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The Embedding Theorems on the Semi-Neat-Semi-Subgroups (SNSS) of Semi-Groups

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Abstract: This study presents the general results of the embedding theorems on the Semi-Pure-Semi-Subgroups (SPSS) of an abelian group. We give the definition of a new semi-subgroups which is called, Semi-Neat-Semi-Subgroups of semi-group G (denote it by SNSS) and investigate some embedding problems on semi-neat-subgroup.

Key words: Embedding theorems, semi-pure-semi-subgroups, semi-neat-semi-subgroups, investigate, abelian, definition

INTRODUCTION

In this study, we give the general results of the embedding theorems on the Semi-Pure-Semi-Subgroups (SPSS) of abelian group G studied by Lajos (1961). We take G is a semigroup and give the definition of a new semi-subgroups which is called, Semi-Neat-Semi-Subgroups of semi-group G (denote it by SNSS) and investigate some embedding problems on semi-neat-subgroup. In section 2, we give the necessary and sufficient conditions for Semi-Neat-Semi-Subgroups (SNSS) to be embeddable in right groups. Moreover, we give a criterion for semi-neat-semi-subgroups with zero elements 0 and being unions of disjoint subgroups $G\alpha$, $\alpha \epsilon Y$ such that:

$$G\alpha \cap G\beta = \{0\} \text{ if } \alpha \neq \beta \quad (1)$$

In section 3, we investigate embeddable of commutative semigroups into groups. In particular into torsion-free groups. For this purpose, we define (m, n) separately of Semi-Neat-Semi-Subgroups (SNSS) embedding into completely regular semi-group. In section 4 (m, n) separate semi-neat-semi-subgroups will be studied.

MATERIALS AND METHODS

Embedding into completely regular Semi-Neat-Semi Subgroups (SNSS)

Definition 1: A subgroup S is said to be neat in G, if p|x in G for x in S and all prime number p then p|x in S. We say that a semi-subgroup S is a semi-neat-semi in G for all x in S, if p|x, in G then p|x in S for some p. And we shall denote it by (SNSS). Clearly, any pure is a (SPSS) and every SPSS is a SNSS. And we have $\alpha g = x$ implies $\alpha xo = x$ for some xo in S and some positive integer α. A semi-neat-semi-subgroups is said to be a right group, if it contained no proper left ideals and is right calculative (Lajos, 1976). For the next two theorems the following lemmas are needed (Lajos, 1975).

Lemma A: A right group is the union of a set of isomorphic disjoint groups, if e and f are distinct idem potent a right group S, then the mapping $x \rightarrow xf$ (x in Se) is an isomorphism of the group Se upon the group Sf (Lajos, 1977).

Lemma B: ≙ semi-groups is a right group, if and only if it is the union of disjoint subgroups such that the set of identity elements of the subgroups is a right zero sub semi-pure subgroups (Lajos, 1977). Now, we are ready to show the following results.

Theorem 1: A Semi-Neat-Semi-Subgroup (SNSS) can be embedded in a right group if and only if it is the union of disjoint Semi-Neat-Semi-Subgroups (SNSS) $S\alpha$, α in Y such that each semi-neat-subgroup $S\alpha$ can be embedded in a group $G\alpha$, $G\alpha \cap G\beta = \{0\}$, if $\alpha \neq \beta$ and for every α and β in Y there exists an isomorphism of $G\alpha$ onto $G\beta$ such that:

- $\alpha\psi\alpha\beta\psi\beta\gamma = \alpha\psi\alpha\gamma$ for all $\alpha \epsilon G\alpha$, α, $\beta, \gamma \epsilon Y$
- $\alpha\psi\alpha\beta = \alpha\beta$ for every $\alpha \epsilon S\alpha$, $\beta \epsilon S\beta$, $\alpha\beta \epsilon Y$
- $\psi\alpha\alpha$ is the identity mapping of $G\alpha$, $\alpha \epsilon Y$

Proof: First, we show that the conditions are sufficient. Assume that the Semi-Neat-Semi-Subgroup (SNSS) $S\alpha$

satieties the conditions of the theorem and Let, G = U{Gα: α∈Y} for any elements α, β of G there exist α, β∈Y such that α∈Gα and β∈Gβ. Let αoβ = αψαβ. Thus, we have defined an operation "o" on G. we prove that (G, 0) is a right group. The operation is single-valued because Gα∩Gβ = {0}, if α≠β. In order to show the associatively, let a, b, c are any elements of G. then there exist α, β, γ∈Y such that α∈Gα, β∈Gβ, c∈Gγ, thus:

$$\begin{aligned} & a0(b0c) = a0(b\psi\beta\gamma c) = a\psi\alpha\gamma(b\psi\alpha\gamma c) = \\ & (a\psi\alpha\gamma b\psi\beta\gamma)c = (a\psi\alpha\beta\psi\beta\gamma b\psi\beta\gamma)c = \\ & (a\psi\alpha\gamma b)\psi\beta\gamma c = (a0b)0c \end{aligned} \quad (2)$$

If a, b∈Gα then αob = αψαb = ab that is Gα is a semigroup which is the union of the disjoint subgroups (semi-subgroups). If e∈Gα and f∈Gβ (are idempotent elements then eψαβ = ff = f. Consequently, the set of the identity element a of the group Gα,α∈Y is a right zero sub-semi-group. Thus, G is a right group, by lemma 2, we show that S can be embedded in G. Since, S is in G, we have only to prove that αob = αb for every α, b∈S.

Let a, b any couple of elements of S. Then there exist α,β∈Y, so that, aSα≤Gα and b∈Sβ≤Gβ, thus aob = αψαβb = ab. By condition (ii), consequently, S is a (SNSS) of G and the first part of the theorem is proved. Conversely, assume that the semigroup S is embeddable in a right group G. By lemma A, G is the union of a distinct of isomorphic subgroups Gα, α∈Y. Thus, S is the union of some sub-semi group Sα, α∈Y where, every Sα can be embedded in Gα. The mapping ψaβ: X→Xf(X) is an isomorphism of the group Gα, upon Gβ. Where, f is the identity of Gβ (see lemma A). Let e, f, h be the identity elements of Gα, Gβ and Gγ, respectively and let a∈Gα, b∈β, x∈Sα, y∈Sβ, α, β, γ∈Y; then:

$$\begin{aligned} & a\psi\alpha\beta\psi\beta\gamma = (af)h = a(fh) = ah = a\psi\alpha\gamma, \\ & x\psi\alpha\beta y = (xf)y = xy = a\psi\alpha\alpha = ae = a \end{aligned} \quad (3)$$

Thus, the theorem is completely proved.

Theorem 2: Any (SNSS) has a zero element 0 can be embedded in a semi-group which has a zero element ó and IB the union of disjoint subgroups, Gα, α∈Y, so that, Gα, Gβ = o for every α≠β in Y, if and only if it is the union of disjoint Gα, Gβ sub semigroup embedded in groups Sα, α in Y such that Sα, Sβ = 0 for every α, β∈Y.

Proof: Since, the necessity of the coalition is trivial, we have only to show the sufficiency. Assume that S is a (SNSS) with zero element is the union of disjoint (SNSS), Sα, α∈Y such that every Sα is embeddable in a group Gα and Sα, Sβ = 0 for every α and β are different, α and β in Y.

We may assume that Gα is generated by Sα and so, Gα and Gβ are different, Gα∩Gβ = {0}, α≠β. Let G = U {Gα: α∈Y} we define an operation "o" on G as follows: for any elements α∈Gα, β∈Gβ, let aob = ab if α = β and aob = 0 if α≠β. It is evident that G is a semi-group with a zero element and G is the union of the subgroups Gα, α∈Y. Since, Sα is (SNSS) and aob = ab for every a, b in S, the semigroup S is embedded in G (S is a semi-neat-semi-subgroup of G).

RESULTS AND DISCUSSION

Embedding in groups: A commutative semigroup can be embedded in a group if and only if it is calculative. For non-connotative semigroups cancellation is an evidently necessary condition for embed ability in a group but it is far from sufficient. The first necessary and efficient condition is due to Lajos (1976).

In this study, we investigate in groups for several classes of commutative Semi-Neat-Semi-Subgroups (SNSS) and deal with the embedding of commutative semigroups into torsion free groups.

Definition 2: Let S be an arbitrary (SNSS) any element a of S will be called a asymmetry element, if xay = yax for every couple x, y (Petrich, 2005).

Lemma C: The eel of all symmetry elements of a Semi-Neat-Semi-Subgroup S (SNSS) is either empty or an ideal of S.

Proof: Let a "be asymmetry element and x, y be any elements of a semigroups S then for every s∈S:

$$\begin{aligned} & x(as)y = xa(sy) = (sy)ax = s(yax) = s(xay) = \\ & (sx)ay = ya(sx) = y(as)x \end{aligned} \quad (4)$$

And:

$$\begin{aligned} & x(sa)y = (xa)ay = ya(xa) = (yax)s = (xay) = \\ & xa(ye) = (ya)ax = y(sa)x \end{aligned} \quad (5)$$

Consequently, both of a and s belong to the set of all symmetry element a of S. Thus, the lemma is proved.

Theorem 3: A left calculative semigroup which has a symmetry element is commutative.

Corollary: A calculative semigroup which has a symmetry element can be embedded in a group.

Proof: Assume that the left calculative semigroup S has a symmetry element a and let x, y be arbitrary elements of S. By lemma C as is a a symmetry element of S for any a and (sa) (xy) = (sax)y = x(as)y = y(as)x = (yas)x = (sa) (yx). Thus, xy = yx because it is a left calculative.

Definition 3: A Semigroup S is said to be separative, if implies for every couple.

Theorem 4: Let, S be a commutative separative (SNSS), then S can be embeddable in a group if and only if some power n of S^n (n>1) is embeddable in a group.

Proof: If a semigroup is embeddable in a group then every power of it is so. Conversely, let S be a commutative separtive (SNSS). Assume that there exists a positive integer n, so that, S^n is embeddable in a group. Since, there is a positive integer w such $2^w \geq n^m$, $S^n \geq S^{2w}$ and so, S^{2w} is embeddable in a group. Consequently, it sufficed to show that S (SNSS) is embeddable in a semigroup S^2 because by applying this particular proposition general times, we get in succession that S^{2w-1}, S^{2w-2}, ..., S^2 and finally, S can be embeddable in a group. Let S be embeddable in a group then S^2 is calculative. We prove that S is calculative too let a, x, y∈S such that ax = ay. Then, ax = $a^2x^2 = a^2yx$ and ${}^2xy = a^2y^2$. The elements a^2, x^2, xy∈S, hence, $x^3 = y^2 = xy$ because S^2 is calculative and is commutative, since, S is separative, we get x = y this means that S (SNSS) is left calculative. Similarly, S is right calculative, thus, S can be embedded in a group and the theorem is proved. Before dealing with the embedding of commutative semigroups in torsion-free groups, we need to prove the following theorem.

Theorem 5: A Semi-Neat-Semi-Subgroup (SNSS) which is not a group cannot be embedded in a torsion group.

Proof: Let, G be a torsion group. Then for any element a of G there exists a positive integer n, so that, $a^n = 1$ (the identity of G). We prove that every Semi-Pure-Semi-Subgroup (SNSS) of G is left simple and right simple. First, we show that if K is a (SNSS) of G, then, the left idealizer of K, Id = {x∈S:xK≤K} and the right idealizer of K equal to K. Since, G is a torsion group, the identity 1 belongs to K. Thus, for any x∈G, x = x1∈K and x = 1x∈Kx. Hence, xK≤K (Kx≤K) and x≤K.

Therefore, Idk = K = dKI. Let S be a Semi-PureSemi-Subgroup (SNSS) of G. Then a semi-pure-semi-subgroup K of S is a left [right] ideal of S iff S≤IdK = K and S≤dKI = K. Thus, if K is (SNSS) is a left (right) ideal of S, then K = S because S≤IdK = K.

Consequently, S has no proper left and right ideals, hence, S is subgroup of G. Thus, if a semi-neat-semi-subgroup S can be embedded in a torsion group G then S is a group. The theorem is proved.

Definition 4: Let m and n be fixed positive integer. A semi pure semi-subgroup S will be called (m, n)-separative, if $a^m b^n = a^n b^m$ implies a = b for all a, b belong to S.

Theorem 6: A commutative Semi-Neat-Semi-Subgroup (SNSS) can be embedded in a torsion-free group, if and only if it is a calculative (m, n)-separative semigroup for all positive integer m>n.

Proof: Assume that the commutative (SNSS), S can be embedded in a torsion-free group G. We may assume that G is generated by S. Then G is a commutative group. Let m and n be positive integers, so that, m>n and a, b∈S with the assumption $a^m b^n = a^n b^m$. We show that a = b. Since, S is cancellative, $\overset{m-n}{ab} = \overset{m-n}{b}$. Since a, b are elements of G, there exists an element x of G such that ax = b. Thus, $b^{m-n} = (ax)^{m-n} = x^{m-n} = b^{m-n} x^{m-n}$ because G is abelian. It followed that x^{m-n} is the identity of G. Hence, a = b.

Conversely, assume that a commutative (SNSS), S is cancellative and for every positive integer m and n, $a^m b^n = a^n b^m$, a, b∈S, implies a = b. Since, S is commutative (SNSS) and calculative it is embeddable in a group G. By making use the usual construction of G [G = SXS where (a, b) R(c, d) iff ad = cb; a, b, c, d∈S, we show that G is a torsion-free. A assume $(a, b)^m$ (c, c) (c∈S) for the element (a,b) of G and for some positive integer m≥2. Then $(a^m b^m) = (c,c)$ that is, $(a^m c, c b^m)$, since, S is commutative, $(a^m c, b^m c)$, since, S is cancellative $a^m = b^m$. Thus, for any couple n>m of positive integers, $b^{n-m} a = b^{n-m} b^m a^{n-m}$, that is b^{n-m}, hence, a = b, by the assumption for S. Consequently, (a, b) = (c, c), c∈S and so, G is a torsion-free group.

On (m.n)-separaive semigroups: Theorem 5 shown that (m, n)-separativity is a useful condition for embedding in torsion-free groups. In this study, we investigate the (m, n)-separative Semi Neat-Semi-Subgroups (SNSS).

Lemma D: If an (m, n)-separative Semi-Neat-Semi Subgroup (SNSS) S, contains an idempotent e, then e is the identity of S.

Proof: Let, S be an (m, n)-separative (SNSS), m>n and e an idempotent of then for every x∈S, $(xe)^n (ex)^m = (xe)^{n-1} (xex(ex)^{m-1} = (xe)^{m-1}xex(ex)^{n-1} = (xe)^m(ex)^n$. Which implise that:

$$Xe = ex \tag{6}$$

Thus, (xe) = x. By the some method we can show that ex = x. Thus, e is the identity element of S.

Lemma E: If a semigroup is a union of disjoint (n, n-2)-separative (SNSS) Semi-Neat-Semi-Subgroups Sα, α∈Y, N>2, then S is a separative.

Proof: First, we show that every Sα is separative. Let a and b be any elements of Sα satisfying: $a^2 = b^2 = ab$. Then, for any positive integer n>2, $a^n b^{n-2} = a^{n-1} b^{n-1}$ and $a^{n-2} b^n = a^{n-1} b^{n-1}$, we obtain that: if x, y∈S, then:

$$x^2 = y^2 = xy. \text{If } x \in S\alpha,\ \alpha \in Y,\ \text{So, } Y = S\alpha \quad (7)$$

We get that S is a separative.

Lemma F: Every (2,1)-separative (SNSS) semigroup is separative. If a semi-neat-semi-subgroup is the union of disjoint (2, 1)-separative semi-groups then it is separative.

Proof: Let S be a (2, 1)-separative (SNSS) semigroup and a, b∈S such that:

$$a^2 = ab = b^2, \text{then } a^2 b = ab^2 \quad (8)$$

which implies a = b. Let S be a (SNSS) Semi-Neat-Semi Subgroup is the union of disjoint (2,1)-separative semigroups, then by lemmas (D, E) and above prove (i) of lemma F, we get that S is a separative.

Theorem 7: A (3, 1)-separative(SNSS) Semi-Neat-Semi Subgroup is cancellative if some power of it is cancellative.

Proof: Assume that S is a in a (3,1) separative (SNSS) and there is a positive integer n, so that, S^n is cancellative. We may assume that n = 2 as we have proved in the proof of theorem 4. Then, S^2 is cancellative. Let a, x, y∈S such that ax = ay. Then:

$$a^2xy = a^2y^2,\ a^2yx = a^2x^2 \quad (9)$$

Consequently, $y^2 = xy$ and $y^2 = x^2$ because S a is cancellative thus:

$$a^3y = xyxy = xy^3 \quad (10)$$

Since, S is (3,1)-separative, so, x = y we can prove similarly that xa = ya implies, y = x for any element a, x, y∈S. Thus, S is calculative and the theorem is completely proved.

CONSLUSION

In this study, we investigate in groups for several classes of commutative Semi-Neat-Semi-Subgroups (SNSS) and deal with the embedding of commutative semigroups into torsion-free groups.

REFERENCES

Lajos, S., 1961. Generalized ideals in semigroups. Acta. Sci. Math. Szeged, 22: 217-222.

Lajos, S., 1975. On the cbaracteriaatitm of completely regular semigroups. Math. Japon, 20: 33-39.

Lajos, S., 1976. A characterization of Cliffordian semigroups. Proc. Japan Acad., 52: 496-497.

Lajos, S., 1977. A characterization of certain elements in semigroupa. Math. Sem. Notes, 5: 257-260.

Petrich, M., 2005. Introduction to Semi-Group. Merrill Publishers, Columbus, Ohio, ISBN:9780675090629, Pages: 198.