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Numerical Integration of Functions from Holder Classes H^s [0, 1] by Linear Legendre Multi Wavelets

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Abstract: In the previous research, a direct computational method based on linear Legendre multi-wavelets has been applied for solving definite integrals. However, the error analysis to show the convergence of the method has not been discussed. Therefore, error analysis of the approximation method is established in the Holder classes $H^s[0, 1]$ to show the efficiency of the method. The connections of the module of difference smoothness of the function is also established. Finally, some numerical examples of the implementation the method for the functions from Holder classes are presented.

Key words: Numerical integration, Holder classes, linear Legendre multi-wavelets, error analysis, computational method, efficiency

INTRODUCTION

In recent years, a great deal of interest has arisen in the numerical solution of integration. It is well known that a lot of work has been done in this area in terms of quadrature rule of numerical integration (Rathod *et al*., 2007; Rokhlin, 1990; Alpert, 1999; Place and Stach, 1999). Unfortunately, quadrature method bears some drawbacks, such as Newton-Cotes quadrature rule for large number of equally spaced nodes may cause erratic behaviour with high degree polynomial interpolation. For the Gaussian quadrature rule, it can be derived by the method of undetermined coefficients but the resulting equations for the unknown nodes and weights are nonlinear. This procedure is complicated to find the unknown nodes and weights. In order to overcome these disadvantages of the previous method we already proposed a new method bases on linear Legendre multi-wavelets for solving numerical integration. This research is the extension of the earlier method (Siraj-ul-Islam *et al*., 2010a, b) where Haar wavelets and hybrid functions are used to find numerical solution of definite integrals with constant limits. Due to the similarity of Haar wavelets and and linear Legendre multi-wavelets, this method are able to solve the integrals easily. Moreover, wavelets have been successfully used in the field of numerical approximations in finding numerical solutions of integral equations and numerical integration (Maleknejad *et al*., 2007; Ahmedov and Bin Abd Sathar, 2013), ordinary differential equations (Dehghan and Lakestani, 2008; Siraj-ul-Islam *et al*., 2010a, b), partial differential equations (Comincioli *et al*., 2000) and fractional partial differential equations (Wu, 2009). However, the present methods only established the error estimations by assuming functions belonging to the $C^1(R)$ and $C^2(R)$ (Babolian and Shahsavaran, 2009; Adibi and Assari, 2010) while others only provide new method without considering the error estimation. Studying continuous but nowhere differentiable functions was emphasized a long time ago and this function play an important role in application problems. If the functions are irregular or lack of differentiability it is possible the graph for continuous function will be fractal. The irregular functions appear in various branches of physics. Passive scalars adverted by a turbulent fluid can have is scalar surfaces which are highly irregular, in the limit of the diffusion constant going to zero. Graphs of projections of Brownian paths are one of the well-known graphs which are nowhere differentiable and have dimension 3/2. Fractional Brownian motion is a generalization of Brownian motion that gives rise to graphs having dimension between 1 and 2. Feynmann paths is similarly, like the Brownian paths which are continuous but nowhere differentiable. Moreover, attractors of some dynamical systems have been shown to be continuous

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but nowhere differentiable. All the irregular functions are characterized at every point by a local Holder exponent usually located between 0 and 1. In this present study, we have consider this method as suggested by Abd Sathar but the error analysis, for the class of function is estimated in the Holder classes.

Numerical integration using Legendre Multi Wavelets (LLMW): The formula for the LLMW are defined as:

$$\psi^0(x)\begin{cases}-\sqrt{3}(4x\text{-}1), & 0\le x<\frac{1}{2}\\ \sqrt{3}(4x\text{-}3), & \frac{1}{2}\le x<1\end{cases}$$

$$\psi^1(x)\begin{cases}6x\text{-}1, & 0\le x<\frac{1}{2}\\ 6x\text{-}5, & \frac{1}{2}\le x<1\end{cases}$$

$$\psi^j_{kn}(x)=2^{\frac{k}{2}}\begin{cases}\psi^j(2^kx\text{-}n), & n2^{\text{-}k}\le x<(n+1)2^{\text{-}k}\\ 0, & \text{otherwise}\end{cases}$$

where, n = 0, 1, ..., 2^k-1, k, $\in\mathbb{Z}^+$ and j = 0, 1 on the interval [0, 1]. The scaling functions is denoted as $\phi_0(x) = 1$ and $\phi_1(x) = \sqrt{3}(2x\text{-}1)$ such that x∈[0, 1). Any function f(x)∈$L^2(\mathbb{R})$ in the interval [0, 1) can be approximate as:

$$f_M(x)=c_0\phi_0(x)+c_1\phi_1(x)+\sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1}c^j_{kn}\psi^j_{kn}(x)=C^T\Psi(x)$$

where, C and Ψ(x) are matrices given by:

$$C=[c_0, c_1, c^0_{00}, c^1_{00}, ..., c^0_{M0}, c^0_{M1}, ..., c^0_{M(2^M\text{-}1)}, c^1_{M0}, c^1_{M1}, ..., c^1_{M(2^M\text{-}1)}]^T$$

And:

$$\Psi=[\phi_0, \phi_1, \psi^0_{00}, \psi^1_{00}, ..., \psi^0_{M0}, \psi^0_{M1}, ..., \psi^0_{M(2^M\text{-}1)}, \psi^1_{M0}, \psi^1_{M1}, ..., \psi^1_{M(2^M\text{-}1)}]^T$$

Consider the integral:

$$\int_a^b f(x)dx$$

The function f(x)∈$L^2(\mathbb{R})$ can be expanded as:

$$f(x)\approx C^T\Psi(x)$$

where, $M\in\mathbb{Z}_+$ is the maximum dilation k = 0, ..., M for LLMV. In this study, the same formula and notation are used by Abd Sathar to derive the numerical integration for single integrals with constant limit as:

$$\int_a^b f(x)dx\approx\frac{1}{2^{k+2}}\sum_{i=0}^{2^{k+2}\text{-}1}f(x_i)=\frac{(b\text{-}a)}{2^{k+2}}\sum_{i=0}^{2^{k+2}\text{-}1}f\left(a+(b\text{-}a)\frac{i+0.5}{2^{k+2}}\right)$$

where, k = 0, 1, ..., M.

Holder classes H^s[0, 1]: The definition of the Holder classes of order s∈(0, 1) is the set of all continuous functions on [0, 1] which satisfies the inequality as follow:

$$|f(x)\text{-}f(y)|\leqslant c|x\text{-}y|^s, \forall x, y\in[0, 1]$$

The Holder classes of order s is denoted by H^s[0, 1]. The Holder classes are nested as the follows:

$$H^\alpha\subset H^\beta,\ \alpha<\beta$$

Moreover, Holder classes are medium spaces between C[0, 1] and C^1[0, 1] such that:

$$C^1[0, 1]\subset H^s[0, 1]\subset C[0, 1],\ 0<s<1$$

MATERIALS AND METHODS

In this study, the error analysis of the approximation by LLMW is worked out, to show the convergence of the method.

Theorem 1: Let f(t)∈H^s[0, 1], 0<s<1, then:

$$\|f\text{-}f_M\|_{L2[0,1]}\le\frac{L^2}{4^{(M+1)s}(4^s\text{-}1)}\left(\frac{3}{16}+\frac{4^s}{9}\left[\frac{8\left(\frac{2}{3}\right)^s\text{-}3+3s}{(s+2)(s+1)}\right]^2\right)$$

Proof: Suppose $f_M(x)$ is the following approximation of f(x):

$$f_M(x)=c_0\phi_0(x)+c_1\phi_1(x)+\sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k\text{-}1}c^j_{kn}\psi^j_{kn}(x)$$

where, $M\in\mathbb{Z}_+$ is the maximum level of dilation k = 0, ..., M for LLMV:

$$f(t)\text{-}f_M(t)=\sum_{k=M+1}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{2^k\text{-}1}c^j_{kn}\psi^j_{kn}(x)=$$

$$\sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k\text{-}1}\left(c^0_{kn}\psi^0_{kn}(x)+c^1_{kn}\psi^1_{kn}(x)\right)$$

so,

$$\|f-f_M\|_{L^2[0,1]} = \int_0^1 |f(x)-f_M(x)|^2 dt,$$

$$= \sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\left[\left(c_{kn}^0\right)^2\int_0^1\left(\psi_{kn}^0(x)\right)^2dx+\left(c_{kn}^1\right)^2\int_0^1\left(\psi_{kn}^1(x)\right)^2dx\right],$$

$$= \sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\left(c_{kn}^0\right)^2+\sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\left(c_{kn}^1\right)^2$$

where, we denote:

$$c_{kn}^0 = \langle f, \psi_{kn}^0\rangle = \int_0^1 f(t)\,\psi_{kn}^0(x)\,dx$$

And:

$$c_{kn}^1 = \langle f, \psi_{kn}^1\rangle = \int_0^1 f(t)\,\psi_{kn}^1(x)\,dx$$

From the definition of LLMW:

$$\psi_{kn}^0(x) = 2^{\frac{k}{2}}\begin{cases} -\sqrt{3}(4(2^k x-n)-1), & n2^{-k} \le x < \left(n+\frac{1}{2}\right)2^{-k}, \\ \sqrt{3}(4(2^k x-n)-3), & \left(n+\frac{1}{2}\right)2^{-k} \le x < (n+1)2^{-k}, \\ 0, & \text{otherwise} \end{cases}$$

And:

$$\psi_{kn}^1(x) = 2^{\frac{k}{2}}\begin{cases} 6(2^k x-n)-1, & n2^{-k} \le x < \left(n+\frac{1}{2}\right)2^{-k}, \\ 6(2^k x-n)-5, & \left(n+\frac{1}{2}\right)2^{-k} \le x < (n+1)2^{-k}, \\ 0, & \text{otherwise} \end{cases}$$

Then the coefficient c^0_{kn} will be estimate as:

$$c_{kn}^0 = \int_0^1 f(x)\psi_{kn}^0(x)dx = 2^{\frac{k}{2}}\left\{\int_{n2^{-k}}^{(n+\frac{1}{2})2^{-k}} -\sqrt{3}(4(2^k x-n)-1)f(x)dx + \int_{(n+\frac{1}{2})2^{-k}}^{(n+1)2^{-k}} \sqrt{3}(4(2^k x-n)-3)f(x)dx\right\},$$

$$= 2^{\frac{k}{2}}\left\{\begin{array}{l}\int_{n2^{-k}}^{(n+\frac{1}{2})2^{-k}} -\sqrt{3}(2^{k+2}x-4n-1)f(t)dt+ \\ \int_{(n+\frac{1}{2})2^{-k}}^{(n+1)2^{-k}} \sqrt{3}(2^{k+2}x-4n-3)f(x)dx\end{array}\right\}$$

$$= 2^{\frac{k}{2}}\int_{n2^{-k}}^{(n+\frac{1}{2})2^{-k}} \sqrt{3}(2^{k+2}x-4n-1)\left[f(x+2^{-k-1})-f(x)\right]dx$$

In regard to this condition of Holder classes, if $f \in H^s[0, 1]$ then:

$$|f(x+h)-f(x)| \leqslant L|h|^s,\ h\in[0, 1],\ s\in(0, 1),\ L>0$$

Hence:

$$\left|c_{kn}^0\right| \le 2^{\frac{k}{2}}\int_{n2^{-k}}^{(n+\frac{1}{2})2^{-k}} \sqrt{3}\left|2^{k+2}x-4n-1\right|\left|f(x+2^{-k-1})-f(x)\right|dx,$$

$$\le 2^{\frac{k}{2}}\sqrt{3}L\,2^{s(-k-1)}\int_{n2^{-k}}^{(n+\frac{1}{2})2^{-k}} \left|2^{k+2}x-4n-1\right|dx,$$

$$= 2^{\frac{-k}{2}}\sqrt{3}L\,2^{s(-k-1)}\int_0^{\frac{1}{2}}\left|2^{k+2}\left[2^{-k}(x+n)\right]-4n-1\right|dx$$

$$= \frac{\sqrt{3}}{4}2^{-\frac{k}{2}}L\,2^{s(-k-1)}$$

And:

$$\left(c_{kn}^0\right)^2 \le \frac{3}{16}2^{-k}L^2\,2^{2s(-k-1)}$$

Then:

$$\sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\left(c_{kn}^0\right)^2 \le \sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\frac{3}{16}2^{-k}L^2\,2^{2s(-k-1)} = \frac{3}{16}L^2\sum_{k=M+1}^{\infty}2^{2s(-k-1)}$$

$$= \frac{3}{16}\frac{L^2}{4^{(M+1)s}(4^s-1)}$$

Now, we proceed to compute coefficients c^1_{kn}:

$$c_{kn}^1 = \int_0^1 f(x)\psi_{kn}^1(x)dx = 2^{\frac{k}{2}}\left\{\int_{n2^{-k}}^{(n+\frac{1}{2})2^{-k}}\left[6(2^k x-n)-1\right]f(x)dx + \int_{(n+\frac{1}{2})2^{-k}}^{(n+1)2^{-k}}\left[6(2^k x-n)-5\right]f(x)dx\right\}$$

Next separate interval of integration:

$$c_{kn}^1 = 2^{\frac{k}{2}}\left\{\begin{array}{l}\int_{n2^{-k}}^{(n+\frac{1}{6})2^{-k}}\left[6(2^k x-n)-1\right]f(x)dx+ \\ \int_{(n+\frac{1}{6})2^{-k}}^{(n+\frac{1}{2})2^{-k}}\left[6(2^k x-n)-1\right]f(x)dx+ \\ \int_{(n+\frac{1}{2})2^{-k}}^{(n+\frac{5}{6})2^{-k}}\left[6(2^k x-n)-5\right]f(x)dx+ \\ \int_{(n+\frac{5}{6})2^{-k}}^{(n+1)2^{-k}}\left[6(2^k x-n)-5\right]f(x)dx\end{array}\right\} \quad (1)$$

And estimate the integrals:

$$\int_{n2^{-k}}^{(n+\frac{1}{6})2^{-k}}\left[6(2^k x-n)-1\right]f(x)dx+\int_{(n+\frac{5}{6})2^{-k}}^{(n+1)2^{-k}}\left[6(2^k x-n)-5\right]f(x)dx \quad (2)$$

$$\int_{(n+\frac{1}{6})2^{-k}}^{(n+\frac{1}{2})2^{-k}}\left[6(2^k x-n)-1\right]f(x)dx+\int_{(n+\frac{1}{2})2^{-k}}^{(n+\frac{5}{6})2^{-k}}\left[6(2^k x-n)-5\right]f(x)dx \quad (3)$$

Respectively, as follows. For Eq. 2, let $-t = 6(2^k x-n)-1$ and $t = 6(2^k x-n)-5$, then:

$$\int_{n2^{-k}}^{(n+\frac{1}{6})2^{-k}}\left[6(2^k x-n)-1\right]f(x)dx +$$
$$\int_{(n+\frac{5}{6})2^{-k}}^{(n+1)2^{-k}}\left[6(2^k x\text{-}n)\text{-}5\right]f(x)dx,$$
$$=\frac{2^{-k}}{6}\left\{\int_0^1 -tf\left[\left(\frac{1-t}{6}+n\right)2^{-k}\right]dt+\int_0^1 tf\left[\left(\frac{t+5}{6}+n\right)2^{-k}\right]dt\right\},$$
$$=\frac{2^{-k}}{6}\int_0^1 t\left(f\left[\left(\frac{t+5}{6}+n\right)2^{-k}\right]-f\left[\left(\frac{1-t}{6}+n\right)2^{-k}\right]\right)dt$$

Because f(x) is from Holder class $H^s[0, 1]$, it must satisfy the inequalities such that:

$$\frac{2^{-k}}{6}\int_0^1 t\left(f\left[\left(\frac{t+5}{6}+n\right)2^{-k}\right]-f\left[\left(\frac{1-t}{6}+n\right)2^{-k}\right]\right)dt \le$$
$$\frac{2^{-k}L}{6}\int_0^1 t\left|2^{-k}\left(\frac{2t+4}{6}\right)\right|^s dt,$$
$$=\frac{2^{-k-ks}L}{6}\,\frac{4\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}$$

The Eq. 3 will be estimated similarly as above, except for replacing $t = 6(2^k x\text{-}n)\text{-}1$ and $-t = 6(2^k x\text{-}n)\text{-}5$:

$$\int_{(n+\frac{1}{6})2^{-k}}^{(n+\frac{1}{2})2^{-k}}\left[6(2^k x\text{-}n)\text{-}1\right]f(x)dx+\int_{(n+\frac{1}{2})2^{-k}}^{(n+\frac{5}{6})2^{-k}}\left[6(2^k x\text{-}n)\text{-}5\right]f(x)dx,$$
$$=\frac{2^{-k}}{6}\left\{\int_0^2 tf\left[\left(\frac{t+1}{6}+n\right)2^{-k}\right]dt+\int_0^2 -tf\left[\left(\frac{5-t}{6}+n\right)2^{-k}\right]dt\right\},$$
$$\le\frac{2^{-k}L}{6}\int_0^2 t\left|2^{-k}\left(\frac{2t-4}{6}\right)\right|^s dt=\frac{2^{-k-ks}L}{6}\,\frac{4\left(\frac{2}{3}\right)^s}{(s+2)(s+1)}$$

Then, substitute Eq. 2 and 3 into Eq. 1 to obtain:

$$c^1_{kn}\le\frac{2^{\frac{-k}{2}-ks}}{6}\left[\frac{4\left(\frac{2}{3}\right)^s}{(s+2)(s+1)}+\frac{4\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]=\frac{2^{\frac{-k}{2}-ks}}{3}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]$$

And:

$$\left(c^1_{kn}\right)^2\le\frac{2^{-k-2ks}L^2}{9}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]^2$$

Thus:

$$\sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\left(c^1_{kn}\right)^2\le\sum_{k=M+1}^{\infty}\sum_{n=0}^{2^k-1}\frac{2^{-k-2ks}L^2}{9}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]^2,$$
$$=\frac{L^2}{9}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]^2\sum_{k=M+1}^{\infty}2^{-2ks}$$
$$=\frac{1}{9}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]^2\frac{4^sL^2}{4^{(M+1)s}(4^s-1)}$$

Finally, the proof is equivalent to the theorem:

$$\|f\text{-}f_M\|_{L^2[0,1]}=\sum_{k=M}^{\infty}\sum_{n=0}^{2^k-1}\left(c^0_{kn}\right)^2+\sum_{k=M}^{\infty}\sum_{n=0}^{2^k-1}\left(c^1_{kn}\right)^2,$$
$$\le\frac{3}{16}\frac{L^2}{4^{(M+1)s}(4^s-1)}+\frac{1}{9}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]^2\frac{4^sL^2}{4^{(M+1)s}(4^s-1)},$$
$$=\frac{L^2}{4^{(M+1)s}(4^s-1)}\left(\frac{3}{16}+\frac{4^s}{9}\left[\frac{8\left(\frac{2}{3}\right)^s-3+3s}{(s+2)(s+1)}\right]^2\right)$$

Numerical examples: In this study, we wish to show the efficiency of LLMW Abd Sathar by comparing it to the previous research Siraj-ul-Islam *et al*. (2010a, b) where they applies Haar wavelets. All the examples functions are belong in the Holder classes $H^s[0, 1]$ (Ahmedov *et al*., 2017). Comparing the numerical results for Haar wavelets and LLMW with the same level of dilation (J is the dilation for Haar wavelets and k = M is the maximum dilation value of LLMW) to validate the error estimation.

Test problems: All the function for the integrals problem are consider in the Holder classes with the exact solution given in Table 1.

Table 1: Integrals function from Holder classes

Examples	Problem	Exact solution
1	$\int_0^1 x^{\frac{1}{2}}(1\text{-}x)^{\frac{1}{2}}dx$	$\frac{1}{8}\pi$
2	$\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}(x)dx$	$-\sqrt{2}\text{EllipticK}\left(\frac{\sqrt{2}}{2}\right)+2\sqrt{2}\text{EllipticE}\left(\frac{\sqrt{2}}{2}\right)$
3	$\int_0^1 x^{\frac{1}{2}}(1\text{-}x^{\frac{2}{3}})^{\frac{3}{4}}dx$	$\frac{15}{256}\pi\sqrt{2}$

Table 2: Absolute errors of example 1

LLMW	Absolute errors	Haar	Absolute errors
k = 4	2.373434E-04	J = 4	6.698944E-04
k = 5	8.40023E-05	J = 5	2.373434E-04
k = 6	2.97151E-05	J = 6	8.40024E-05
k = 7	1.05084E-05	J = 7	2.97147E-05

Table 3: Absolute errors of example 2

LLMW	Absolute errors	Haar	Absolute errors
k = 4	2.34124E-04	J = 4	6.62217E-04
k = 5	8.2775E-05	J = 5	2.34124E-04
k = 6	2.9266E-05	J = 6	8.2775E-05
k = 7	1.0344E-05	J = 7	2.9265E-05

Table 4: Absolute errors of example 3

LLMW	Absolute errors	Haar	Absolute errors
k = 4	1.435076E-04	J = 4	4.156067E-04
k = 5	4.95815E-05	J = 5	1.435076E-04
k = 6	1.71563E-05	J = 6	4.95815E-05
k = 7	5.9484E-06	J = 7	1.71562E-05

RESULTS AND DISCUSSION

In Table 2-4 a comparative analysis between Haar wavelets and LLMW in terms of absolute error is obtained. The error analysis for LLMW method is examined by considering the functions belonging in Holder space $H^s[0, 1]$. It is clearly observed that the error estimation by linear Legendre multi-wavelets performed a better result than the Haar wavelets in all the examples (Table 2-4).

CONCLUSION

In this research, the LLMW is presented to solve the definite integrals. The error analysis theorem for the approximate solution of the functions belong in the Holder classes based on LLMW are established. Numerical example revealed that the accuracy of LLMW are better than the Haar wavelets.

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