

## Results for Normal Matrices and Majorization Inequalities for their Eigenvalues

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**Abstract:** In this study, we introduce and study some of the classical results to normal matrix case and give generalization for Lemma 2.5.2 by Horn and Johnson and put special emphasis on majorization inequalities for eigenvalues of normal matrices.

**Key words:** Normal matrices, majorization, doubly stochastic matrices, generalization, eigenvalues, special emphasis

### INTRODUCTION

There are several differences between Hermitian matrices and normal matrices. For example, the sum of two Hermitian matrices is Hermitian and the product of two Hermitian matrices is Hermitian if and only if the matrices commute but the sum (or difference) of two normal matrices is not necessarily normal. The principal submatrices of a normal matrix are not necessarily normal. In this study, we will give generalization of Lemma 2.5.2 by Horn and Johnson (1990) which prove that if  $A \in M_n$ ,  $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$  be partitioned such that the diagonal blocks are square, then, the upper triangular matrix is normal, if and only, if  $A_{11}$  and  $A_{22}$  are normal and  $A_{12} = 0$ , then, we will discuss some majorization inequalities for normal matrices. Marshall and Olkin (1979), Zhang (2011), Knyazev and Argentati (2010), Marshall *et al.* (2011) and Ando (1994). All through this study,  $M_n$  will stand for space of all  $n \times n$  complex matrices. The direct sum of two matrices  $A \in M_m$  and  $B \in M_n$  is a larger block matrix  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ , denoted by  $A \oplus B$ .

#### Preliminaries

**Definition 3.1:** Let,  $A \in M_n$ . Then:

- $A$  is called Hermitian (or self adjoint) if  $A^* = A$
- $A$  is called normal if  $A^*A = AA^*$
- $A$  is called unitary if  $A^*A = AA^* = I_n$  where,  $I_n$  is the identity matrix of order  $n$

**Definition 3.2:** Let,  $x$  and  $y$  be real  $n$ -vectors and let  $x_1 \geq x_2 \geq \dots, \geq x_n$  and  $y_1 \geq y_2 \geq \dots, \geq y_n$  be their coordinates arranged in the decreasing order. We say that,  $x$  is majorized by  $y$  if  $\sum_{j=1}^k x_j \leq \sum_{j=1}^k y_j$  for all  $1 \leq k \leq n$  and  $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$ .

If  $x$  is majorized by  $y$ , we write  $x < y$ . When the last equality condition is not required,  $x$  is said to weakly majorized (or submajorized) by  $y$  and this weak relation is denoted by  $x <_w y$ .

#### Theorem 3.3: Schur's Unitary Triangularization

**Theorem:** Let,  $A \in M_n$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then there is a unitary matrix  $U \in M_n$  such that  $U^*AU = T = [t_{ij}]$  is an upper triangular matrix with diagonal entries  $t_{ii} = \lambda_i$ ,  $i = 1, 2, \dots, n$ .

**Theorem 3.4:** A necessary and sufficient condition that  $x < y$  is that there exists a doubly stochastic matrix  $P$  such that  $x = Py$ . Here, we regard vectors as column vectors, i.e.,  $n \times 1$  matrices.

**Proposition 3.5; Schur:** If  $A \in M_n$  is Hermitian matrix, then,  $\text{Diag}(A) < \lambda(A)$ . Where  $\text{Diag}(A)$  is the diagonal part of a square matrix  $A$ .

**Proposition 3.6; Fan:** If  $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in M_{m+n}$ , then,  $\lambda(A \oplus B) < \lambda \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ . Where,  $H$  is Hermitian matrix.

**Proposition 3.7; Fan:** Let,  $A, B \in M_n$  are Hermitian matrices, then,  $\lambda(A+B) < \lambda(A) + \lambda(B)$ .

### MAJORIZATION AND NORMAL MATRICES

**Proposition 4.1:** If  $A_1, A_2 \in M_n$  are such that,  $A_1^*A_1 + A_2^*A_2 = A_1A_1^* + A_2A_2^* = I$ . Then,  $A_1^*A_2$  is normal.

**Proof:** Since,  $A_1A_1^* + A_2A_2^* = I$ , we will multiply pre and post of this equation by  $A_1^*$  and  $A_1$ , respectively, we get:

$$(A_1^*A_1)^2 + A_1^*A_2A_2^*A_1 = A_1^*A_1$$

Then:

$$(A_1^*A_1)^2 - A_1^*A_1 = -A_1^*A_2A_2^*A_1$$

Now, we will multiply pre and post of this equation  $A_1A_1^* + A_2A_2^* = I$  by  $A_2^*$  and  $A_2$ , respectively, we get:

$$A_2^*A_1A_1^*A_2 + (A_2^*A_2)^2 = A_2^*A_2$$

Then:

$$(A_2^* A_2)^2 - A_2^* A_2 = -A_2^* A_1 A_1^* A_2$$

Pre multiplying of this equation  $A_1^* A_1 + A_2^* A_2 = I$  by  $A_1^* A_1$ , we get:

$$(A_1^* A_1)^2 - A_1^* A_1 = -A_1^* A_1 A_2^* A_2$$

Post multiplying of this equation  $A_1^* A_1 + A_2^* A_2 = I$  by  $A_2^* A_2$ , we get:

$$(A_2^* A_2)^2 - A_2^* A_2 = -A_1^* A_1 A_2^* A_2$$

Thus,  $A_2^* A_1 A_1^* A_2 = A_1^* A_2 A_2^* A_1$ , i.e.,  $A_1^* A_2$  is normal.

**Proposition 4.2:** Let,  $A \in M_n$ , then, the matrix  $X = \begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}$  is normal.

**Proof:** We get the proof from comparing  $X^* X$  and  $XX^*$ . Thus, any square matrix can be a principal submatrix of a normal matrix. Now, we will give the generalization for Lemma 2.5.2 by Horn and Johnson (1990).

**Proposition 4.3:** Let  $A$  be a normal matrix and  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

be partitioned such that the diagonal blocks are square, then,  $A_{11}$  (resp.  $A_{22}$ ) is normal, if and only if  $A_{12} A_{12}^* = A_{21}^* A_{21}$  (resp.  $A_{21} A_{21}^* = A_{12}^* A_{12}$ ).

**Proof:** We get the proof from comparing:

$$A^* A = \begin{pmatrix} A_{11}^* A_{11} + A_{21}^* A_{21} & A_{11}^* A_{12} + A_{21}^* A_{22} \\ A_{12}^* A_{11} + A_{22}^* A_{21} & A_{12}^* A_{12} + A_{22}^* A_{22} \end{pmatrix}$$

And:

$$A A^* = \begin{pmatrix} A_{11} A_{11}^* + A_{12} A_{12}^* & A_{11} A_{21}^* + A_{12} A_{22}^* \\ A_{21} A_{11}^* + A_{22} A_{12}^* & A_{21} A_{21}^* + A_{22} A_{22}^* \end{pmatrix}$$

Note that  $A_{12} A_{12}^* = A_{21}^* A_{21}$  is not always true for normal matrices, for example: Let:

$$A_{11} = A_{22} = A_{12}^* = A_{21}^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then,  $A_{12} A_{12}^* \neq A_{21}^* A_{21}$ .

**Lemma 4.4:** Let,  $A \in M_n$ , then,  $\text{Re } \lambda(A) < \lambda(\text{Re } A)$ .

**Proof:** We know from Theorem 3.3 that there is a unitary matrix  $U$  such that  $U^* A U = T$ , where,  $T$  is an upper Triangle matrix.

So, we will use the fact that real part of the eigenvalues of  $T$  coincide with the diagonal entries of  $T + T^*/2$ . Thus,  $\text{Re } \lambda(A) = \text{Re } \lambda(T) < \lambda(T + T^*/2)$  (by using Proposition 3.5)  $= \lambda(A + A^*/2) = \lambda(\text{Re } A)$ .

**Proposition 4.5:** Let  $A, B \in M_n$  be normal matrices, then,  $\text{Re } \lambda(A+B) < \text{Re } \lambda(A) + \text{Re } \lambda(B)$ .

**Proof:**  $\text{Re } \lambda(A+B) < \lambda(\text{Re}(A+B))$  (by Lemma 4.4)  $= \lambda(\text{Re } A + \text{Re } B) < \lambda(\text{Re}(A)) + \lambda(\text{Re}(B))$  (by Proposition 3.7)  $= \text{Re } \lambda(A) + \text{Re } \lambda(B)$ .

Note that Proposition 4.5 shows that Proposition 3.7 can be extended to the case of normal matrices. Note that, if  $N$  is a normal matrix, then,  $\text{Re } \lambda(N) = \lambda(\text{Re } N)$ .

**Proposition 4.6:** Let,  $N$  be a normal matrix and  $N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$  be partitioned such that the diagonal blocks are square, then,  $\text{Re } \lambda(N_{11} \oplus N_{22}) < \text{Re } \lambda(N)$ .

**Proof:**  $\text{Re } \lambda(N_{11} \oplus N_{22}) < \lambda(\text{Re } (N_{11} \oplus N_{22}))$  (by Lemma 4.4)  $= \lambda(\text{Re } N_{11} \oplus \text{Re } N_{22})$ :

$$< \lambda \left( \begin{pmatrix} \text{Re } N_{11} & \frac{(N_{12} + N_{21}^*)}{2} \\ \frac{(N_{21} + N_{12}^*)}{2} & \text{Re } N_{22} \end{pmatrix} \right)$$

by Proposition 3.6  $= \lambda(\text{Re } N) = \text{Re } \lambda(N)$ .

## CONCLUSION

This study will give the generalization for Lemma 2.5.2 by Horn and Johnson (1990).

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