

## On Notion Of $\beta\delta$ -Reduction for Main Canonical Notion of $\delta$ -Reduction

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**Abstract:** In this study, the notion of  $\beta\delta$ -reduction for main canonical notion of  $\delta$ -reduction is considered. Typed  $\lambda$ -terms use variables of any order and constants of order  $\leq 1$  where constants of order 1 are strongly computable, monotonic functions with indeterminate values of arguments. The canonical notion of  $\delta$ -reduction is the notion of  $\delta$ -reduction that is used in the implementation of functional programming languages. For main canonical notion of  $\delta$ -reduction the uniqueness of  $\beta\delta$ -normal form of typed  $\lambda$ -terms is shown.

**Key words:** Typed  $\lambda$ -terms, main canonical notion, uniqueness of  $\beta\delta$ -normal form, monotonic functions, canonical notion,  $\delta$ -reduction

### INTRODUCTION

The canonical notion of  $\delta$ -reduction as well as the main canonical notion of  $\delta$ -reductions are introduced by Nigiyanyan and Khondkaryan (2017) where also shown that the main canonical notion of  $\delta$ -reduction is a canonical notion of  $\delta$ -reduction. In this research, we examine the uniqueness of  $\beta\delta$ -normal form of typed  $\lambda$ -terms for main canonical notion of  $\delta$ -reduction.

### MATERIALS AND METHODS

Definitions of this study are take from the researches of Nigiyanyan and Khondkaryan (2017), Nigiyanyan (1992, 1993) and Budaghyan (2002). Let,  $M$  be a partially ordered set which has a least element  $\perp$  which corresponds to the indeterminate value and each element of  $M$  is comparable only with  $\perp$  and itself. Let us define the set of types (denoted by types):

- $M \in \text{Types}$ ,
- If  $\beta, \alpha_1, \dots, \alpha_k \in \text{Types}$  ( $k > 0$ ), then, the set of all monotonic mappings from  $\alpha_1 \times, \dots, \times \alpha_k$  into  $\beta$  (denoted by  $[\alpha_1 \times, \dots, \times \alpha_k \rightarrow \beta]$ ) belongs to types

Let  $\alpha \in \text{Types}$ , then, the order of type  $\alpha$  (denoted by  $\text{ord}(\alpha)$ ) will be natural number which is defined in the following way: if  $\alpha = M$  then  $\text{ord}(\alpha) = 0$ , if  $\alpha = [\alpha_1 \times, \dots, \times \alpha_k \rightarrow \beta]$  where,  $\beta, \alpha_1, \dots, \alpha_k \in \text{Types}$ ,  $k > 0$  then  $\text{ord}(\alpha) = 1 + \max(\text{ord}(\alpha_1), \dots, \text{ord}(\alpha_k), \text{ord}(\beta))$ . If  $x$  is a variable of type  $\alpha$  and constant  $c \in \alpha$ , then,  $\text{ord}(x) = \text{ord}(c) = \text{ord}(\alpha)$ .

Let  $\alpha \in \text{Types}$  and  $V_\alpha$  be a countable set of variables of type  $\alpha$ , then,  $V = \bigcup_{\alpha \in \text{Types}} V_\alpha$  is the set of all variables. The set of all terms, denoted by  $\Lambda = \bigcup_{\alpha \in \text{Types}} \Lambda_\alpha$  where,  $\Lambda_\alpha$  is the set of terms of type  $\alpha$  is defined the following way:

- If  $c \in \alpha$ ,  $\alpha \in \text{Types}$ , then  $c \in \Lambda_\alpha$
- If  $x \in V_\alpha$ ,  $\alpha \in \text{Types}$ , then  $x \in \Lambda_\alpha$
- If  $\tau \in \Lambda_{[\alpha_1 \times, \dots, \times \alpha_k \rightarrow \beta]}$ ,  $t_i \in \Lambda_{\alpha_i}$  where,  $\beta, \alpha_i \in \text{Types}$ ,  $i = 1, \dots, k$ ,  $k \geq 1$ , then,  $\tau(t_1, \dots, t_k) \in \Lambda_\beta$  (the operation of application,  $(t_1, \dots, t_k)$  is the scope of the applicator  $\tau$ )
- If  $\tau \in \Lambda_\beta$ ,  $x_i \in V_{\alpha_i}$  where,  $\beta, \alpha_i \in \text{Types}$ ,  $i \neq j$ ,  $x_i \neq x_j$ ,  $i, j = 1, \dots, k$ ,  $k \geq 1$ , then,  $\lambda x_1, \dots, x_k [\tau] \in \Lambda_{[\alpha_1 \times, \dots, \times \alpha_k \rightarrow \beta]}$  (the operation of abstraction,  $\tau$  is the scope of the abstractor  $\lambda x_1, \dots, x_k$ )

The notion of free and bound occurrences of variables as well as free and bound variable are introduced in the conventional way. The set of all free variables in the term  $t$  is denoted by  $FV(t)$ . Terms  $t_1$  and  $t_2$  are said to be congruent (which is denoted by  $t_1 \equiv t_2$ ), if one term can be obtained from the other by renaming bound variables. The free occurrence of a variable in the term is called internal, if it does not enter in the applicator, the scope of which contains a free occurrence of some variable. The free occurrence of a variable in the term is called external, if it does not enter in the scope of the applicator that contains a free occurrence of some variable.

Let,  $t \in \Lambda_\alpha$ ,  $\alpha \in \text{Types}$  and  $FV(t) \subseteq \{y_1, \dots, y_n\}$   $\bar{y}_0 = (y_1^0, \dots, y_n^0)$  where,  $y_i \in V_{\beta_i}$ ,  $y_i^0 \in \beta_i$ ,  $\beta_i \in \text{Types}$ ,  $i = 1, \dots, n$ ,  $n \geq 0$ . The value of the term  $t$  for the values of the variables  $y_1, \dots, y_n$  equal to  $\bar{y}_0 = (y_1^0, \dots, y_n^0)$  is denoted by  $\text{val}_{\bar{y}_0}(t)$  and is defined in the conventional way.

Let, terms  $t_1, t_2 \in \Lambda_\alpha$ ,  $\alpha \in \text{Types}$ ,  $FV(t_1) \cup FV(t_2) = \{y_1, \dots, y_n\}$ ,  $y_i \in V_{\beta_i}$ ,  $y_i^0 \in \beta_i$ ,  $\beta_i \in \text{Types}$ ,  $i = 1, \dots, n$ ,  $n \geq 0$  then, terms  $t_1$  and  $t_2$  are called equivalent (denoted by  $t_1 \sim t_2$ ), if for any  $\bar{y}_0 = (y_1^0, \dots, y_n^0)$  where,  $y_i^0 \in V_{\beta_i}$ ,  $i = 1, \dots, n$  we have the following:  $\text{val}_{\bar{y}_0}(t_1) = \text{val}_{\bar{y}_0}(t_2)$ . A term  $t \in \Lambda_\alpha$ ,  $\alpha \in \text{Types}$  is called a constant term with value  $a \in \alpha$ , if  $t \sim a$ .

Further, we assume that  $M$  is a recursive set and considered terms use variables of any order and

constants of order  $\leq 1$  where constants of order 1 are strongly computable, monotonic functions with indeterminate values of arguments. A function  $f: M^k \rightarrow M$ ,  $k \geq 1$  with indeterminate values of arguments is said to be strongly computable, if there exists an algorithm which stops with value  $f(m_1, \dots, m_k) \in M$  for all  $m_1, \dots, m_k \in M$  (Nigiyani, 2015).

To show mutually different variables of interest  $x_1, \dots, x_k$ ,  $k \geq 1$  of a term  $t$ , the notation  $t(x_1, \dots, x_k)$  is used. The notation  $t(t_1, \dots, t_k)$  denotes the term obtained by the simultaneous substitution of the terms  $t_1, \dots, t_k$  for all free occurrences of the variables  $x_1, \dots, x_k$ , respectively where  $x_i \in V_{\alpha_i}$ ,  $i \neq j \Rightarrow x_i \neq x_j$ ,  $t_i \in \Lambda_{\alpha_i}$ ,  $\alpha_i \in \text{Types}$ ,  $i, j = 1, \dots, k$ ,  $k \geq 1$ .

A substitution is said to be admissible, if all free variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

A term  $t$  with a different fixed occurrences of subterms  $\tau_1, \tau_2$  where,  $\tau_1$  is not a subterm of  $\tau_2$  and  $\tau_2$  is not a subterm of  $\tau_1$  and  $\tau_i \in \Lambda_{\alpha_i}$ ,  $\alpha_i \in \text{Types}$ ,  $i = 1, 2$  is denoted by  $t_{\tau_1, \tau_2}$ . A term with the fixed occurrences of the terms  $\tau'_1, \tau'_2$  replaced by the terms  $\tau''_1, \tau''_2$ , respectively is denoted by  $t_{\tau'_1, \tau'_2}$  where,  $\tau'_i \in \Lambda_{\alpha_i}$ ,  $i = 1, 2$ .

A term of the form  $\lambda x_1, \dots, x_k [\tau [x_1, \dots, x_k]] (t_1, \dots, t_k)$  where,  $x_i \in V_{\alpha_i}$ ,  $i \neq j \Rightarrow x_i \neq x_j$ ,  $\tau \in \Lambda$ ,  $t_i \in \Lambda_{\alpha_i}$ ,  $\alpha_i \in \text{Types}$ ,  $i, j = 1, \dots, k$ ,  $k \geq 1$  is called a  $\beta$ -redex, its convolution is the term  $\tau_i [t_1, \dots, t_k]$ . The set of all pairs  $(\tau_0, \tau_1)$  where,  $\tau_0$  is a  $\beta$ -redex and  $\tau_1$  is its convolution is called a notion of  $\beta$ -reduction and is denoted by  $\beta$ . A one-step  $\beta$ -reduction ( $\rightarrow \beta$ ) and  $\beta$ -reduction ( $\rightarrow \beta$ ) are defined in the conventional way. A term containing no  $\beta$ -redexes is called a  $\beta$ -normal form. The set of all  $\beta$ -normal forms is denoted by  $\beta\text{-NF}$ .

$\delta$ -redex has a form  $f(t_1, \dots, t_k)$  where,  $f \in [M^k \rightarrow M]$ ,  $t_i \in \Lambda_{\alpha_i}$ ,  $i = 1, \dots, k$ ,  $k \geq 1$  its convolution is either  $m \in M$  and in this case  $f(t_1, \dots, t_k) \sim m$  or a subterm  $t_i$  and in this case  $f(t_1, \dots, t_k) \sim t_i$ ,  $i = 1, \dots, k$ . A fixed set of term pairs  $(\tau_0, \tau_1)$  where,  $\tau_0$  is a  $\delta$ -redex and  $\tau_1$  is its convolution is called a notion of  $\delta$ -reduction and is denoted by  $\delta$ . A one-step  $\delta$ -reduction ( $\rightarrow \delta$ ) and  $\delta$ -reduction ( $\rightarrow \delta$ ) are defined in the conventional way.

A one-step  $\beta\delta$ -reduction ( $\rightarrow$ ) and  $\beta\delta$ -reduction ( $\rightarrow$ ) defined in the conventional way. A term containing no  $\beta\delta$ -redexes is called normal form. The set of all normal forms is denoted by  $\text{NF}$ .

A notion of  $\delta$ -reduction is called a single-valued notion of  $\delta$ -reduction, if  $\delta$  is a single-valued relation, i.e., if  $(\tau_0, \tau_1) \in \delta$  and  $(\tau_0, \tau_2) \in \delta$ , then,  $\tau_1 \equiv \tau_2$  where  $\tau_0, \tau_1, \tau_2 \in \Lambda_M$ . A notion of  $\delta$ -reduction is called an effective notion of  $\delta$ -reduction if there exists an algorithm which for any term  $f(t_1, \dots, t_k)$  where,  $f \in [M^k \rightarrow M]$ ,  $t_i \in \Lambda_{\alpha_i}$ ,  $i = 1, \dots, k$ ,  $k \geq 1$ , gives its convolution, if  $f(t_1, \dots, t_k)$  is a  $\delta$ -redex and stops with a negative answer otherwise (Barendregt, 1981).

**Definition 1; Nigiyani and Khondkaryan (2017):** An effective, single-valued notion of  $\delta$ -reduction is called a canonical notion of  $\delta$ -reduction if:

- $t \in \beta\text{-NF}$ ,  $t \sim m$ ,  $m \in M \setminus \{\perp\} \Rightarrow t \rightarrow_{\delta} m$
- $t \in \beta\text{-NF}$ ,  $FV(t) = \emptyset$ ,  $t \sim \perp \Rightarrow t \rightarrow_{\delta} \perp$

**Main canonical notion of  $\delta$ -reduction, the uniqueness of the  $\beta\delta$ -normal form**

**Definition 2:** Let,  $C$  be a recursive set of strongly computable, monotonic functions with indeterminate values of arguments. The following notion of  $\delta$ -reduction is called main canonical notion of  $\delta$ -reduction if for every  $f \in C$ ,  $f: M^k \rightarrow M$ ,  $k \geq 1$ , we have:

- If  $f(m_1, \dots, m_k) = m$  where  $m, m_1, \dots, m_k \in M$ ,  $m \neq \perp$ , then  $(f((\mu_1, \dots, \mu_k)) \in \delta$  where,  $\mu_i = m_i$ , if  $m_i \neq \perp$  and  $\mu_i \equiv t_i$ ,  $t_i \in \Lambda_M$  if  $m_i = \perp$ ,  $i = 1, \dots, k$ ,  $k \geq 1$
- If  $f(m_1, \dots, m_k) = \perp$  where,  $m_1, \dots, m_k \in M$ , then  $(f(m_1, \dots, m_k), \perp) \in \delta$

Nigiyani and Khondkaryan (2017) showed that the  $\delta$  is a canonical notion of  $\delta$ -reduction.

**Definition 3:** The term  $t \in \Lambda$  is said to be strongly normalizable, if the length of each  $\beta\delta$ -reduction chain from the term  $t$  is finite.

## RESULTS AND DISCUSSION

**Theorem 1; Budaghyan (2002):** Every term is strongly normalizable.

**Theorem 2; Budaghyan (2002):** For every term  $t \in \Lambda$ , if  $t \rightarrow_{\beta} t'$ ,  $t \rightarrow_{\delta} t''$  and  $t', t'' \in \beta\text{-NF}$  then,  $t' \equiv t''$ .

**Definition 4:** Let,  $t \in \Lambda_{\alpha}$ ,  $\alpha \in \text{Types}$  and  $t \equiv t_1 \rightarrow \dots \rightarrow t_n$ ,  $n \geq 1$  where,  $t_i \in \Lambda_{\alpha}$ ,  $i = 1, \dots, n$ , then, the sequence  $t_1, \dots, t_n$  is called the inference of the term  $t_n$  from the term  $t$  and  $n$  is called the length of that inference.

**Definition 5:** The inference tree of the term  $t$  is an oriented tree with the root  $t$  and if a term  $\tau$  is some node of the tree and  $\tau_1, \dots, \tau_k$ ,  $k \geq 0$  are all  $\beta\delta$ -redexes of  $\tau$ , then,  $\tau_{\tau_1}, \dots, \tau_{\tau_k}$ ,  $k \geq 0$  are all descendants of the node  $\tau$  where,  $\tau'_i$  is the convolution of  $\tau_i$ ,  $i = 1, \dots, k$ .

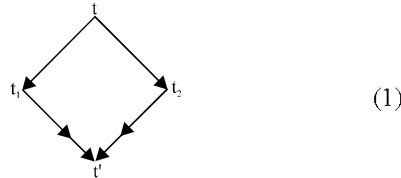
It is easy to see that each node in the inference tree of the term  $t$  has finite number of descendants and if  $\tau$  is a leaf of that tree then  $\tau \in \text{NF}$ .

**Theorem 3:** For the main canonical notion of  $\delta$ -reduction  $\delta$  and for every term  $t \in \Lambda$ , if  $t \rightarrow t'$ ,  $t \rightarrow t''$  and  $t', t'' \in \text{NF}$ , then,  $t' \equiv t''$ . To prove theorem 3 let us prove lemma 1-3.

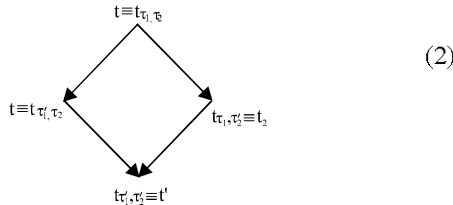
**Lemma 1:** Let,  $\delta$  be the main canonical notion of  $\delta$ -reduction,  $t$  be a term with a fixed occurrence of the term  $\tau$ . If  $t$  is a  $\delta$ -redex,  $\tau$  is a  $\beta\delta$ -redex then there exists  $m \in M$ ,  $m \neq \perp$  such that  $t \rightarrow_{\delta} m$  and  $t_{\tau} \rightarrow_{\delta} m$  where,  $\tau'$  is the convolution of the  $\tau$  which is a  $\beta\delta$ -redex.

**Proof:** Let,  $t_i \equiv f(t_1, \dots, t_i, \dots, t_k)$ ,  $f \in [M^k \rightarrow M]$ ,  $t_i \in \Lambda_M$ ,  $i = 1, \dots, k$ ,  $1 \leq i \leq k$ . Since,  $t$  is a  $\beta\delta$ -redex, then,  $t_i \notin M$  and since,  $t$  is a  $\delta$ -redex then from the definition 2 follows that there exists  $m \in M$ ,  $m \neq \perp$  such that  $(t, m) \in \delta$ . Therefore,  $t \rightarrow_{\delta} m$ . Since,  $t_i \notin M$  and  $(t, m) \in \delta$  where,  $m \neq \perp$ , then, from the definition 2 follows that  $f(t_1, \dots, \mu, \dots, t_k) \in \delta$  for every term  $\mu \in \Lambda_M$ . Therefore,  $f(t_1, \dots, t_{\tau}, \dots, t_k), m) \in \delta$  and  $t_{\tau} \rightarrow_{\delta} m$ .

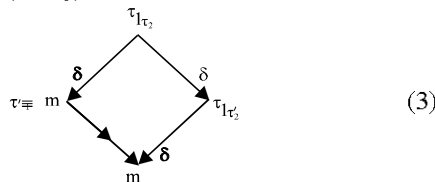
**Lemma 2:** For the main canonical notion of  $\delta$ -reduction  $\delta$  and for every term  $t \in \Lambda_{\alpha}$ ,  $\alpha \in \text{Types}$ , if  $t \rightarrow t_1$ ,  $t \rightarrow t_2$ ,  $t_1, t_2 \in \Lambda_{\alpha}$  then, there exists a term  $t' \in \Lambda_{\alpha}$  such that  $t_1 \rightarrow t'$  and  $t_2 \rightarrow t'$ :



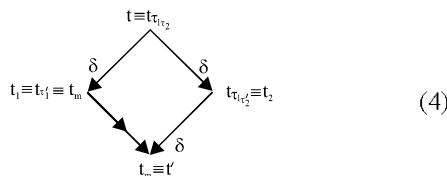
**Proof:** If  $t_1 \equiv t_2$ , then,  $t' \equiv t_1 \equiv t_2$ . If  $t_1 \neq t_2$  then, there exist  $\beta\delta$ -redexes  $\tau_1, \tau_2 \in \Lambda$  such that  $t \equiv t_{\tau_1} \equiv t_{\tau_2}$ ,  $t_{\tau_1} \equiv t_{\tau_1}$  and  $t_2 \equiv t_{\tau_2}$  where terms  $\tau'_1, \tau'_2$  are the convolutions of  $\tau_1$  and  $\tau_2$  accordingly. If  $\tau_1$  is not a subterm of  $\tau_2$  and  $\tau_2$  is not a subterm of  $\tau_1$ , then from Eq. 2 follows that  $t' \equiv t_{\tau'_1 \tau'_2}$ :



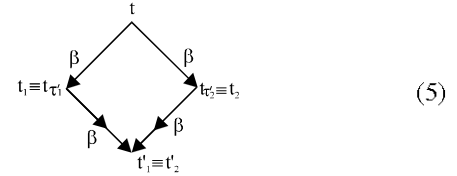
If  $\tau_2$  is a subterm of  $\tau_1$  or  $\tau_1$  is a subterm of  $\tau_2$ , then, the following cases are possible:  $\tau_1$  and  $\tau_2$  are both  $\delta$ -redexes. Without loss of generality we suppose that  $\delta_2$  is a subterm of  $\tau_1 = (t_1 \equiv t_{\tau_2})$ . From lemma Eq. 1 follows Eq. 3:



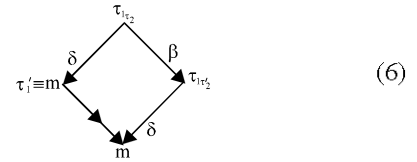
where,  $m \in M$ ,  $m \neq \perp$  and  $m$  is the convolution of the term  $\tau_1$ . Therefore, from Eq. 4 follows that  $t' \equiv t_m$ :



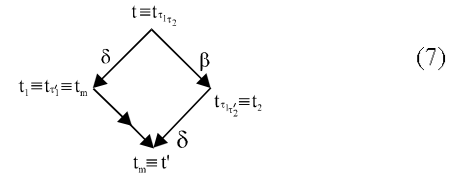
$\tau_1$  and  $\tau_2$  are both  $\beta$ -redexes. From theorem 2 and Eq. 5 follows that  $t' \equiv t'_{\tau_1} \equiv t'_{\tau_2}$ :



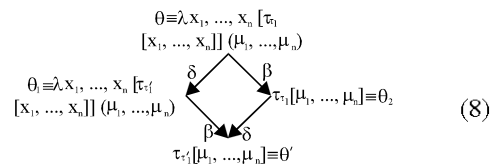
$\tau_1$  is  $\delta$ -redex and  $\tau_2$  is  $\beta$ -redex or  $\tau_1$  is  $\beta$ -redex and  $\tau_2$  is  $\delta$ -redex. Without loss of generality, we suppose that  $\tau_1$  is  $\delta$ -redex and  $\tau_2$  is  $\beta$ -redex. Let,  $\tau_2 \equiv \lambda x_1, \dots, x_n [\tau [x_1, \dots, x_n]] (\mu_1, \dots, \mu_n) \tau \in \Lambda$ ,  $x_i \in \Lambda$ ,  $x_i, \forall \alpha_i, \alpha_i \in \text{Types}$ ,  $i = 1, \dots, n$ . If  $\tau_1 \equiv \tau_{\tau_2}$ , the from Lemma 1 follows Eq. 6:



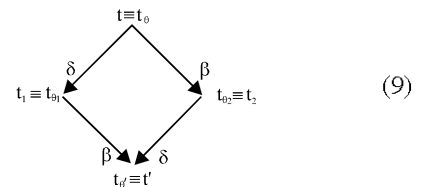
where  $m \in M$ ,  $m \neq \perp$  and  $m$  is the convolution of the term  $\tau_1$ . Therefore, from Eq. 7 follows that  $t' \equiv t'_m$ :



If  $\tau_2 \equiv \lambda x_1, \dots, x_n [\tau_{\tau_1} [x_1, \dots, x_n]] (\mu_1, \dots, \mu_n)$ , then, it is easy to see that if  $\tau_1 [x_1, \dots, x_k] \rightarrow_{\delta} \tau'_1$ , then,  $\tau_1 [\mu_1, \dots, \mu_n] \rightarrow_{\delta} \tau'_1$  and we have:



Therefore, Eq. 9 follows that  $t' \equiv t_{\tau_1 [\mu_1, \dots, \mu_n]}$ :



If  $\lambda x_1, \dots, x_n [\tau [x_1, \dots, x_n]] (\mu_1, \dots, \mu_{i-1}, \dots, \mu_n)$ , then, without loss of generality, we suppose that  $i = 1$  and Eq. 10 and 11 follows that  $t' \equiv t_{\tau [\mu_1, \dots, \mu_n]}$ :

$$\begin{array}{c}
 \theta \equiv \lambda x_1, \dots, x_n [\tau \\
 [x_1, \dots, x_n]] (\mu_{\mu_1}, \dots, \mu_n) \\
 \delta \quad \beta \\
 \theta \equiv \lambda x_1, \dots, x_n [\tau \\
 [x_1, \dots, x_n]] (\mu_{\mu_1}, \dots, \mu_n) \quad \tau[\mu_{\mu_1}, \dots, \mu_n] \equiv \theta_2 \\
 \beta \quad \delta \\
 \tau[\mu_{\mu_1}, \dots, \mu_n] \equiv \theta'
 \end{array} \quad (10)$$

$$\begin{array}{c}
 t \equiv t_0 \\
 \delta \quad \beta \\
 t_1 \equiv t_0 \quad t_2 \equiv t_2 \\
 \delta \quad \beta \\
 t' \equiv t'
 \end{array} \quad (11)$$

In conclusion, we showed that in all cases there exists a term  $t'$  such that  $t_1 \rightarrow t'$  and  $t_2 \rightarrow t'$ .

**Lemma 3:** For every term, the number of inferences to normal forms from that term is finite.

**Proof:** We consider the inference tree of the term  $t$ . Let us suppose that the number of inferences to normal forms from the term  $t$  is infinite which means that the number of paths from root  $t$  to leafs is also infinite. Since, every node in the inference tree has finite number of descendants, then from the Konig's lemma follows that there exists an infinite path that starts from the root  $t$  which contradicts to the theorem 1. Therefore, the number of paths from the root  $t$  to leafs is finite which means that the number of inferences to normal forms from the term  $t$  is also finite.

It follows from lemma 3 that for every term  $t$  the inference tree of the term  $t$  is a finite tree. The height of an inference tree of the term  $t$  is the length of the longest path from the root  $t$  to a leaf.

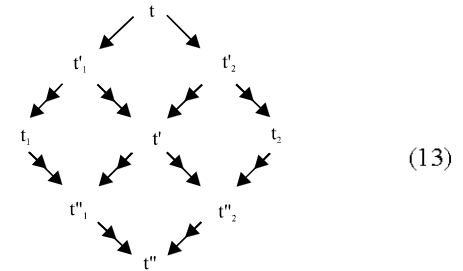
**Definition 6:** The set of all terms the height of the inference tree of which is equal to  $n-1$  is denoted by  $\Lambda^{(n)}$ ,  $n \geq 1$ .

**Lemma 4:** For every term  $t \in \Lambda$ , if  $t \rightarrow t_1$  and  $t \rightarrow t_2$ ,  $t_1, t_2 \in \Lambda$ , then, there exists a term  $t' \in \Lambda$  such that  $t_1 \rightarrow t'$  and  $t_2 \rightarrow t'$ :

$$\begin{array}{c}
 t \\
 \delta \quad \beta \\
 t_1 \quad t_2 \\
 \delta \quad \beta \\
 t'
 \end{array} \quad (12)$$

**Proof:** Let,  $t \in \Lambda^{(0)}$ , then,  $t \in \text{NF}$  and  $t \equiv t_1 \equiv t_2 \equiv t'$ . Now, let us suppose that the lemma 4 holds for every term  $\tau \in \Lambda^{(k)}$ ,

$k \leq n-1$ ,  $n \leq 2$  and show that it holds for every term  $t \in \Lambda^{(n)}$ . If  $t \equiv t_1$ , then,  $t_1 \rightarrow t_2$  and  $t' \equiv t_2$ . If  $t \equiv t_2$  then  $t_2 \rightarrow t_1$  and  $t' \equiv t_1$ . If  $t_1 \neq t$  and  $t_2 \neq t$ , then, there exist terms  $t'_1, t'_2 \in \Lambda$  such that  $t \rightarrow t'_1 \rightarrow t_1$  and  $t \rightarrow t'_2 \rightarrow t_2$ . Therefore, from lemma 2 follows that there exists a term  $t'$  such that  $t'_1 \rightarrow t'$  and  $t'_2 \rightarrow t'$ . Since,  $t'_1 \rightarrow t_1$ ,  $t'_1 \rightarrow t'$ , and  $t'_2 \in \Lambda^{(k)}$ ,  $1 \leq k_1 \leq n-1$ , then, from the induction hypothesis follows that there exists a term  $t''_1$  such that  $t'_1 \rightarrow t''_1$  and  $t''_1 \rightarrow t'_1$ . Since,  $t'_2 \rightarrow t_2$ ,  $t'_2 \rightarrow t'$  and  $t'_2 \in \Lambda^{(k)}$ ,  $1 \leq k_2 \leq n-1$ , then, from the induction hypothesis follows that there exists a term  $t''_2$  such that  $t'_2 \rightarrow t''_2$  and  $t''_2 \rightarrow t'_2$ . Since,  $t''_1 \rightarrow t'_1$ ,  $t''_1 \rightarrow t''_2$  and  $t''_2 \in \Lambda^{(k)}$ ,  $1 \leq k_3 \leq n-1$ , then, from the induction hypothesis follows that there exists a term  $t''$  such that  $t''_1 \rightarrow t''$  and  $t''_2 \rightarrow t''$ . Therefore,  $t_1 \rightarrow t''$  and  $t_2 \rightarrow t''$ :



**Proof of theorem 3:** Let us suppose that the original statement is false and  $t' \neq t''$ . It follows from lemma 4 that there exists a term  $t''' \in \Lambda$  such that  $t' \rightarrow t'''$  and  $t'' \rightarrow t'''$ . Since,  $t', t'' \in \text{NF}$  then  $t' \equiv t'' \equiv t'''$ . Therefore, we have a contradiction and the original statement is true.

### CONCLUSION

In this study, the uniqueness of  $\beta\delta$ -normal form of typed  $\lambda$ -terms for main canonical notion of  $\delta$ -reduction has shown.

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