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Existence and Uniqueness Solution of Linear Fractional Volterra Integro-Differential Equations

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Abstract: We study the existence and uniqueness of the solutions of linear fractional Volterra integro-differential equation with initial conditions. For existence the unique solution our analysis is based on an application of Picard iteration method to get uniformly convergent series to exact solution.

Key words: Picard iteration, Riemann fractional operator, Volteraa integro-differential equation, linear, Picard, conditions

INTRODUCTION

Consider the linear fractional Volterra integro-differential equation of the form:

$$D^{\alpha}y(t)=f(t)+\frac{1}{N}\int_0^t k(t,s)y(s)ds,\ 0<\alpha<1 \qquad (1)$$

with initial conditions:

$$y(0)=y_0$$

where, k(t, s), y(s) and f(t) are given continuous functions defined, respectively, on $I = [a, b]\subset R.D^{\alpha}$ denotes the fractional Riemann Liouville derivative of order α.

Now, we recall some puplished works on this subject in 2011, discussed the existence and uniqueness of solution to fractional order ordinary and delay differential equations and they used first banach contraction principle to show the existence and uniqueness of the solution under certain conditions (Abbas, 2011). In 2012 by using a fixed point theorem in banach algebraic proved an existence result for a fractional functional differential equation in the Riemann-Liouville sense (Ammi *et al*., 2012). And in 2012, applying Picards approximation method to prove existence and uniqueness of a system of nonlinear fractional integro-differential equations with initial conditions (Sallo, 2012). Sufficient conditions are given for the existence of solutions for an integral equation of fractional order with multiple time delays in banach space 2012. Compactness types condition is using to obtain local and global existence of solution 2014. Fixed point theorem are used to obtain the existence and uniqueness of solutions for Handamard-type sequential fractional order fractional differential equations 2017.

Definition 1; Podlubny (1999): Suppose that $\alpha>0, t>a, \alpha, a, t\in R$. Then, the Riemann fractional integral operator is:

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds$$

Theorem 1; Diethelm (2004): Assume that $n\geq 0$, $m = \lceil n\rceil$, ($\lceil n\rceil$ smallest integer >m) and $f\in C^m[a, b]$ then:

$$J_a^n D_{*a}^n f(x)=f(x)-\sum_{k=0}^{m-1}\frac{D^k f(a)}{k!}(x-a)^k$$

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Since, k(t, s) and y(s) continuous on I, there exists a constant M>0 such that $|k(t, s)\ y(s)|\leq M$ and $N>M = \max|k(t, s)\ y(s)|$, $(t, s)\in I$ by the fractional integral operator on both side of (Eq. 1), we get:

$$J^{\alpha}D^{\alpha}y(t)=J^{\alpha}f(t)+J^{\alpha}\left[\frac{1}{N}\int_0^t k(t,s)y(s)ds\right]$$

Then:

$$y(t)=J^{\alpha}f(t)+y_0+\frac{1}{\Gamma(\alpha)}\frac{1}{N}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s k(s,z)y(z)dz\right)ds$$

Let:

$$J^{\alpha}f(t)+y_0=F(t)$$

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$$y(t) = F(t) + \frac{1}{\Gamma(\alpha)}\frac{1}{N}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s k(s,z)y(z)dz\right)ds$$

$$\left|y(t)-F(t)\right| \le \frac{1}{\Gamma(\alpha)}\frac{1}{N}\left|\int_0^t (t-s)^{\alpha-1}\int_0^s \left|k(s,z)y(z)\right|dzds\right| \le$$
$$\frac{1}{\Gamma(\alpha)}\frac{M}{N}\left|\int_0^t (t-s)^{\alpha-1}.s\,ds\right| \le \tag{2}$$
$$\frac{1}{\Gamma(\alpha)}\frac{M}{N}\left(\frac{t^{\alpha+1}}{\alpha}+\frac{t^{\alpha+1}}{\alpha(\alpha+1)}\right), t\in[a,b]$$

For a given value of y(0) the Picard iterations for (Eq. 2) is defined by:

$$y_{n+1}(t) = F(t) + \frac{1}{\Gamma(\alpha)}\frac{1}{N}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s k(s,z)y_n(z)dz\right)dsn = 0,1 \tag{3}$$

For n = 1:

$$\left|y_2(t)-y_1(t)\right| \le \frac{M}{N\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\int_0^s \left|y_1(z)-y_0(z)\right|dzds \le$$
$$\frac{M}{N\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|.\left\|y_1-y_0\right\|ds \le \tag{4}$$
$$\frac{M}{N\Gamma(\alpha)}\left\|y_1-y_0\right\|\int_0^t (t-s)^{\alpha-1}ds \le$$
$$\frac{M}{N\Gamma(\alpha)}\left\|y_1-y_0\right\|.\frac{t^{\alpha}}{\alpha}$$

For n = m, let the inequality holds, i.e., Eq. 5:

$$\left|y_{m+1}(t)-y_m(t)\right| \le \left(\frac{M}{N\Gamma(\alpha)}\right)^m \left\|y_m-y_{m-1}\right\|\frac{t^{ma}}{\alpha m!} \tag{5}$$

For n = m+1, we get:

$$\left|y_{m+2}(t)-y_{m+1}(t)\right| \le \frac{M}{N\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1}\right|\int_0^s \left|y_{m+1}(t)-y_m(t)\right|dzds \le$$
$$\frac{M}{N\Gamma(\alpha)}\left(\frac{M}{N\Gamma(\alpha)}\right)^m \left\|y_{m+1}-y_m\right\|\frac{t^{m\alpha}}{\alpha m!}\int_0^t\int_0^s \left|(t-s)^{\alpha-1}\right|dzds \le$$
$$\left(\frac{M}{N\Gamma(\alpha)}\right)^{m+1}\left\|y_{m+1}-y_m\right\|\frac{t^{\alpha(m+1)}}{\alpha^2 m!}+\frac{t^{\alpha(m+1)}}{\alpha^2(\alpha+1)m!}$$

By using Eq. 5, hence:

$$\left|y_{n+1}(t)-y_n(t)\right| \le \left(\frac{M}{N\Gamma(\alpha)}\right)^n \left\|y_n-y_{n-1}\right\|\frac{t^{n\alpha}}{\alpha n!}, n\in N$$

Which implies that the series:

$$\sum_{n=1}^{\infty}\left[y_{n+1}(t)-y_n(t)\right] \tag{6}$$

is absolutely and uniformly convergent Eq. 6, on the other hand, $y_n(t)$ can be written as:

$$y_n(t) = y_1(t) + \sum_{i=1}^{n-1}\left[y_{i+1}(t)-y_i(t)\right]$$

Then, from uniform convergence of the series (Eq. 6), we conclude that $\lim_{n\to\infty} y_n(t)$ exists for all t∈[0, b]. Let $\lim_{n\to\infty} y_n(t) = y(t)$ then by continuity of k(s, z) y(z) $(t-s)^{\alpha-1}$ in $y_n(z)$, we have:

$$\lim_{n\to\infty} k(s,z)y_n(z)(t-s)^{\alpha-1} = k(s,z)y(z)(t-s)^{\alpha-1}$$

And also:

$$\lim_{n\to\infty} y_n(t) = F(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s k(s,z)y(z)dz\right)ds = y(t)$$

Therefore, y(t) is the unique solution of Eq. 1.

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CONCLUSION

In this study, we prove existence and uniqueness of linear fractional Volterra integro-differential equations by applying Picards iteration and mathematical induction to get a convergent sequence to complete the uniqueness.

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