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# On Cyclic Triple System and Factorization

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**Abstract:** A near-k-factor of a graph G is a spanning subgraph in which exactly one isolated vertex and all other vertices of order k. In this study, we employ the near-four-factor concept and $(m_1, m_2, \dots, m_r)$-cycle system to present a new method for constructing a cyclic 12-fold triple system. Firstly, we would like to propose a new type of cyclic triple system called cyclic triple near factorization, denoted by CTNF(ν). Then, we prove the existence of CTNF(ν) along with an algorithm for starter triples of CTNF(ν) for ν = 12n+2 when, n is even. Finally, we use the construction of CTNF(ν) to demonstrate the existence of [a, b] factorization of $12 K_\nu$ for a a = 8 and b = 4 (ν-1).

**Key words:** Triple system, near-k-factor, complete multigraph, starter triples, demonstrate, factorization

## INTRODUCTION

All graphs under consideration are finite, undirected and without loop. An m-cycle, written $C_m = (c_0, \dots, c_{m-1})$, consists of m distinct vertices $\{c_1, c_2, \dots, c_m\}$ and m edges $\{c_ic_{i+1}\}$, $0 \le i \le m-2$ and $\{c_0c_{m-1}\}$. An $(m_1, m_2, \dots, m_r)$-cycle system of graph G is a pair (V, C) where, V is the vertex set of G and C is the collection of $\{C_{mi}, 1 \le i \le r\}$ for whose edge sets partition of the edge set of G and is called to be a cyclic if $V = Z_\nu$ and we have $C_{mi}+1 \in C$ whenever $C_{mi} \in C$. In particular, if $m_1 = \dots = m_r = m$, it is called cyclic m-cycle system (Wu and Lee, 2008).

Let $d_G(\nu)$ be a degree of ν in G. A graph G is called [a, b]-graph if $a \le d_G(\nu) \le b$ for every $\nu \in V(G)$ and $b \ge a \ge 0$. An [a, b]-factor of G is a spanning [a, b]-subgraph and [k, k]-factor is abbreviated to an k-factor. While a near-k-factor is a spanning subgraph in which all but one vertex has a degree k with the remaining vertex having degree 0 that is said isolated vertex. The decomposition of edge set of G into [a, b]-factors (respectively, near-k- factor) is called a [a, b]-factorization (respectively, near-k- factorization) (Mao-Cheng, 1991).

A balanced incomplete block design, denoted by (ν, k, λ)-BIBD is a pair (V, B) where, V is a finite set of ν points and B is a collection of k-subsets of V such that each pair of distinct points of V is contained in precisely λ blocks. It easy to see that any (ν, k, λ)-BIBD can be viewed as a decomposition of complete multigraph $\lambda K_\nu$, the graph with ν vertices in which every two vertices are joined by λ parallel edges into copies of $K_k$. An automorphism of BIBD (V, B) is a permutation τ on V such that $B = \{b_1, \dots, b_k\} \in B$ if and only if $\tau(B) = \{\tau b_1, \dots, \tau b_k\} \in B$. If there is an automorphism τ of order ν, then, BIBD is called cyclic. Thus, the automorphism will be represented by τ: (0, 1, …, ν-1). The orbit of the block B, denoted by orb (B) is the set of all distinct blocks in the collection $\{B+i | i \in Z_\nu\}$. The orbit of block B is said to be a full if orb (B) = ν and otherwise is said a short (Mathon, 1987). A cyclic λ-fold triple system of order ν, denoted by CTS (ν, λ) is cyclic (ν, 3, λ)-BIBD. A CTS(ν, λ) is simple, if it contains no repeated triple. When $\nu \neq 0 \pmod 3$ then, there is no short orbit of block (Tian and Wei, 2013).

The existence of CTS (ν, λ) for ν ≡ 1, 3 (mod 6) have been studied by Colbourn and Colbourn (1981). While Colbourn and Rosa (1999) have given the necessary conditions for the existence of CTS (ν, λ). The existence of CTS (ν, λ) for any possible parameters ν and λ is an interesting problem, since, this kind of design has a nice combinatorics and algebraic properties and also has a connection to the optical orthogonal codes (Chen and Wei, 2012).

Motivated by construction these designs, Tian and Wei (2010) provided a new method for decomposing all the triples of $Z_\nu$ into simple CTS(ν, $\lambda_i$) where, $1 \le \lambda_i < \nu-2$, i = 1, 2, …, n for odd cases ν ≡ 1, 3, 5 (mod 6). They defined the large set of cyclic triple systems to be a decomposition of all triples of $Z_\nu$ into indecomposable cyclic triple systems. On the other hand, the near-one-factorization concept was employed to build up a simple 3-fold triple system for the odd cases

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(Ibrahim, 2006). Moreover, Matarneh and Ibrahim (2014) utilized the near-two-factorization of a complete multigraph to introduce a CTS (ν, 6), called array cyclic design.

In this study, we propose a direct method to construct a new type of CTS (ν, 12) with some conditions for ν = 12n+2 when n is even. This design is called cyclic triple near factorization (briefly CTNF(ν)). Then, the construction of CTNF(ν) will be used to prove the existence of [a, b]-factorization of 12 $K_v$.

## MATERIALS AND METHODS

**Definitions and preliminaries:** Throughout of this study whenever we say $\lambda K_v$, (or m-cycle), we understand that their vertices are in $Z_v$ with even order and $Z^*_v = Z_v - \{0\}$. In this study, we will obtain the main results by using the difference set method. That already detected to be successful in several cases for the construction of the cyclic $(m_1, m_2, \ldots, m_r)$-cycle system and cyclic triple.

For $\{a, b\} \subset Z_v$ and $a \neq b$, the difference d of a pair {a, b} is defined d = min {|a-b|, ν-|a-b|}, hence, $1 \leq d \leq \nu/2$. Let, B is a k-subset of $Z_v$, the differences of B is the multiset D(B) = {min{|a-b|, ν-|a-b|}, a≠b∈B. In general, the list of differences of multiset A = $\{B_1, B_2, \ldots, B_r\}$ of k-subsets of $Z_v$ is defined as D(A) = $D(B_1) \cup D(B_2) \cup, \ldots, \cup D(B_r)$ such that the union must be understood between multisets (elements must be counted with their respective, multiplicities) (Tian and Wei, 2013).

**Definition 1; Abel and Buratti (2006):** Let, A be a multiset of k-subsets of $Z_v$. An A is a (ν, k, λ)-difference system if D(A) covers each element of $Z^*_{v+2/2}$ exactly λ times except for the middle difference (ν/2) appears λ/2 times.

**Theorem 1; Abel and Buratti (2006):** Let, A be a multiset of k-subsets of $Z_v$. Then A is a starter of cyclic (ν, k, λ)-BIBD if and only if A is the (ν, k, λ)-difference system.

**Definition 2; Tian and Wei (2010):** A (ν, k, λ)-BIBD (V, B) is indecomposable if there does not exist $\lambda_1 < \lambda$ such that $(V, B_1)$ is $(\nu, k, \lambda_1)$-BIBD and $B_1 \subset B$.

**Lemma 1; Colbourn and Rosa (1999):** Let, ν ≡ 2 (mod 4), then, the necessary condition for the existence CTS(ν, λ) is λ≡ 0 (mod 12).

**Definition 3; Alqadri and Ibrahim (2017):** A $(m^*_1, m^*_2, \ldots, m^*_r)$-cycle system of G is a decomposition of G into cycles that have the lengths $\{m_1, m_2, \ldots, m_r\}$.

**Definition 4; Alqadri and Ibrahim (2017):** A set of cycles that generates all cycles of a cyclic $(m^*_1, m^*_2, \ldots, m^*_r)$-cycle system of G by repeated addition of 1 modular ν which is called a starter set.

The superscript notation shall be used to describe a starter set of cyclic design. Therefore, $\alpha = \{C^{n_1}_{m_1}, C^{n_2}_{m_2}, C^{n_r}_{m_r}\}$ means that there are $n_1$ cycles of length $m_1$, $n_2$ cycles of length $m_2$, etc. as well as we consider that $C_{m_i}$ be the ith m-cycle in starter set δ (Tian and Wei, 2010).

**Definition 5; Alqadri and Ibrahim (2017):** A differences list from the cycle graph $C_m = (c_0, \ldots, c_{m-1})$ is the multiset $D(C_m) = \{\min\{|c_i - c_{i-1}|, \nu - |c_i - c_{i-1}|\}, i = 1, 2, \ldots, m\}$ where, $c_0 = c_m$.

**Theorem 2:** Let, $\delta = \{C_{m_1}, C_{m_2}, \ldots, C_{m_r}\}$ be a set of cycles. A δ is a starter of $(m_1, m_2, \ldots, m_r)$-cycle system $\lambda K_v$ if and only if $D(\delta) = \cup^r_{i=1} D(C_{m_i})$ covers each element of $Z^*_{v+2/2}$ exactly λ times except for the middle difference (ν/2) appears λ/2 times.

## RESULTS AND DISCUSSION

**Cyclic triple near factorization:** In this study, we define a new concepts as a base towards constructing a new types of cyclic 12-fold triple system and [a, b]-factorization.

**Definition 6:** A cyclic $(m^*_1, m^*_2, \ldots, m^*_r)$-cycle factorization of $\lambda K_v$ is a cyclic $(m^*_1, m^*_2, \ldots, m^*_r)$-cycle system in which its starter (briefly α) satisfies a near-four-factor denoted by CCF($\lambda K_v$, α).

**Definition 7:** A cyclic triple factorization of order ν, denoted by CTNF(ν) is a ν×2(ν-1) array that satisfies the following conditions:

- The entries in row r form a near-four-factor with focus r
- The triples associated with any row contain no repetitions

The cyclic $(m_1^*, m_2^*, \ldots, m_r^*)$-cycle factorization of 4 $K_v$ will be employed to construct CTNF(ν). Example 1 explains the construction method of CTNF(ν) for ν = 26.

**Example 1:** Suppose G = 4 $K_{26}$ and α = $\{C_4^6, C_7^2, C_6^2\}$ is a cycles set of G such that:

Table 1: The list of differences of 4-cycles in $\alpha = \{C_4^6, C_7^2, C_6^2\}$

| Cycles | Difference set |
|---|---|
| (1, 25, 14, 12) | {2, 24, 11, 15, 2, 24, 11, 15} |
| (2, 24, 15, 11) | {4, 22, 9, 17, 4, 22, 9, 17} |
| (3, 23, 16, 10) | {6, 20, 7, 19, 6, 20, 7, 19} |
| (4, 22, 17, 9) | {8, 18, 5, 21, 8, 18, 5, 21} |
| (5, 21, 18, 8) | {10, 16, 3, 23, 10, 16, 3, 23} |
| (6, 7, 20, 19) | {1, 25, 13, 13, 1, 25, 13, 13} |

Table 2: The list of differences of 7 and 6-cycles in $\alpha = \{C^6_4, C^2_7, C^2_6\}$

| Cycles | Difference set |
|---|---|
| (13, 2, 12, 3, 11, 4, 10) | {11, 15, 10, 16, 9, 17, 8, 18, 7, 19, 6, 20, 3, 23} |
| (13, 24, 14, 23, 15, 22, 16) | {11, 15, 10, 16, 9, 17, 8, 18, 7, 19, 6, 20, 3, 23} |
| (6, 1, 5, 17, 19, 18) | {5, 21, 4, 22, 12, 14, 2, 24, 1, 25, 12, 14} |
| (20, 25, 21, 9, 7, 8) | {5, 21, 4, 22, 12, 14, 2, 24, 1, 25, 12, 14} |

Table 3: Construction of CTNF (26)

| $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | ... | $C_{49}$ | $C_{50}$ | $C_{51}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1, 25 | 25, 14 | 14, 12 | 12, 1 | ... | 9, 7 | 7, 8 | 8, 20 |
| 1 | 2, 0 | 0, 15 | 15, 13 | 13, 2 | ... | 10, 8 | 8, 9 | 9, 21 |
| 2 | 3, 1 | 1, 16 | 16, 14 | 14, 3 | ... | 11, 9 | 9, 10 | 10, 22 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 24 | 25, 23 | 23, 12 | 12, 10 | 10, 25 | ... | 7, 5 | 5, 6 | 6, 18 |
| 25 | 0, 24 | 24, 13 | 13, 11 | 11, 0 | ... | 8, 6 | 6, 7 | 7, 19 |

$$C_{4_1} = (1, 25, 14, 12), C_{4_2} = (2, 24, 15, 11), C_{4_3} = (3, 23, 16, 10)$$
$$C_{4_4} = (4, 22, 17, 9), C_{4_5} = (5, 21, 18, 8), C_{4_6} = (6, 7, 20, 19)$$
$$C_7^* = (13, 2, 12, 3, 11, 4, 10), C_7^{**} = (13, 24, 14, 23, 15, 22, 16)$$
$$C_6^* = (6, 1, 5, 17, 19, 18), C_6^{**} = (20, 25, 21, 9, 7, 8)$$

An easy verification shows that each nonzero integers in $Z_{26}$ appears twice in the cycles of α. Since, a cycle graph is the 2-regular graph, then each vertex in $Z_{26}$-{0} has a 4°. Hence, the cycles of α satisfy a near-four-factor with zero isolated. Furthermore, the differences lists of the cycles in α are listed in Table 1 and 2.

As can be seen from Table 1 and 2, the differences list of α, D(α), covers each integer in $Z^*_{13}$ exactly four times and the middle difference 13 occurs twice. Based on theorem 2, the set of cycles α is a starter of a cyclic ($4^*$, $7^*$, $6^*$)-cycle factorization of $4K_{26}$, CCF(4 $K_{26}$, α). Therefore, the construction of CCF ($4K_{26}$, α) can be viewed as (26×10) array in which $\alpha = \{C_4^6, C_7^2, C_6^2\}$ generates all of its cycles by repeated addition of 1 modular (26).

In order to form CTNF (26), we place the zero element, isolated vertex in the first column then we split the edges of the cycles in each row of CCF($4K_{26}$, α) into separated edges by setting each edge in a specific column. Here, we have 26 rows and 50 columns with a column that has an isolated vertex as illustrated in Table 3.

To form triples, we append $C_1$ with $C_i$ for $2 \le i \le 52$. For example, in the first row, we have {0, 1, 25}, {0, 25, 14}, {0, 14, 12} and by continuing in the same fashion in the remaining rows, we will generate the desired design.

In fact, the construction of CCF($4K_v$, α) is the fundamental question of the existence of CTNF(v). To address this question, Alqadri and Ibrahim (2017) provided the starter of CCF($4K_v$, α). They constructed the m-cycles for m>4 by connected paths as $C_m = (v_0, P_{2n}, P_{2r})$ such that $v_0$ is a vertex and $\{P_{2n}, P_{2r}\}$ are a paths of even order. For even paths $\{P_{2n}, P_{2r}\}$ in $C_m$, it can be written as:

$$P_{2n} = [a_1, b_1, a_2, b_2, ..., a_n, b_n] = [\cup_{i=1}^{n} a_i, b_i]$$

$$P_{2r} = [c_1, d_1, c_2, d_2, ..., c_r, d_r] = [\cup_{i=1}^{r} c_i, d_i]$$

where, the edge set of $P_{2n}$ will be represented as:

$$E(P_{2n}) = \begin{cases} \{a_i, b_i\}, & 1 \le i \le n, \\ \{a_{i+1}, b_i\}, & 1 \le i \le n\text{-}1 \end{cases}$$

$$E(P_{2r}) = \begin{cases} \{c_i, d_i\}, & 1 \le i \le r, \\ \{c_{i+1}, d_i\}, & 1 \le i \le r\text{-}1 \end{cases}$$

Thus, the edge set of $C_m = (v_0, P_{2n}, P_{2r})$ will expressed as follows:

$$E(C_m) = E(P_{2n}) \cup E(P_{2r}) \cup \{v_0, a_1\} \cup \{b_n, c_1\} \cup \{v_0, d_r\}$$

The starter set of CCF ($4K_v$, α) is shown in construction I below.

**Construction I:** Let, n be an even integer, n>2. Suppose $\alpha = \{C_4^{3n}, C_{4n-1}^{2}, C_{2n+2}^{2}\}$ is a cycles set of $4K_{12n+2}$ where the list of 4-cycles is:

$$C_{4_i} = (i, 12n+2\text{-}i, 6n+1+i, 6n+1\text{-}i), 1 \le i \le 3n, i \ne \frac{5n+4}{2}$$

when i = 5n+4/2, let:

$$C_{4_i} = \left(\frac{5n+4}{2}, 6n+1\text{-}\frac{5n+4}{2}, 12n+2\text{-}\frac{5n+4}{2}, 6n+1+\frac{5n+4}{2}\right)$$

Whereas $C_{4n-1}^{*} = (4n+2, P_{4n-2}^{*})$ and $C_{4n-1}^{**} = (8n, P_{4n-2}^{**})$ are (4n-1)-cycles in which the paths $\{P_{4n-2}^{*}, P_{4n-2}^{**}\}$ are represented as follows:

$$P_{4n-2}^{*} = [6n+1, 2, 6n, 3, ..., 4n+3, 2n] = [\cup_{i=1}^{2n-1} 6n+2-i, i+1]$$
$$P_{4n-2}^{**} = [6n+1, 12n, 6n+2, 12n\text{-}1, ..., 8n\text{-}1, 10n+2] = [\cup_{i=1}^{2n-1} 6n+i, 12n+1\text{-}i]$$

Furthermore, $C_{2n+2}^{*}$ = (9n, $P_2^{*}$, $P_{2n-2}^{*}$) and $C_{2n+2}^{**}$ = (3n+2, $P_3^{**}$, $P_{2n-2}^{**}$) are (2n+2)-cycles such that the paths {$P^{*}_{3}$, $P^{*}_{2n-2}$, $P^{**}_{3}$, $P^{**}_{2n-2}$} are represented as follows:

$$P_3^{*} = [2n+2, 1, 2n+1]$$

$$P_{2n-2}^{**} = [8n+1, 10n-1, 8n+2, 10n-2, ..., 9n-1, 9n+1] = \left[\cup_{i=1}^{n-1} 8n+i, 10n-i\right]$$

$$P_3^{**} = [10n, 12n+1, 10n+1]$$

$$P_{2n-2}^{**} = [4n+1, 2n+3, 4n, 2n+4, ..., 3n+3, 3n+1] \left[\cup_{i=1}^{n-1} 4n+2-i, 2n+2+i\right]$$

We will use the cycles in construction I to prove the next theorem and also to formulate an algorithm for a starter of 12-fold triple system.

**Theorem 3:** There exists a cyclic triple factorization of order 12n+2 for n ≡ 0 (mod 2).

**Proof:** We split the proof into two cases as follows:

**Case 1:** n = 2, see example 1.

**Case 2:** For n>2. There exists CCF ($4K_v$, α) for ν = 12n+2. Then, the construction of CCF ($4K_{12n+2}$, α) can be represented as (ν×|α|) array in which the cycles in each row satisfy the near-four-factor. For producing the CTNF (12n+2), we need to have 12n+2 rows and 2 (12n+1) columns and a column with an isolated vertex. In the construction of CCF ($4K_{12n+2}$, α), we place the isolated vertex in the first column and then partition the edge set of the cycles in each row into separated edges by setting every edge in a column. On the other hand, the number of columns equal to the edges cycles of α = {$C_4^{3n}$, $C_{4n-1}^{2}$, $C_{2n+2}^{2}$} which is calculated as follows:

$$4\times(3n)+2\times(4n-1)+2\times(2n+2) = 2(12n+1)$$

In order to form triples, appending the first column with each other columns. Since, there are no identical edges in each row, then all the associated triples in each row will be distinct. The triples in the first row in the construction of CTNF (12n+2) is considered a starter triples (briefly A).

**Theorem 4:** For ν = 12n+2 where, n ≡ 0 (mod 2), there exists a cyclic 12-fold triple system has cyclic triple near factorization of order ν.

**Proof:** In order to prove this theorem, we need to calculate the differences list of the starter triples A of CTNF(12n+2). Depending on theorem 3, all triples in A have formed as {0, $e_i$}, 1 ≤i≤2 (12n+1) where $e_i$ = {$c_{1,i}$, $c_{2,i}$} is the edge set of α. Then, the differences list of A is calculated as follows:

$$D(A) = \cup \left\{ \begin{matrix} d(e_i) \cup d(0, c_{j,i}), 1 \le i \le \\ 2(12n+1) \text{ and } 1 \le j \le 2 \end{matrix} \right\} \quad (1)$$

Indeed, the cycles of α in the construction I is the starter of cyclic (4*, (4n-1)*, (2n+2)*)-cycle system of $4K_{12n+2}$ (Alqadri and Ibrahim, 2017). Based on theorem 2 $D(\alpha) = \cup_{i=1}^{2(12n+1)} d(e_i)$ covers each nonzero element of $Z_{6n+1}$ four times and the middle difference {6n+1} occurs twice.

Furthermore, α satisfies the near-four-factor of $4K_{12n+2}$ with isolated zero integer, i.e., {1, 2, …, ν/2, (ν/2+1), (ν/2+2), …, 12n+1} appear four times as end points of the edge set of α. Hence, for any end point $c_{j,i}$ of an edge $e_i$, 1 ≤i≤2 (12n+1)and 1 ≤j≤2, we find out:

$$d\{0, c_{j,i}\} = \min\{c_{j,i}, \nu\text{-}c_{j,i}\} = \begin{cases} c_{j,i} & c_{j,i} \ge \frac{\nu}{2} \\ \nu\text{-}c_{j,i} & c_{j,i} < \frac{\nu}{2} \end{cases}$$

Then, every integer of {1, 2, …, ν/2, (ν/2-1), (ν/2-2), ..., 2, 1} occur four times in $\cup_{i,j\in I} d(0, c_{j,i})$ and this means $\cup_{i,j\in I} d(0, c_{j,i})$ contained all integers of $Z^{*}_{6n+1}$ 8 times and the integer {6n+1} four times. From Eq. 1, we conclude that D(A) covers each nonzero element of $Z_{6n+1}$ 12 times and the middle difference {6n+1} occurs 6 times. Therefore, A is a starter of cyclic 12-fold triple system based on theorem 1.

**Lemma 2:** A CTNF(ν) is indecomposable (ν, 3, 12)-BIBD for ν = 12n+2.

**Proof:** Based on lemma 1, the smallest λ for the existence of λ-fold triple system of order ν = 12n+2 is 12. Then, there is no λ<12 such that CTS(ν, λ) exists.

**Algorithm of starter triples of CTNF(12n+2):** In this study, we use the starter cycles of CCF($4K_{12n+2}$, α) to construct and formulate the algorithm of starter triples A of CTNF(12n+2). The process of formulating an algorithm for the starter triples A will be split into two cases depending on n = 2 or n>2.

**Case 1:** For n = 2. By virtue example 1, it follows that a starter triples of CTNF (26) is:

$$A = A_1 \cup A_2$$

Such that:

$$A_1 = \begin{cases} \{0, i, 26\text{-}i\} \cup \{0, 26\text{-}i, 13+i\} & 1 \le i \le 5 \\ \{0, 13+i, 13\text{-}i\} \cup \{0, 13\text{-}i, i\} & 1 \le i \le 5 \\ \{0, 14\text{-}i, i+1\} \cup \{0, 13\text{-}i, i+1\} & 1 \le i \le 3 \\ \{0, 12+i, 25\text{-}i\} \cup \{0, 13+i, 25\text{-}i\} & 1 \le i \le 3 \end{cases}$$

$$A_2 = \begin{Bmatrix} \{0, 6, 7\}, \{0, 7, 20\}, \{0, 20, 19\}, \\ \{0, 19, 6\}, \{0, 13, 10\}, \{0, 13, 16\} \end{Bmatrix}$$

**Case 2:** n = 0 (mod 2) where, n≥4. The generated triples by appending zero integer isolated vertex, to each edge in the edge set of the cycles in the construction I are formed as subsets. We begin with the 4-cycles ($C_4^{3n}$) as follows:

$$S_1 = \begin{Bmatrix} \left\{0, \frac{5n+4}{2}, \frac{7n-2}{2}\right\}, \left\{0, \frac{7n+2}{2}, \frac{19n}{2}\right\}, \\ \left\{0, \frac{19n}{2}, \frac{17n-6}{2}\right\}, \left\{0, \frac{17n+6}{2}, \frac{5n-4}{2}\right\} \end{Bmatrix}$$

$$S_2 = \left\{\{0, i, 12n+2\text{-}i\} \quad 1 \le i \le 3n, i \ne \frac{5n+4}{2}\right\}$$

$$S_3 = \left\{\{0, 12n+2\text{-}i, 6n+1+i\} \quad 1 \le i \le 3n, i \ne \frac{5n+4}{2}\right\}$$

$$S_4 = \left\{\{0, 6n+1\text{-}i, 6n+1+i\} \quad 1 \le i \le 3n, i \ne \frac{5n+4}{2}\right\}$$

$$S_5 = \left\{\{0, i, 6n+1\text{-}i\} \quad 1 \le i \le 3n, i \ne \frac{5n+4}{2}\right\}$$

According to, the edge set of (4n-1) -cycles, the list of generated triples from $\{C_{4n-1}^*, C_{4n-1}^*\}$ will be as follows:

$$S_6 = \{\{0, i+1, 6n+2\text{-}i\} \quad 1 \le i \le 2n\text{-}1\}$$

$$S_7 = \{\{0, i+1, 6n+1\text{-}i\} \quad 1 \le i \le 2n\text{-}2\}$$

$$S_8 = \{\{0, 12n+1\text{-}i, 6n+1+i\} \quad 1 \le i \le 2n\text{-}2\}$$

$$S_9 = \{\{0, 12n+1\text{-}i\ 6n+i\} \quad 1 \le i \le 2n\text{-}1\}$$

$$S_{10} = \begin{Bmatrix} \{0, 4n+2, 2n\}, \{0, 4n+2, 6n+1\}, \\ \{0, 10n+2, 8n\}, \{0, 8n+6n+1\} \end{Bmatrix}$$

Meanwhile, the following subsets represent the produced triples from the cycles $\{C_{2n+2}^*, C_{2n+2}^{**}\}$:

$$S_{11} = \{\{0, 10n\text{-}i, 8n+i\} \quad 1 \le i \le n\text{-}1\}$$

$$S_{12} = \{\{0, 10n\text{-}i, 8n+1+i\} \quad 1 \le i \le n\text{-}2\}$$

$$S_{13} = \{\{0, 2n+2+i, 4n+2\text{-}i\} \quad 1 \le i \le n\text{-}1\}$$

$$S_{14} = \{\{0, 2n+2+i, 4n+1\text{-}i\} \quad 1 \le i \le n\text{-}2\}$$

$$S_{15} = \begin{Bmatrix} \{0, 9n+1, 9n\}, \{0, 9n, 2n+2\}, \{0, 2n+2, 1\}, \{0, 1, 2n+1\}, \\ \{0, 2n+1, 8n+1\}, \{0, 3n, 3n+2\}, \{0, 3n2, 10n\}, \\ \{0, 10n, 12n+1\}, \{0, 12n+1, 10+1\}, \{0, 10n+1, 4n+1\} \end{Bmatrix}$$

For simplicity, we will combine the subsets together which have a relationship between their triples. As a result, the algorithm of the starter triples A of CTNF (12n+2) can be formulated as:

$$A = A_1 \cup A_2$$

Such that $A_1$ and $A_2$ are computed below:

$$A_1 = \begin{cases} \{0, i, 12n+2\text{-}i\} & 1 \le i \le 6n, \text{ if } i \notin \left\{\frac{5n+4}{2}, \frac{7n-2}{2}\right\} \\ \{0, 12n+2\text{-}i, 6n+1+i\} \cup \{0, 6n+1\text{-}i, i\} & 1 \le i \le 3n, \text{ if } i \notin \left\{\frac{5n+4}{2}\right\} \\ \{0, 6n+i, 12n+1\text{-}i\} & 1 \le i \le 3n\text{-}1, \text{ if } i \notin \{2n, 2n+1\} \\ \{0, 6n+2\text{-}i, i+1\} & 1 \le i \le 3n\text{-}1, \text{ if } i \notin \{2n, 2n+1\} \\ \{0, 6n+1\text{-}i, i+1\} \cup \{0, 6n+1+i, 12n+1\text{-}i\} & 1 \le i \le 2n\text{-}2 \\ \{0, 8n+i, 10n\text{-}i\} \cup \{0, 4n+2\text{-}i, 2n+2+i\} & 1 \le i \le 2n\text{-}1 \end{cases}$$

$$A_2 = \Big\{\left\{0, \frac{5n+4}{2}, \frac{7n-2}{2}\right\}, \left\{0, \frac{7n-2}{2}, \frac{19n}{2}\right\}, \left\{0, \frac{19n}{2}, \frac{17n+6}{2}\right\}, \left\{0, \frac{17n+6}{2}, \frac{5n+4}{2}\right\}, \{0, 4n+2, 2n\}$$
$$\{0, 4n+2, 6n+1\}, \{0, 8n, 6n+1\}, \{0, 8n+10n+2\}, \{0, 9n+1, 9n\}, \{0, 9n, 2n+2\}, \{0, 2n+2, 1\},$$
$$\{0, 1, 2n+2\}, \{0, 2n+1, 8n+1\}, \{0, 3n, 3n+2\}, \{0, 3n+2, 10n\}, \{0, 10n, 12n+1\},$$
$$\{0, 12n+1, 10n+1\}, \{0, 10n+1, 4n+1\}\Big\}$$

**Theorem 5:** There exist [8, 4 (ν-1)]-factorization of $12K_v$ for ν = 12n+2, n ≡ 0 (mod 2).

**Proof:** From the construction of CTNF(12n+2), it can be viewed the triples in CTNF(12n+2) as ν×2 (ν-1) array in which the vertex r is included in 2(ν-1) distinct triples and each another vertex in $Z_{12n+2}$ occurs in exactly four triples in row r. As pointed out in the introduction, the (ν, 3, λ)-BIBD can be regarded as a decomposition of $\lambda K_v$ into copies of $K_3$. Since, any vertex in $K_3$ has a degree two, then any vertex has a degree either 8 or 4 (ν-2) in every row r for 0≤r≤-1. So, we can consider each row $r_i$ in CTNF(12n+2) is [8,4 (ν-1)]-factor of 12K where, the element $\gamma_i$ has a degree 4 (ν-1) and another vertex has a 8°.

## CONCLUSION

In this study, we have defined a new method to construct a new cyclic 12-fold triple system. We employed a cyclic ($m^*_1$, $m^*_2$, …, $m^*_r$)-cycle system and near-four-factorization of $4K_v$ to construct cyclic triple factorization, CTNF(ν). We have also, proven the existence of CTNF(ν) for ν = 12+2 when n is even besides an algorithm for its starter triples. Therefore, we used the related results of CTNF(ν) to demonstrate the existence of [8, 2 (ν-1)]-factorization of $12K_v$. We expect the construction of CTNF(ν) is a simple design and can be extended for ν = 12+2 when n is odd and also for ν ≡ 6, 10 (mod 12).

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