial}{\partial v_t}+ \\ \zeta_{v_x}\frac{\partial}{\partial v_x}+\zeta_{z_t}\frac{\partial}{\partial z_t}+\zeta_{z_x}\frac{\partial}{\partial z_x} \end{aligned} \tag{4}$$

After that, applying the formula given in Eq. 6 for system (Eq. 1) given as:

$$\begin{aligned} &X^{[1]} = \left[u_t+uu_x+v_x\right]_{(1)=0} = 0 \\ &X^{[1]} = \left[v_t+uv_x+vu_x\right]_{(1)=0} = 0 \\ &X^{[1]} = \left[z_t+vz_x+zv_x\right]_{(1)=0} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &\zeta_{u_t}+u\zeta_{u_x}+\zeta_{v_x}+\sigma u_x = 0 \\ &\zeta_{v_t}+u\zeta_{v_x}+v\zeta_{u_x}+\sigma v_x+\tau u_x = 0 \\ &\zeta_{z_t}+v\zeta_{z_x}+z\zeta_{v_x}+\psi_{v_x}+\tau z_x = 0 \end{aligned} \tag{6}$$

Then, the determining equation of system given in Eq. 6

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is:

$$\sigma_t-(uu_x+v_x)(\sigma_u-\zeta_t)-u_x\eta_t+u\begin{bmatrix}\sigma_x+u_x(\sigma_u-\eta_x)+\\(uu_x+v_x)\zeta_x\end{bmatrix}+ \\ \tau_x+v_x(\tau_u-\eta_x)+(uv_x+uv_x)\zeta_x+\sigma u_x=0 \quad (7)$$

$$\tau_t(uv_x+vu_x)(\tau_x-\zeta_t)-v_x\eta_t+u\begin{bmatrix}\tau_x+v_x(\tau_v-\eta_x)+\\(uv_x+vu_x)\zeta_x\end{bmatrix}+ \\ v[\sigma_x+u_x(\sigma_u-\eta_x)+(uu_x+v_x)\zeta_x]+\tau u_x+\sigma v_x=0 \quad (8)$$

$$\psi_t-(vz_x+zv_x)(\psi_z-\zeta_t)-z_x\eta_t+v\begin{bmatrix}\psi_x+z_x(\psi_z-\eta_x)+\\(vz_x+zv_x)\zeta_x\end{bmatrix}+ \\ z[\tau_x+v_x(\tau_v-\eta_x)+(uv_x+vu_x)\zeta_x]+\tau z_x+\psi v_x=0 \quad (9)$$

At last, we separation the coefficient of Eq. 7-9 with respect to u_x, v_x and z_x this attend for coefficient of these variables equal to zero:

$$\begin{aligned}&u_x:-(\sigma_u-\zeta_t)u-\eta_t+(\sigma_u-\eta_x)u+u^2\zeta_x+v\zeta_x+\sigma=0\\&v_x:-(\sigma_u-\zeta_t)+2u\zeta_x+(T_u-\eta_x)=0\\&1:\sigma_t+u\sigma_x+v\sigma_x=0\end{aligned} \quad (10)$$

$$\begin{aligned}&u_x:-v(\tau_v-\zeta_t)+2vu\zeta_x+v(\sigma_u-\eta_x)+\tau=0\\&v_x:-u(\tau_v-\zeta_t)+(\tau_v-\zeta_t)u+u^2\zeta_x-\eta_t+v\zeta_x+\sigma=0\\&1:\tau_t+u\tau_x+v\sigma_x=0\end{aligned} \quad (11)$$

$$\begin{aligned}&v_x:-z(\psi_z-\zeta_t)+vz\zeta_x z(\tau_v-\eta_x)+zu\zeta_x+\psi=0\\&z_x:-v(\psi_z-\zeta_t)+v(\psi_z-\eta_x)-\eta_t+v^2\zeta_x+\tau=0\\&1:\psi_t+v\psi_x+z\tau_x=0\end{aligned} \quad (12)$$

The above system is given the general solution:

$$\zeta(t,x)=c_1t+c_2 \quad (13)$$

$$\eta(t,x)=c_1x+c_4 \quad (14)$$

$$\psi(t,x,z)=c_3z \quad (15)$$

$$\sigma(t,x,u)=\tau(t,x,v)=0 \quad (16)$$

The Lie symmetries are determined as follows:

$$X_1=t\frac{\partial}{\partial t}+\times\frac{\partial}{\partial x} \quad (17)$$

$$X_2=t\frac{\partial}{\partial t} \quad (18)$$

$$X_3=z\frac{\partial}{\partial z} \quad (19)$$

$$X_4=\frac{\partial}{\partial x} \quad (20)$$

Example (2): Solve the following system of PDE by Lie group:

$$\begin{aligned}&\frac{\partial u}{\partial t}+k^2\frac{\partial u}{\partial x}+\left(\frac{\partial v}{\partial x}\right)^2=0\\&\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}+k\left(\frac{\partial u}{\partial x}\right)^2=0\\&\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}+\left(\frac{\partial v}{\partial x}\right)^2=0\end{aligned} \quad (21)$$

Write the vector field as:

$$X=\zeta\text{-}\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial v}+\psi\frac{\partial}{\partial k} \quad (22)$$

Remark that the operator X depend on the variables t, x, u, v, k as follows:

$$\begin{aligned}&\zeta=\zeta(t,x)\\&\eta=\eta(t,x)\\&\sigma=\sigma(t,x,u)\\&\tau=\tau(t,x,v)\\&\psi=\psi(t,x,k)\end{aligned} \quad (23)$$

Now, we must find the 1st for X of style:

$$X^{[1]}=X+\zeta_{u_t}\frac{\partial}{\partial u_t}+\zeta_{u_x}\frac{\partial}{\partial u_x}+\zeta_{v_t}\frac{\partial}{\partial v_t}+ \\ \zeta_{v_x}\frac{\partial}{\partial v_x}+\zeta_{k_t}\frac{\partial}{\partial k_t}+\zeta_{k_x}\frac{\partial}{\partial k_x} \quad (24)$$

After that, applying the equation in Eq. 24 to the system given in Eq. 1:

$$\begin{aligned}&X^{[1]}=[u_t+k^2u_x+v_x^2]_{(1)=0}=0\\&X^{[1]}=[v_t+v_x+ku_x^2]_{(1)=0}=0\\&X^{[1]}=[k_t+uk_x+v_x^2]_{(1)=0}=0\end{aligned} \quad (25)$$

Then:

$$\begin{aligned}&\zeta_{u_t}+2k\psi u_x+k^2\zeta_{u_x}+2v_x\zeta_{v_x}=0\\&\zeta_{v_t}+\zeta_{v_x}+\psi u_x^2+2ku_x\zeta_{u_x}=0\\&\zeta_{k_t}+u\zeta_{k_x}+\sigma k_x+2v_x\zeta_{v_x}=0\end{aligned}\tag{26}$$

Then, the determining equation of system given in Eq. 6 is:

$$\begin{aligned}&\sigma_t-\left(k^2u_x+v_x^2\right)\left(\sigma_u-\zeta_t\right)-u_x\eta_t+2k\psi u_x+\\&k^2\left[\sigma_x+u_x\left(\sigma_u-\eta_x\right)+\left(k^2u_x+v_x^2\right)\zeta_x\right]+\\&2v_x\left[\tau_x+v_x\left(\tau_u-\eta_x\right)+\left(v_x+ku_x^2\right)\zeta_x\right]=0\end{aligned}\tag{27}$$

$$\begin{aligned}&\tau_t-\left(v_x+ku_x^2\right)\left(\tau_v-\zeta_t\right)-v_x\eta_t+v_x\left(\tau_v-\eta_x\right)+\left(v_x+ku_x^2\right)\zeta_x+\\&\psi u_x^2+2ku_x\left[\sigma_x+u_x\left(\sigma_u-\eta_x\right)+\left(k^2u_x+v_x^2\right)\zeta_x\right]=0\end{aligned}\tag{28}$$

$$\begin{aligned}&\psi_t-\left(uk_x+v_x^2\right)\left(\psi_k-\zeta_t\right)-k_x\eta_t+\sigma k_x+\\&u\left[\psi_x+k_x\left(\psi_k-\eta_x\right)+\left(uk_x+v_x^2\right)\zeta_x\right]+\\&2v_x\left[\tau_x+v_x\left(\tau_v-\eta_x\right)+\left(v_x+ku_x^2\right)\zeta_x\right]=0\end{aligned}\tag{29}$$

Solving Eq. 27-29 by separation of the coefficient of the variables:

$$\begin{aligned}&u_x,\ v_x,\ k_x,\ u_x^2,\ v_x^2.\ v_xu_x^2\\&u_x{:}-k^2\left(\sigma_u-\zeta_t\right)-\eta_t+2k\psi+k^2\left(\sigma_u-\eta_x\right)+k^4\zeta_x=0\end{aligned}\tag{30}$$

$$\begin{aligned}&v_x:2\tau_x=0\\&v_x^2:-\left(\sigma_u-\zeta_t\right)+k^2\zeta_x+2\left(\tau_v-\eta_x\right)+\zeta_x=0\\&v_xu_x^2:2k\zeta_x\end{aligned}\tag{31}$$

$$\begin{aligned}&1:\sigma_t+k^2\sigma_x=0\\&u_x:2k\sigma_x=0\\&v_x{:}-\left(\tau_v-\zeta_t\right)+k^2+2\left(\tau_v-\eta_x\right)+\zeta_x+2k\sigma_x=0\\&u_x^2{:}-k\left(\tau_v-\zeta_t\right)+k\zeta_x+\psi+2k\left(\sigma_u-\eta_x\right)+k^2\zeta_x=0\\&u_x\,v_x^2:2k\zeta_x=0\\&1:\tau_t+\tau_x=0\end{aligned}\tag{32}$$

$$\begin{aligned}&k_x{:}-u\left(\psi_k-\zeta_t\right)-\eta_t+u\left(\psi_k-\eta_x\right)+u\zeta_x\sigma=0\\&v_x:2\upsilon_x=0\\&v_x^2{:}-\left(\psi_k-\zeta_t\right)+u\zeta_x+2\zeta_x+2\left(\tau_v-\eta_x\right)=0\\&v_xu_x^2:2k\zeta_x=0\\&1:\psi_t+u\psi_x=0\end{aligned}\tag{33}$$

Then, the general solution of above system:

$$\begin{aligned}&\zeta(t,x)=c_2\\&\eta(t,x)=c_3\\&\tau(t,x,v)=c_1\\&\sigma(t,x,u)=\psi(t,x,k)=0\end{aligned}\tag{34}$$

$$X_1=\frac{\partial}{\partial v}\tag{35}$$

$$X_2=\frac{\partial}{\partial t}\tag{36}$$

$$X_3=\frac{\partial}{\partial x}\tag{37}$$

RESULTS AND DISCUSSION

Example (3): Solve the following system of PDE by Lie group:

$$\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial k}{\partial x}=0\\&\frac{\partial v}{\partial t}+k\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0\\&\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}+k\frac{\partial u}{\partial x}=0\end{aligned}\tag{38}$$

First write the vector field gives:

$$X=\zeta\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial x}+\sigma\frac{\partial}{\partial u}+\tau\frac{\partial}{\partial v}+\psi\frac{\partial}{\partial k}\tag{39}$$

The operator X depend on the variables as follows:

$$\begin{aligned}&\zeta=\zeta(t,x)\\&\eta=\eta(t,x)\\&\sigma=\sigma(t,x,u)\\&\tau=\tau(t,x,v)\\&\psi=\psi(t,x,k)\end{aligned}\tag{40}$$

Now, we need find the 1st prolongation of the operator X in the style:

$$\begin{aligned}&X^{[1]}=X+\zeta_{u_t}\frac{\partial}{\partial u_t}+\zeta_{u_x}\frac{\partial}{\partial u_x}+\zeta_{v_t}\frac{\partial}{\partial v_t}+\\&\zeta_{v_x}\frac{\partial}{\partial v_x}+\zeta_{k_t}\frac{\partial}{\partial k_t}+\zeta_{k_x}\frac{\partial}{\partial k_x}\end{aligned}\tag{41}$$

Next, applying the formula given in Eq. 38-41, we obtain the following:

$$\begin{aligned} &X^{[1]}\left(u_t+uu_x+vk_x\right)\Big|_{(1)=0}=0\\ &X^{[1]}\left(v_t+kv_x+vu_x\right)\Big|_{(1)=0}=0\\ &X^{[1]}\left(k_t+uk_x+ku_x\right)\Big|_{(1)=0}=0 \end{aligned} \tag{42}$$

We result the following:

$$\begin{aligned} &\zeta_{u_t}+\sigma u_x+u\zeta_{u_x}+\tau k_x+v\zeta_{k_x}=0\\ &\zeta_{v_t}+\psi v_x+k\zeta_{v_x}+\tau u_x+v\zeta_{u_x}=0\\ &\zeta_{k_t}+\tau k_x+u\zeta_{k_x}+\psi u_x+k\zeta_{u_x}=0 \end{aligned} \tag{43}$$

Then, the determining equation of above system given in Eq. 6, we obtain:

$$\begin{aligned} &\sigma_t-\left(uu_x+vk_x\right)\left(\sigma_u-\zeta_t\right)-u_x\eta_t+\sigma u_x+\\ &u\left[\sigma_x+u_x\left(\sigma_u-\eta_x\right)+\left(uu_x+vk_x\right)\zeta_x\right]+\\ &\tau k_x+v\left[\psi_x+k_x\left(\psi_k-\eta_x\right)+\left(vk_x+ku_x\right)\right]=0 \end{aligned} \tag{44}$$

$$\begin{aligned} &\tau_t-\left(kv_x+vu_x\right)\left(\tau_v-\zeta_t\right)-v_x\eta_t+\psi v_x+k\begin{bmatrix}\left(\tau_x-\eta_k\right)+\\ \left(kv_x+vu_x\right)\end{bmatrix}+\\ &\tau u_x+v\left[\sigma_x+u_x\left(\sigma_u-\eta_x\right)+\left(uu_x+vk_x\right)\zeta_x\right]=0 \end{aligned} \tag{45}$$

$$\begin{aligned} &u\left[\psi_x+k_x\left(\psi_k-\eta_x\right)+\left(uk_x+ku_x\right)\zeta_x\right]+\psi u_x+\\ &k\left[\sigma_x+u_x\left(\sigma_u-\eta_x\right)+\left(uu_x+vk_x\right)\zeta_x\right]=0 \end{aligned} \tag{46}$$

Lastly, solve above system with separation of the coefficient u_x, v_x and k_x:

$$\begin{aligned} &u_x{:}-u\left(\sigma_u-\zeta_t\right)-\eta_t+u\left(\sigma_u-\eta_x\right)+u^2\zeta_x+vk+\sigma=0\\ &k_x{:}-v\left(\sigma_u-\zeta_t\right)+uv\zeta_x+v\left(\psi_k-\eta_x\right)+\tau+u=0\\ &1{:}\ \sigma_t+u\sigma_x+v\psi_x=0 \end{aligned} \tag{47}$$

$$\begin{aligned} &u_x{:}-v\left(\tau_u-\zeta_t\right)+v\left(\sigma_u-\eta_x\right)+vu+kv+\tau=0\\ &v_x{:}-k\left(\tau_v-\zeta_t\right)-\eta_t+\psi+k\left(\tau_k-\eta_x\right)+k^2+v^2\zeta_x=0\\ &1{:}\ \tau_t+k\tau_x+v\sigma_x=0 \end{aligned} \tag{48}$$

$$\begin{aligned} &u_x{:}-k\left(\psi_v-\zeta_t\right)+ku\zeta_x+\psi+k\left(\sigma_v-\eta_x\right)+ku\zeta_x=0\\ &k_x{:}-u\left(\psi_v-\zeta_t\right)-\eta_t+\sigma+u\left(\psi_k-\eta_x\right)+u^2\zeta_x+kv\zeta_x=0\\ &1{:}\psi_t+u\psi_x+k\sigma_x=0 \end{aligned} \tag{49}$$

The above system is introduce the general solution as:

$$\begin{aligned} &\zeta(t,x)=c_3t+c_4\\ &\eta(t,x)=c_1t+c_2\\ &\sigma(t,x,u)=\left(c_1-c_3\right)u\\ &\tau(t,x,v)=\left(c_1-c_3\right)v\\ &\psi(t,x,v)=\left(c_1-c_3\right)k \end{aligned} \tag{50}$$

The Lie symmetries of above system as:

$$X_1=x\frac{\partial}{\partial v}+u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+k\frac{\partial}{\partial k} \tag{51}$$

$$X_2=\frac{\partial}{\partial x} \tag{52}$$

$$X_3=t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}-k\frac{\partial}{\partial k} \tag{53}$$

$$X_4=\frac{\partial}{\partial t} \tag{54}$$

CONCLUSION

In this research, we introduced steps of algorithm for transformation invariance system 1st order partial differential equations to find the general solution.

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