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On Chromatic Polynomial of Elementary h-Uniform Hyper Cycles

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Abstract: One of the most popular and useful areas of graph theory is graph colorings. Frequently, we are concerned with determining the least number of colors with which we can achieve a proper coloring on a hypergraph. Furthermore, we want to count the possible number of different proper colorings on a hygraph with a given number of colors which is called the chromatic polynomial.

Key words: Graph theory, graph colorings, hygraph, polynomial, chromatic, h-uniform, hypercycle

INTRODUCTION

Let P(G, λ) denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, written G~H, if P (G, λ) = P (H, λ). A graph G is chromatically unique, written X-unique, if for any graph H, G~H implies that G is isomorphic with H. Many papers have been published about chromaticity of graphs which study the chromatically equivalent and the chromatically unique of graphs, see, for example (Khalaf and Peng, 2009, 2010). For more details about chromatic polynomials and chromaticity of graphs, we refer to the monograph (Dong *et al*., 2005). Hypergraph theory is a part of general study of combinatorial properties of finite families of finite sets. A hypergraph is defined to be a family of hypergraphes which are sets of vertices of cardinality not necessarily 2 (as for graphs).

The concept of a hypergraph was first introduced by C. Berge at Tihany colloquium in 1960 to extend, simplify and unify the results a bout graph and he developed many properties of vertex coloring of a hypergraph (Berge, 1973, 1989; Barchadt and Lazuka, 2007).

MATERIALS AND METHODS

Definition 1.1; Berge (1973, 1989): The couple H = (V, ε) where $\{v_1, v_2, ..., v_n\}$ (n∈N) is a finite set of vertices. And ε (H) = $\{e_1, e_2, ..., e_m\}$ (m∈N)is an edge-sets with $e_i \subseteq V$ (H) and $|e_i| \geq 1$ $(1 \leq i \leq m)$, then we say that H is a hypergraph of order |V| = n and size |ε| = m where e_i is called an edge or hyperedge.

Definition 1.2; Zhao (2009): Let H = (V, ε) be a hypergraph. If no hyperedge of ε(H) is subset of any other hyperedge of ε(H), then we say that H is sperner.

Definition 1.3; Allagan (2014): Let H = (V, ε) be a hypergraph then we say that H is a simple hypergraph, if all hyperedges are differents.

Definition 1.4; Borowieceki and Ewa (2000): Let H = (V, ε) be a hypergraph of size m∈N and Let e_i, $e_j \in \varepsilon(H)$ $(1 \leq i, j \leq m)$, if $e_i \cap e_j = \phi$ or $e_i \cap e_j = \{v\}$ where, v∈V (H) then we say that H is a linear hypergraph.

Definition 1.5; Walter (2009): Let H = (V, ε) be a hypergraph of size m∈N and Let $e_i \in \varepsilon(H)$ $(1 \leq i \leq m)$ be a hyperedge if $|e_i| = h$ for each i, then we say that H is h-uniform hypergraph, notice that 2-uniform hypergraph is just a graph. The number of hyperedges containing a vertex v∈V (H) is its degree d_H (v).

Definition 1.6: Let H = (V, ε) be a hypergraph and let u, v∈V (H) where u≠v, then u,v are lying in the same component, if there are vertices $u = v_0, v_1, .., v_k = v \in V$ (H) and hyperedges $e_1, ..., e_k \in \varepsilon(H)$ such that $v_{i-1}, v_i \in e_i$ for each i $(1 \leq i \leq k)$. Thus, if H has only one component then H is connected, otherwise H has r≥2 (r∈N) connected components.

Definition 1.7; West (2005): Let H = (V, ε) be a hypergraph. A hyperpath (or path) of length L joining two vertices u and v in H is a subhypergraph consisting of L+1 distinct vertices $u = v_0, v_1, ..., v_L = v$ and L distinct hyperedges $e_1, e_2, ..., e_L$ of and such that $v_{j-1}, v_j \in e_j$ for each j $(1 \leq j \leq L)$. An example is given in Fig. 1.

Fig. 1: A hyperpath

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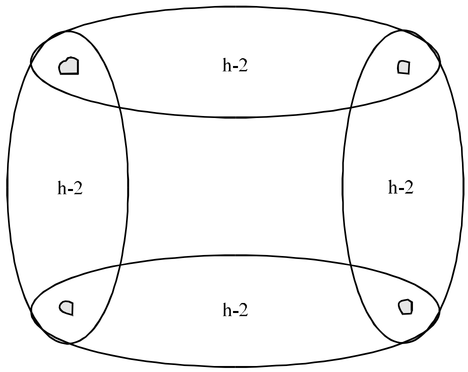

Fig. 2: C^h_4 an elementary hypercycle

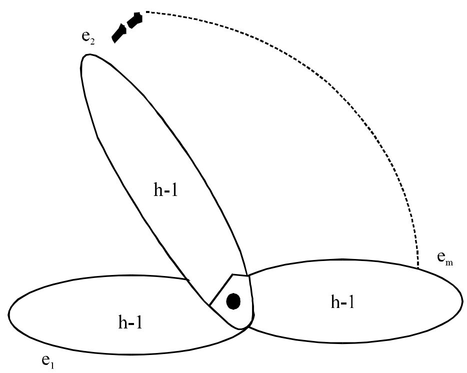

Fig. 3: A hyperstar

Definition 1.8: Let H = (V, ε) be a hypergraph. A hypercycle (cycle) C of Length L in H is a subhypergraph connecting L different vertices $v_1, ..., v_L$ and L different hyperedges $e_1, e_2, ..., e_L$ such that $v_j, v_{j+1} \in e_j$ for each j $(1 \leq j \leq L\text{-}1)$ and $v_1, v_L \in e_L$.

If $d_c(v_j) = 2$ for each j and $d_c(y) = 1$ for each other vertex $y \in \bigcup_{j=1}^{L} e_j$, then we say that C is an elementary hypercycle. An elementary h-uniform hypercycle with $m \geq 3$ is denoted by C^h_m and with 2 hyperedges by C^h_2 or C (2, h). If we join the end vertices of an hyperedge, then we get a hypercycle of length 1 denoted by C^h_1. Elementary hypercycle C^h_4 is shown in Fig. 2.

Definition 1.9; Donmen (1993): Let H = (V, ε) be a connected h-uniform hypergraph without hypercycles. Then, H is called h-uniform hypertree and denoted T^h_m where, m is the number of its hyperedges. If all hyperedge of T^h_m are intersecting in only one vertex then we say that T^h_m is a hyperstar as shown in Fig. 3.

Definition 1.10; Borowiecki and Lazuka (2000) and Drgas-Burchardt and Lazuka (2006): Let H = (V (H) (V, ε)) be a hypergraph and $\lambda \in N$, a λ-coloring of H is a function f: V (H)→{1, ..., λ} such that for each e∈ε (H), here exists x, y in e for which f (x)≠f (y).

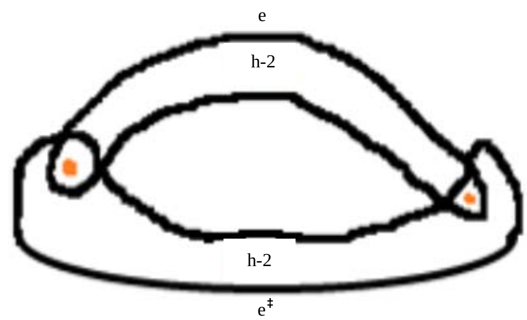

Fig. 4: e* inactive hyperedge

The number of different λ-coloring of H is given by a polynomial P (H, λ) of degree |V (H)| in λ called the chromatic polynomial of H.

Definition 1.11; Barchadt and Lazuka (2007): Two hypergraphs H_1 and H_2 are said to be chromatically equivalent, denoted by $(H_1 \sim H_2)$, if P $(H_1, \lambda) = (H_2, \lambda)$. A simple hypergraph H_1 is chromatically unique, if $H_1 \cong H_2$ for every simple hypergraph H_2 such that $H_1 \sim H_2$; that is the structure of H_1 is uniquely determined up to isomorphism by its chromatic polynomial.

Definition 1.12; Barchadt and Lazuka (2007): Let H be a hypergraph. If we dropp an edge e∈ε (H) we get a hypergraph H^- which is chromatically equivalent to H, then e is called chromatically inactive. Otherwise, e is said to be chromatically active. The edge e* in Fig. 4 is chromatically inactive H.

RESULTS AND DISCUSSION

Some known results on enumeration of P (H, λ)

Theorem 2.1; Borowiecki and Lazuka (2000): Let H be a hypergraph with order n and size m, then:

$$P(H, \lambda) = \lambda^n + a_{n-1}\lambda^{n-1} +, ..., a_2\lambda^2 + a_1\lambda$$

where, $a_i = \Sigma_{j\geq 0} (-1)^j N(i, j)$ for $1 \leq i \leq n\text{-}1$ and N (i, j) denoted the number of subhypergraphs of H with |V (H)| vertices, i components and j edges.

Theorem 2.2; Kashif (2011): Let H = (V, ε) be a hypergraph and let H-e be the hypergraph obtained by deleting some edge e∈ε(H), H.e be the hypergraph obtained by contracting all vertices in e to a common vertex v and dropping all chromatically inactive edges then:

$$P(H, \lambda) = P(H\text{-}e, \lambda)\text{-}P(H.e, \lambda)$$

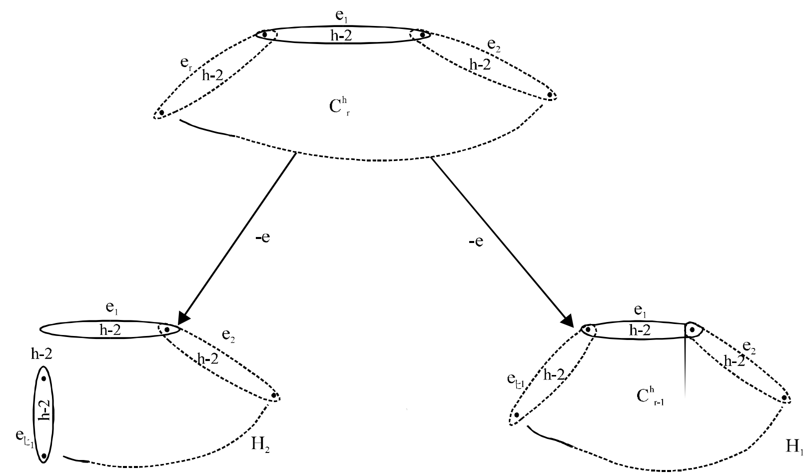

Fig. 5: Hypercycle

Theorem 2.3; West (2005): Assume that $H_1, H_2, ..., H_m$ be a connected components of hypergraph H, then:

$$P(H, \lambda) = \prod_{i=1}^{m} P(H_i, \lambda)$$

Theorem 2.4; Bokhary *et al.* (2009): Let H be a hypergraph such that:

$$H = H_1 \cup H_2, ..., \cup H_m \text{ for } m \geq 2$$

Where:

$$H_i \cap H_j = k_p, i \neq j \text{ and } \cap_{i=1}^{m} H_i = k_p$$

where, k_p is a complete graph with $P \geq 1$ vertices, then:

$$P(H, \lambda) = \left[P(k_p, \lambda)\right]^{1-m} \prod_{i=1}^{m} P(H_i, \lambda)$$

Theorem 2.5; Borowiecki and Ewa (2007): Let T^h_m be a linear h-uniform hypertree with size m and $h \geq 2$ then:

$$P(T^h_m, \lambda) = \lambda(\lambda^{h-1} - 1)^m$$

Theorem 2.6; Tomescu (1998): Let C^h_m is elementary h-uniform hypertree with size m then:

$$P(C^h_m, \lambda) = (\lambda^{h-1}-1)^m + (-1)^m (\lambda-1)$$

Elementary h-uniform hypercycle: In this section, we give the recurrence relation for the chromatic polynomial of an elementary h-uniform hypercycle.

Example 3.1: Let C^3_3, C^3_4, C^3_5 be three elementary h = uniform hypercycles. Apply the theorem (3.3.6) on C^3_3, C^3_4, C^3_5 resp:

$$\begin{aligned} P(C^3_3, \lambda) &= (\lambda^2-1)^3 + (-1)^3(\lambda-1) = \\ &\lambda^6-3\lambda^4+3\lambda^2-1-(\lambda-1) = \lambda^6-3\lambda^4+3\lambda^2-1-\lambda+1 = \\ &\lambda^6-3\lambda^4+3\lambda^2-\lambda \end{aligned} \quad (1)$$

And:

$$\begin{aligned} P(C^3_4, \lambda) &= (\lambda^2-1)^4 + (-1)^4(\lambda-1) = \\ &\lambda^8-4\lambda^6+6\lambda^4-4\lambda^2+1+\lambda-1 = \lambda^8-4\lambda^6+4\lambda^4-4\lambda^2+\lambda \end{aligned} \quad (2)$$

Also:

$$\begin{aligned} P(C^3_5, \lambda) &= (\lambda^2-1)^5 + (-1)^5(\lambda-1) = \\ &\lambda^{10}-5\lambda^8+10\lambda^6-10\lambda^4+5\lambda^2-1-\lambda+1 = \\ &\lambda^{10}-5\lambda^8+10\lambda^6-10\lambda^4+5\lambda^2-\lambda \end{aligned} \quad (3)$$

To introduce the relation, we need this proposition.

Proposition 3.2: For $r \geq 2$, $h \geq 3$ we have:

$$P(C^h_r, \lambda) = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - P(C^h_{r-1}, \lambda)$$

Proof: Let C^h_r be an elementary h-uniform hypercycle as shown in Fig. 5. If we delete any hyperedge and contract it we obtain two hypergraphs H_1 and H_2, thus by the theorem (2.2) we get:

$$P(C^h_r, \lambda) = P(H_1, \lambda) - P(H_2, \lambda) \quad (4)$$

where, H_1 composed from h-2 isolated vertices and hypertree T^h_{r-1}, thus, by the theorems (2.3 and 2.5):

$$P(H_1, \lambda) = \lambda^{h-2}\lambda(\lambda^{h-1}-1)^{r-1} = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} \quad (5)$$

And H_2 is just hypercycle C^h_{r-1}, thus:

$$P(H_2, \lambda) = P(C^h_{r-1}, \lambda) \qquad (6)$$

Substitute 5 and 6 in 4 we get:

$$P(C^h_r, \lambda) = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - P(C^h_{r-1}, \lambda)$$

As required:

Example 3.3: Consider C^3_3, C^3_2, C^3_1 . By the theorem (2.6):

$$C^3_3 = (\lambda^2-1)^3-(\lambda-1) = \lambda^6-3\lambda^4+3\lambda^2-\lambda \qquad (7)$$

$$C^3_2 = (\lambda^2-1)^2+(\lambda-1) = \lambda^4-2\lambda^2+\lambda \qquad (8)$$

Now by applying our result (3.2) and the theorem (2.6):

$$P(C^3_3, \lambda) = \lambda^2(\lambda^2-1)^2-P(C^2_2, \lambda) = \lambda^2(\lambda^4-2\lambda^2+1)-\left[(\lambda^2-1)^2+(\lambda-1)\right] = \lambda^6-3\lambda^4+3\lambda^2-\lambda$$

Which is equal to 7. Also:

$$P(C^3_1, \lambda) = (\lambda^2-1)-(\lambda-1) = \lambda^2-\lambda \qquad (9)$$

Then:

$$P(C^3_2, \lambda) = \lambda^2(\lambda^2-1)-P(C^3_1, \lambda) = \lambda^4-\lambda^2-\left((\lambda^2-1)-(\lambda-1)\right) = \lambda^4-\lambda^2-(\lambda^2-\lambda) = \lambda^4-2\lambda^2+\lambda$$

Which is equal to 7. Now, we can introduce the general recurrence formula for any elementary h-uniform hypercycle.

Theorem 3.4: For r≥2, h≥3, the recurrence relation for the chromatic polynomial of an elementary h-uniform hypercycle C^h_r is given by:

$$P(C^h_r, \lambda) = \lambda^{h-1}P(C^h_{r-1}, \lambda)-(\lambda^{h-1}-1)P(C^h_{r-2}, \lambda)- \lambda^{h-1}(\lambda-1)\left[(-1)^{r-1}-(-1)^{r-2}\right]$$

Notice: If r is odd, then:

$$P(C^h_r, \lambda) = \lambda^{h-1}P(C^h_{r-1}, \lambda)-(\lambda^{h-1}-1)P(C^h_{r-2}, \lambda)-2\lambda^{h-1}(\lambda-1)$$

If r is even, then:

$$P(C^h_r, \lambda) = \lambda^{h-1}P(C^h_{r-1}, \lambda)-(\lambda^{h-1}-1)P(C^h_{r-2}, \lambda)-2\lambda^{h-1}(\lambda-1)$$

Proof: Let C^h_r be an elementary h-uniform hypercycle. By applying proposition (3.2) on C^h_r and C^h_{r-1} then we get:

$$P(C^h_r, \lambda) = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1}-P(C^h_{r-1}, \lambda)$$

But:

$$P(C^h_{r-1}, \lambda) = \left[\lambda^{h-1}(\lambda^{h-1}-1)^{r-2}-P(C^h_{r-2}, \lambda)\right.$$

Substitute above result we get:

$$= \lambda^{h-1}(\lambda^{h-1}-1)^{r-1}-\left[\lambda^{h-1}(\lambda^{h-1}-1)^{r-2}-P(C^h_{r-2}, \lambda)\right] = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1}-\lambda^{h-1}(\lambda^{h-1}-1)^{r-2}+P(C^h_{r-2}, \lambda)$$

By adding and subtracting the terms $(-1)^{r-1}\lambda^{h-1}(\lambda-1)$ and $(-1)^{r-2}\lambda^{h-1}(\lambda-1)$ we get:

$$P(C^h_r, \lambda) = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1}-\lambda^{h-1}(\lambda^{h-1}-1)^{r-2}+P(C^h_{r-2}, \lambda)+ (-1)^{r-1}\lambda^{h-1}(\lambda-1)-(-1)^{r-1}\lambda^{h-1}(\lambda-1)+(-1)^{r-2}\lambda^{h-1}(\lambda-1)-(-1)^{r-2}\lambda^{h-1}(\lambda-1) = \left[\lambda^{h-1}(\lambda^{h-1}-1)^{r-1}+(-1)^{r-1}\lambda^{h-1}(\lambda-1)\right]-\left[\lambda^{h-1}(\lambda^{h-1}-1)^{r-2}+(-1)^{r-2}\lambda^{h-1}(\lambda-1)\right]+ P(C^h_{r-2}, \lambda)-\left[(-1)^{r-1}\lambda^{h-1}(\lambda-1)-(-1)^{r-2}\lambda^{h-1}(\lambda-1)\right] = \lambda^{h-1}\left[(\lambda^{h-1}-1)^{r-1}+(-1)^{r-1}(\lambda-1)\right]-\lambda^{h-1}\left[(\lambda^{h-1}-1)^{r-2}+(-1)^{r-2}(\lambda-1)\right]+ P(C^h_{r-2}, \lambda)-\lambda^{h-1}(\lambda-1)\left[(-1)^{r-1}-(-1)^{r-2}\right] = \lambda^{h-1}P(C^h_{r-1}, \lambda)-\lambda^{h-1}P(C^h_{r-2}, \lambda)+ P(C^h_{r-2}, \lambda)-\lambda^{h-1}(\lambda-1)\left[(-1)^{r-1}-(-1)^{r-2}\right] = \lambda^{h-1}P(C^h_{r-1}, \lambda)- \lambda^{h-1}(\lambda-1)P(C^h_{r-2}, \lambda)-\lambda^{h-1}(\lambda-1)\left[(-1)^{r-1}-(-1)^{r-2}\right]$$

As required:

Example 3.5: From example (3.1) we know that:

$$P(C^3_3, \lambda) = \lambda^6-3\lambda^4+3\lambda^2-\lambda \qquad (10)$$

$$P(C^3_4, \lambda) = \lambda^8-4\lambda^6+6\lambda^4-4\lambda^2+\lambda \qquad (11)$$

$$P(C^3_5, \lambda) = \lambda^{10}-5\lambda^8+10\lambda^6-10\lambda^4+5\lambda^2-\lambda \qquad (12)$$

Now, if we apply our result (3.4) on C^3_5 we get:

$$P(C^3_5, \lambda) = \lambda^2P(C^3_4, \lambda)-(\lambda^2-1)P(C^3_3, \lambda)-\lambda^2(\lambda-1)\left[(-1)^4-(-1)^3\right] = \lambda^2\left[\lambda^8-4\lambda^6+6\lambda^4-4\lambda^2+\lambda\right]-(\lambda^2-1)\left[\lambda^6-3\lambda^4+3\lambda^2-\lambda\right]-2\lambda^2(\lambda-1) = \lambda^2\left[\lambda^8-4\lambda^6+6\lambda^4-4\lambda^2+\lambda\right]-(\lambda^2-1)\left[\lambda^6-3\lambda^4+3\lambda^2-\lambda\right]-2\lambda^2(\lambda-1) = \lambda^{10}-4\lambda^8+6\lambda^6-4\lambda^4+\lambda^3-\lambda^8+3\lambda^6-3\lambda^4+\lambda^3+\lambda^6-3\lambda^4+3\lambda^2-\lambda-2\lambda^3+ 2\lambda^2 = \lambda^{10}-5\lambda^8+10\lambda^6-10\lambda^4+5\lambda^2-\lambda$$

Which is equal to 12.

Example 3.6: Consider C_2^4, C_3^4, C_4^4. By applying the theorem (2.6):

$$P\left(C_2^4, \lambda\right) = \left(\lambda^3-1\right)^2 + \left(\lambda-1\right) = \lambda^6-2\lambda^3+\lambda \quad (13)$$

$$P\left(C_3^4, \lambda\right) = \left(\lambda^3-1\right)^3 + \left(\lambda-1\right) = \lambda^9-3\lambda^6+3\lambda^2-\lambda \quad (14)$$

$$P\left(C_4^4, \lambda\right) = \left(\lambda^3-1\right)^4 + \left(\lambda-1\right) = \lambda^{12}-4\lambda^9+6\lambda^6-4\lambda^3+\lambda \quad (15)$$

By applying our result (3.4):

$$P\left(C_4^4, \lambda\right) = \lambda^3 P\left(C_3^4, \lambda\right)-\left(\lambda^3-1\right)P\left(C_2^4, \lambda\right)-\lambda^3\left(\lambda-1\right)\left[\left(-1\right)^3-\left(-1\right)^2\right] =$$
$$\lambda^3\left[\lambda^9-3\lambda^6+3\lambda^3-\lambda\right]-\left(\lambda^3-1\right)\left[\lambda^6-2\lambda^3+\lambda\right]+2\lambda^4-2\lambda^3 = \lambda^{12}-3\lambda^9+3\lambda^6-\lambda^4+$$
$$2\lambda^6-\lambda^4+\lambda^6-2\lambda^3+\lambda+2\lambda^4-2\lambda^3 = \lambda^{12}-4\lambda^9+6\lambda^6-4\lambda^3+\lambda$$

Which is equal to 15.

CONCLUSION

In this study we introduce the recurrence relation for the chromatic polynomial of an elementary h-uniform hypercycle.

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