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Development of A-Stable Block Method for the Solution of Stiff Ordinary Differential Equations

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Abstract: A fixed step-size multistep block method for stiff Ordinary Differential Equations (ODEs) using the 2-point Block Backward Differentiation Formulas (BBDF) with improved efficiency is established. The method is developed using Taylor's series expansion. The order and the error constant of the method are determined. To validate the new method is suitable for solving stiff ODEs, the stability and convergence properties are discussed. Numerical results indicate that the new method produced better accuracy than the existing methods when sloving the same problems.

Key words: Block method, backward differentiation formula, stiff, stability, ordinary differential equations, efficiency

INTRODUCTION

Many fields of applications such as science and engineering are often modelled by a system of Ordinary Differential Equations (ODEs). These ODEs can be classified into two classes which are stiff and non-stiff ODEs. The non-stiff problems usually solved using the explicit method while stiff problems are solved via. the implicit method. In this study, we consider the first order ODEs in the form of:

$$y' = f(x, y), y(a) = y_0, x \in [a, b] \tag{1}$$

The system of Eq. 1 is called stiff, if the real components of all the eigenvalues of Jacobian matrix $\partial f/\partial y$ are negative (Semenov, 2011), i.e., Re $(\lambda_i)<0$, I = 1, 2, ..., N and the ratio $s = \frac{\max\{|Re(\lambda_i)|, i=1,2,\ldots,N\}}{\min\{|Re(\lambda_i)|, i=1,2,\ldots,N\}}$ is large where parameter s is called stiffness ratio.

In the earlier researches, stiff problems are solved using implicit one-step methods which consist of implicit and semi-implicit Runge Kutta (RK) methods and the trapezoidal rule (Lapidus and Seinfeld, 1971). Rosser (1967) proposed the RK in a block of N steps to reduce the number of function evaluation when compare with the classical RK method. Uses of the block method are then extended by Gear (1988) to solve parallel solution of ODEs. Studying about the block method is continued by developing the one-step fourth order of block method based on the composite Simpson rule (Voss and Abbas, 1997). The results obtained shows that the maximum absolute error of the method is more accurate as compared to the block predictor-corrector using composite trapezoidal rule. The one-step block method is then modified to the multi-step block method using the idea from Gear's method and block method to create a new formula called fully implicit r-point Block Backward Differentiation Formula (rBBDF) method (Ibrahim *et al*., 2007). The advantage of rBBDF method is the method can evaluate the solution more than one point at one time and may have several points in each block depending on the structure of that block (Nasir *et al*., 2012). Regarding to the statement an implicit 2-point block method with an extra future point is introduced by upgrading the idea of rBBDF method called 2-point Improved Block Backward Differentiation Formula (2IBBDF) (Musa *et al*., 2013). Below we give the basic definition of a block method described by Chu and Hamilton (1987).

Definition 1: If r denotes the block size and h is the step size then block size in time is rh. Let m represent the block number when m = 0, 1, 2, ... and let n = mr then the general b-block, r-point method is a matrix finite difference Eq. 2 of the form:

$$Y_m = \sum_{s=1}^{b} A_s Y_{m-s} + h\sum_{s=0}^{b} B_s F_{m-s} \tag{2}$$

where, $Y_m = [y_n, y_{n+1}, ..., y_{n+r-1}]^T$ and $f_m = [f_n, f_{n+1}, ..., f_{n+r-1}]^T$ are vectors, A_s and B_s are r×r coefficient matrices. If r = 1, then Eq. 2 is the classical method. In this study, we proposed

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a new implicit 2-point block method where adopted from the definition of Linear Multistep Method (LMM) (Lambert, 1973) as:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{3}$$

where, α_j and β_j are constant; by assuming that $\alpha_k \neq 0$ and that not both α_0 and β_0 are zero. If the $\beta_k \neq 0$, then the method is said to be implicit. The purposes of derived method are to approximate the solutions of problems related to stiffness and to establish the improvement in terms of accuracy and execution time. Therefore, this study is organized as follows. In the following study, derivation of the method is briefly explained by extending the strategy of the 2IBBDF method (Musa *et al.*, 2013). The method is extended by increasing the order of the method to improve the accuracy when compared with the existing methods of same order. The following Eq. 4 show the corrector formula of the 2IBBDF method.

$$y_{n+1} = \frac{5}{4}y_n - \frac{1}{4}y_{n+2} + \frac{1}{2}hf_n + hf_{n+1}$$

$$y_{n+2} = \frac{1}{8}y_{n-1} - \frac{1}{2}y_n + \frac{11}{8}y_{n+1} + \frac{1}{4}hf_{n+1} + \frac{1}{2}hf_{n+2} \tag{4}$$

We will investigate the stability properties of proposed method for the validity to solve stiff problems in the subsection stability analysis of the method. The convergence properties of derived method are also discussed by determining its consistency and zero-stable. Then, implementation of the method will be presented the application of Newton's iteration towards the proposed method to solve stiff problems. To prove the effectiveness of the method, we choose a few test problems related to the first order stiff ODEs and the numerical results obtained will be compared with the existing methods. The discussion will be made and a simple conclusion will be prepared in the last chapter.

MATERIALS AND METHODS

Construction of the method: A detailed description on the construction of implicit 2-point block method will be presented in this section. The method will evaluate the solution using four starting values which are y_{n-3}, y_{n-2}, y_{n-1} and y_n. The general formula of the proposed method takes the form of LMM:

$$\sum_{j=0}^{5}\alpha_{j,i} y_{n+j-3} = h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-1}\right),\ k = i = 1,2 \tag{5}$$

where, $\alpha_{j,i}$ and $\beta_{k,i}$ are the parameters to be determined, h is the fixed step size and ρ is a value that must be chosen in between -1 and 1 for the stability purposes (Vijitha-Kumara, 1985). The linear difference operator, L_i associated with the general formula of the proposed method Eq. 5 will be defined as:

$$\begin{aligned} L_i[y(x_n);h] &= \sum_{j=0}^{5}\alpha_{j,i} y_{n+j-3} = h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-1}\right) = \alpha_{0,i}y_{n-3} + \\ &\alpha_{1,i}y_{n-2} + \alpha_{2,i}y_{n-1} + \alpha_{4,i}y_n + \alpha_{4,i}y_{n+1} + \alpha_{5,i}y_{n+2} - \\ &h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-1}\right) \end{aligned} \tag{6}$$

where, k = i = 1, 2, y_n is an arbitrary function and $\rho = -7/8$. The Taylor's series expansions are then applied to Eq. 6 and expand the y_n and f_n in terms of x_n which yields:

$$\begin{aligned} y_{n-3} &= y(x_n) - 3hy'(x_n) + \frac{(-3h)^2}{2!}y''(x_n) +, ..., + \frac{(-3h)^q}{q!}y^{(q)}(x_n), \\ y_{n-2} &= y(x_n) - 2hy'(x_n) + \frac{(-2h)^2}{2!}y''(x_n) +, ..., + \frac{(-2h)^q}{q!}y^{(q)}(x_n), \\ y_{n-1} &= y(x_n) - hy'(x_n) + \frac{(-h)^2}{2!}y''(x_n) +, ..., + \frac{(-h)^q}{q!}y^{(q)}(x_n), \\ y_n &= y(x_n), \\ y_{n+1} &= y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) +, ..., + \frac{h^q}{q!}y^{(q)}(x_n), \\ y_{n+2} &= y(x_n) + 2hy'(x_n) + \frac{(2h)^2}{2!}y''(x_n) +, ..., + \frac{(2h)^q}{q!}y^{(q)}(x_n), \\ f_{n+k} &= y'(x_n) + khy''(x_n) + \frac{(kh)^2}{2!}y'''(x_n) +, ..., + \frac{(kh)^q}{q!}y^{(q)}(x_n), \\ f_{n+(k-1)} &= y'(x_n) + (k-1)hy''(x_n) + \frac{((k-1)h)^2}{2!}y'''(x_n) +, ..., + \frac{((k-1)h)^q}{q!}y^{(q)}(x_n) \end{aligned} \tag{7}$$

where, q = 3, 4, ..., N. Based on the set of Eq. 7, we collect the coefficients according to y_n and its derivative as follows:

$$\begin{aligned} &C_{0,i}y(x_n) + C_{1,i}hy'(x_n) + C_{2,i}h^2y''(x_n) + \\ &C_{3,i}h^3y'''(x_n) +, ..., + C_{q,i}h^q y^{(q)}(x_n) = 0 \end{aligned} \tag{8}$$

Where:

$$\begin{aligned} C_{0,i} &= \alpha_{0,i} + \alpha_{1,i} + \alpha_{2,i} + \alpha_{3,i} + \alpha_{4,i} + \alpha_{5,i}, \\ C_{1,i} &= -3\alpha_{0,i} - 2\alpha_{1,i} - \alpha_{2,i} + \alpha_{4,i} + 2\alpha_{5,i} - (k-(k-1)\rho)\beta_{k,i}, \\ C_{q,i} &= \frac{(-3)^q}{q!}\alpha_{0,i} + \frac{(-2)^q}{q!}\alpha_{1,i} + \frac{(-1)^q}{q!}\alpha_{2,i} + \frac{1^q}{q!}\alpha_{4,i} + \\ &\frac{2^q}{q!}\alpha_{5,i} - \left(\frac{k^{(q-1)}}{(q-1)!} - \frac{k-1^{(q-1)}}{(q-1)!}\rho\right)\beta_{k,i} \\ q &= 2,3, ..., k = 1,2 \end{aligned} \tag{9}$$

To obtain the values of α and β for the first point, y_{n+1}, we let k = i = 1 and $\alpha_{4,1} = 1$, into Eq. 9. We then obtained the following:

$$\begin{aligned}
C_{0,1} &= \alpha_{0,1}+\alpha_{1,1}+\alpha_{2,1}+\alpha_{3,1}+\alpha_{5,1} = -1,\\
C_{1,1} &= -3\alpha_{0,1}-2\alpha_{1,1}-\alpha_{2,1}+\alpha_{4,1}+2\alpha_{5,1}-(1-\rho)\beta_{1,1} = -1,\\
C_{2,1} &= -\frac{9}{2}\alpha_{0,1}+2\alpha_{1,1}+\frac{1}{2}\alpha_{2,1}+2\alpha_{5,1}-\beta_{1,1} = -\frac{1}{2},\\
C_{3,1} &= -\frac{9}{2}\alpha_{0,1}-\frac{4}{3}\alpha_{1,1}-\frac{1}{6}\alpha_{2,1}+\frac{4}{3}\alpha_{5,1}-\frac{1}{2}\beta_{1,1} = -\frac{1}{6},\\
C_{4,1} &= \frac{27}{8}\alpha_{0,1}+\frac{2}{3}\alpha_{1,1}+\frac{1}{24}\alpha_{2,1}+\frac{2}{3}\alpha_{5,1}-\frac{1}{6}\beta_{1,1} = -\frac{1}{24},\\
C_{5,1} &= \frac{81}{40}\alpha_{0,1}-\frac{4}{15}\alpha_{1,1}-\frac{1}{120}\alpha_{2,1}+\frac{4}{15}\alpha_{5,1}-\frac{1}{24}\beta_{1,1} = -\frac{1}{120}
\end{aligned} \tag{10}$$

Then, the set of Eq. 10 is solved simultaneously which produces:

$$\alpha_{0,1} = \frac{1}{73}, \alpha_{1,1} = -\frac{11}{146}, \alpha_{2,1}\frac{6}{73}, \alpha_{3,1} = -\frac{82}{73}, \quad \alpha_{5,1} = -\frac{15}{146}, \beta_{1,1} = -\frac{48}{73} \tag{11}$$

Similar procedure is applied to obtain the coefficients of the second point, y_{n+2} by substituting k = i = 2 and $\alpha_{5,2} = 1$ into Eq. 9, the values of α and β are obtained as:

$$\alpha_{0,2} = \frac{15}{236}, \alpha_{1,2} = \frac{23}{59}, \alpha_{2,2} = -1, \alpha_{3,2} = \frac{78}{59}, \quad \alpha_{4,2} = -\frac{389}{236}, \beta_{2,2} = \frac{24}{59} \tag{12}$$

Hence, the corrector formula of implicit 2-point block method is written as:

$$\begin{aligned}
y_{n+1} &= -\frac{1}{73}y_{n-3}+\frac{11}{146}y_{n-2}-\frac{6}{73}y_{n-1}+\frac{82}{73}y_n-\frac{15}{146}y_{n+2}+\frac{42}{73}hf_n+\frac{48}{73}hf_{n+1},\\
y_{n+2} &= \frac{15}{236}y_{n-3}-\frac{23}{59}y_{n-2}+y_{n-1}-\frac{78}{59}y_n+\frac{389}{236}y_{n+1}+\frac{21}{59}hf_{n+1}+\frac{24}{59}hf_{n+2}
\end{aligned} \tag{13}$$

when the values of $\alpha_{j,i}$ and $\beta_{k,i}$ in Eq. 11 and 12 are substituted into Eq. 6. The method Eq. 13 is called an Improved 2-point Block Backward Differentiation Formula of fifth order (I2BBDF(5)).

Stability analysis of the method: In this subsection, stability properties of the derived method are analysed. The linear stability properties of the corrector Eq. 13 are obtained by applying the scalar test Eq. 14:

$$y' = \lambda y, \lambda<0 \tag{14}$$

And gives the following matrix Eq. 15:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix} = \begin{bmatrix}-\frac{1}{73} & \frac{11}{146}\\ \frac{15}{236} & -\frac{23}{59}\end{bmatrix}\begin{bmatrix}y_{n-3}\\y_{n-2}\end{bmatrix}+\begin{bmatrix}-\frac{6}{73} & \frac{82}{73}\\ 1 & -\frac{78}{59}\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}+\begin{bmatrix}0 & -\frac{15}{146}\\ \frac{389}{236} & 0\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}+ h\begin{bmatrix}0 & \frac{42}{73}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\lambda y_{n-1}\\ \lambda y_n\end{bmatrix}+h\begin{bmatrix}\frac{48}{73} & 0\\ \frac{21}{59} & \frac{24}{59}\end{bmatrix}\begin{bmatrix}\lambda y_{n+1}\\ \lambda y_{n+2}\end{bmatrix} \tag{15}$$

By applying $h\lambda = \bar{h}$, into Eq. 15 we have:

$$\begin{bmatrix}1-\frac{48}{73}\bar{h} & \frac{15}{146}\\ -\frac{389}{236}-\frac{21}{59}\bar{h} & 1-\frac{24}{59}\bar{h}\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix} = \begin{bmatrix}-\frac{6}{73} & \frac{82}{73}+\frac{42}{73}\bar{h}\\ 1 & -\frac{78}{59}\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}+ \begin{bmatrix}-\frac{1}{73} & \frac{11}{146}\\ \frac{15}{236} & -\frac{23}{59}\end{bmatrix}\begin{bmatrix}y_{n-3}\\y_{n-2}\end{bmatrix} \tag{16}$$

which is equivalent to $AY_m = BY_{m-1}+CY_{m-2}$ where:

$$A = \begin{bmatrix}1-\frac{48}{73}\bar{h} & \frac{15}{146}\\ -\frac{389}{236}-\frac{21}{59}\bar{h} & 1-\frac{24}{59}\bar{h}\end{bmatrix}, Y_m\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix}, B = \begin{bmatrix}-\frac{6}{73} & \frac{82}{73}+\frac{42}{73}\bar{h}\\ 1 & -\frac{78}{59}\end{bmatrix},$$
$$Y_{m-1} = \begin{bmatrix}y_{n-1}\\y_n\end{bmatrix}, C = \begin{bmatrix}-\frac{1}{73} & \frac{11}{146}\\ \frac{15}{236} & -\frac{23}{59}\end{bmatrix}, Y_{m-2} = \begin{bmatrix}y_{n-3}\\y_{n-2}\end{bmatrix} \tag{17}$$

Stability polynomial of the I2BBDF(5) method can be obtained by substituting the matrix A, B and C into the following formula:

$$R(t,\bar{h}) = \det(At^2-Bt-C) \tag{18}$$

And yield:

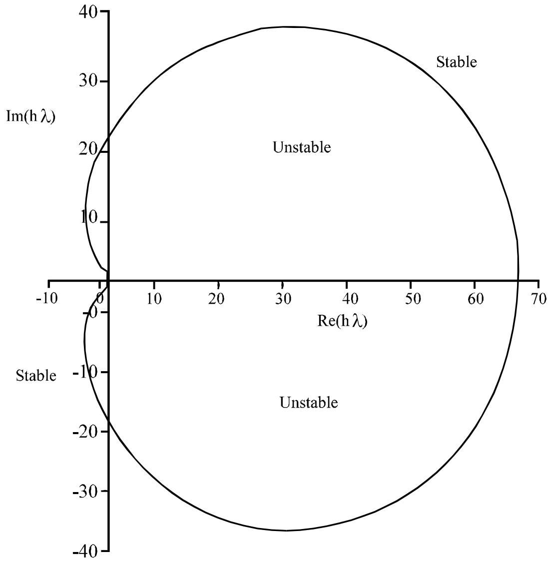

Fig. 1: Stability region of the I2BBDF(5) method

$$R(t,\bar{h}) = \det\left(\left(\begin{bmatrix} 1-\frac{48}{73}\bar{h} & \frac{15}{146} \\ -\frac{389}{236}-\frac{21}{59}\bar{h} & 1-\frac{24}{59}\bar{h} \end{bmatrix}\right) t^2 - \begin{bmatrix} -\frac{6}{73} & \frac{82}{73}+\frac{42}{73}\bar{h} \\ 1 & -\frac{78}{59} \end{bmatrix} t - \begin{bmatrix} -\frac{1}{73} & \frac{11}{146} \\ \frac{15}{236} & -\frac{23}{59} \end{bmatrix}\right) =$$
$$\frac{40291}{34456}t^4 - \frac{8853}{8614}t^4\bar{h} + \frac{1152}{4307}t^4\bar{h}^2 - \frac{1484}{4307}t^3 - \frac{19389}{8614}t^3\bar{h} - \frac{882}{4307}t^3\bar{h}^2 - \frac{12555}{17228}t^2 - \frac{7443}{8614}t^2\bar{h} - \frac{416}{4307}t - \frac{315}{8614}t\bar{h} + \frac{19}{34456} \quad (19)$$

Then, the boundary of stability is plotted by letting, $t = e^{I\theta}$, $0 \leq \theta \leq 2\pi$ into Eq. 19. The stability region is given in Fig. 1. As presented in Fig. 1, the absolute stability lies on the entire outer part of the circle and the regions of stability covers the left-half plane. It can be concluded that the method is A-stable, since, it satisfies the definition below (Ibrahim *et al*., 2003).

Definition 2: A numerical method is said to be A-stable, if its region of absolute stability contains whole left plane. To determine the convergence of method, we consider the theorem of convergent (Ken *et al*., 2011) in this study.

Theorem 1: The necessary and sufficient conditions for a LMM (3) to be convergent are that it be consistent and zero-stable.

Definition 3: The LMM (3) is said to be consistent, if it has order, $p \geq 1$.

Definition 4: The difference operator (6) and the associated LMM (3) are said to be order p, if $C_0 = C_1 = , \ldots , = C_p = 0$ and $C_{p+1} \neq 0$ where:

$$\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_{j,i}, \; C_1 = \sum_{j=0}^{k}\left(j\alpha_{j,i} - \beta_{k,i}\right), \\ &\vdots \\ C_6 &= \sum_{j=0}^{k}\left(\frac{j^q}{q!}\alpha_{j,i} - \frac{j^{(q-1)}}{(q-1)!}\beta_{k,i}\right).q = 2,3, \ldots, \end{aligned} \quad (20)$$

By substituting the coefficients of α and β into Eq. 20, we determined the order of the method as follows:

$$\begin{aligned} C_0 &= \sum_{j=0}^{5}\alpha_{j,i} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ C_1 &= \sum_{j=0}^{5}\left(j\alpha_{j,i} - \beta_{k,i}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ C_2 &= \sum_{j=0}^{5}\left(\frac{j^2}{2!}\alpha_{j,i} - j\beta_{k,i}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ C_3 &= \sum_{j=0}^{5}\left(\frac{j^3}{3!}\alpha_{j,i} - \frac{j^2}{2!}\beta_{k,i}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ C_4 &= \sum_{j=0}^{5}\left(\frac{j^4}{4!}\alpha_{j,i} - \frac{j^3}{3!}\beta_{k,i}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ C_5 &= \sum_{j=0}^{5}\left(\frac{j^5}{5!}\alpha_{j,i} - \frac{j^4}{4!}\beta_{k,i}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ C_6 &= \sum_{j=0}^{5}\left(\frac{j^6}{6!}\alpha_{j,i} - \frac{j^5}{5!}\beta_{k,i}\right) = \begin{bmatrix} \frac{9}{730} \\ -\frac{33}{590} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \quad (21)$$

Therefore, the Eq. 13 is of order five with the error constant as displayed in $C_6 = C_{p+1} \neq 0$. Hence, its proved that the method is consistent, since, the order of the method, $p = 5 \geq 1$. Next, we further the analysis by determining the zero-stable of the method. Equation 13 is said to be zero-stable, if no root of the first characteristic polynomial, $R(t, \bar{h})$ is having a modulus greater than one and every root of modulus one is simple (Kuboye and Omar, 2015). By referring to the first characteristic polynomial of the Eq. 19, we replaced $\bar{h} = 0$ and solve for t to determine the roots of the stability polynomial as follows: $t_1 = 1$, $t_2 = 0.0055\text{-}0.0004i$, $t_3 = 0.5563\text{-}0.000146i$ and $t_4 = -0.1547+0.000546i$.

Therefore, the I2BBDF(5) method is zero-stable, since, all the values of t satisfied the definition of zero stability stated by Kuboye and Omar (2015).

From the above discussion, the proposed method is consistent and zero stable, hence, we conclude that the I2BBDF(5) method is converged by theorem 1.

Implementation of the method: This subsection deals with the effect of Newton's iteration to approximate the solutions of y_{n+1} and y_{n+2} concurrently in each step size. The method (Eq. 13) are expressed in the form of:

$$\begin{aligned} F_1 &= y_{n+1}+\frac{15}{146}y_{n+2}-\frac{42}{73}hf_n-\frac{48}{73}hf_{n+1}-\mu_1, \\ F_2 &= y_{n+2}-\frac{389}{236}y_{n+1}-\frac{21}{59}hf_{n+1}-\frac{24}{59}hf_{n+2}-\mu_2, \end{aligned} \quad (22)$$

where, the back values for each point is given below:

$$\begin{aligned} \mu_1 &= \frac{1}{73}y_{n-3}+\frac{11}{146}y_{n-2}-\frac{6}{73}y_{n-1}+\frac{82}{73}y_n, \\ \mu_2 &= \frac{15}{236}y_{n-3}-\frac{23}{59}y_{n-2}+y_{n-1}-\frac{78}{59}y_n, \end{aligned} \quad (23)$$

The application of Newton's iteration for the implicit 2-point block method will take the form:

$$Y^{(i+1)}_{n+1,\,n+2}-Y^{(i)}_{n+1,\,n+2} = \left[F_j'\left(y^{(i)}_{n+1,\,n+2}\right)\right]^{-1}\left[F_j\left(y^{(i)}_{n+1,\,n+2}\right)\right] \quad (24)$$

where, j = 1, 2 and (i+1)th is the iterative value of $y_{n+1,\,n+2}$. We let:

$$Y^{(i+1)}_{n+1,n+2}-Y^{(i)}_{n+1,n+2} = E^{i+1}_{n+1,n+2} \quad (25)$$

The Eq. 24 where $E^{(i+1)}_{n+1,n+2}$ are introduced to be the increment value from (i+1)th to (i)th iterations. By replacing the Eq. 22-24, the method is presented into matrix mode as:

$$\begin{bmatrix} 1-\frac{48}{73}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{15}{146} \\ -\frac{389}{236}-h\frac{\partial f_{n+1}}{\partial y_{n+1}} & 1-\frac{24}{59}h\frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}\begin{bmatrix} e^{(i+1)}_{n+1} \\ e^{(i+1)}_{n+2} \end{bmatrix} = \begin{bmatrix} -y^{(i)}_{n+1}-\frac{15}{146}y^{(i)}_{n+2}+\frac{42}{73}hf^{(i)}_n+\frac{48}{73}hf^{(i)}_{n+1}+\mu^{(i)}_1 \\ -y^{(i)}_{n+2}-\frac{389}{236}y^{(i)}_{n+1}+\frac{21}{59}hf^{(i)}_{n+1}+\frac{24}{59}hf^{(i)}_{n+2}+\mu^{(i)}_2 \end{bmatrix} \quad (26)$$

Then, solve the Eq. 26 for $e^{(i+1)}_{n+1,\,n+2}$ as the values can be approximated and the solutions values of y_{n+1} is computed from $y^{(i+1)}_{n+1} = y^{(i)}_{n+1} + e^{(i+1)}_{n+1}$ and y_{n+2} is figured out from $y^{(i+1)}_{n+2} = y^{(i)}_{n+2} + e^{(i+1)}_{n+2}$. The absolute error of the calculation can be defined as:

$$\text{Error} = \left|y_{exact}-y_{approximate}\right| \quad (27)$$

And the maximum error can be calculated as:

$$\text{MAXE} = \max_{0\le n\le TS}(\text{error}) \quad (28)$$

where, TS is the total step taken.

RESULTS AND DISCUSSION

In order to study the efficiency of the proposed method, we select some numerical examples and the results will be compared with the 2-point implicit block method with an off-stage function with $\zeta = -1/4$ (2P4BBDF) (Zainal, 2013) and fifth order of Block Backward Differentiation Formula (BBDF(5)) (Nasir *et al.*, 2012) methods. The selection of those existing methods is based on the order of method for fair comparison. Below are the selected tested problems that we will consider in this study.

Problem 1: Ibrahim (2006):

$$y' = -10y+10,\ y(0) = 2,\ 0 \le x \le 10 \quad (29)$$

The exact solution is $y(x) = 1+e^{-10x}$. The eigen value is $\lambda = -10$.

Problem 2: Voss and Abbas (1997):

$$y' = \frac{50}{y}-50y,\ \ y(0) = \sqrt{2},\ 0\le x\le 1 \quad (30)$$

The exact solution is $y(x) = \sqrt{1+e^{-10x}}$. The eigenvalue is $\lambda = -50\ (1/1+e^{-100x}+1)$.

Problem 3: Burden and Faires (2004):

$$\begin{aligned} &y_1' = 32y_1+66y_2+\frac{2}{3}x+\frac{2}{3},\ \ y_2' = -66y_1-133y_2-\frac{1}{3}x-\frac{1}{3}, \\ &y_1(0) = \frac{1}{3},\ y_2(0) = \frac{1}{3},\ 0\le x\le 1 \end{aligned} \quad (31)$$

The exact solutions are: $y_1(x) = 2/3x+2/3e^{-x}-1/3e^{-100x}$, $y_2(x) = -1/3x-1/3e^{-x}+2/3e^{-100x}$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -100$.

Table 1: The numerical results for problem 1-3

Problems/h	Methods	TS	MAXE	Time
10^{-3}	2P4BBDF	5,000	1.00128e-02	-
	BBDF(5)	5,000	2.56073e-04	6.53216e-05
	I2BBDF(5)	5,000	2.37551e-04	2.81388e-05
10^{-5}	2P4BBDF	500,000	1.03537e-04	-
	BBDF(5)	500,000	2.69869e-08	3.59375e-03
	I2BBDF(5)	500,000	2.50500e-08	8.22420e-04
10^{-7}	2P4BBDF	50,000,000	1.03571e-06	-
	BBDF(5)	50,000,000	3.60687e-10	1.39051e-01
	I2BBDF(5)	50,000,000	1.92962e-10	5.43618e-02
10^{-3}	2P4BBDF	500	2.82446e-02	-
	BBDF(5)	500	4.88893e-03	1.90019e-05
	I2BBDF(5)	500	4.50402e-03	4.44133e-06
10^{-5}	2P4BBDF	50,000	3.65253e-04	-
	BBDF(5)	50,000	7.13439e-07	2.21668e-04
	I2BBDF(5)	50,000	6.62190e-07	4.99637e-05
10^{-7}	2P4BBDF	5,000,000	3.66172e-06	-
	BBDF(5)	5,000,000	7.15947e-11	1.00759e-02
	I2BBDF(5)	5,000,000	6.64568e-11	7.01645e-03
10^{-3}	2P4BBDF	500	4.93545e-02	-
	BBDF(5)	500	1.04842e-02	1.24915e-05
	I2BBDF(5)	500	9.68471e-03	6.43562e-06
10^{-5}	2P4BBDF	50,000	6.88147e-04	-
	BBDF(5)	50,000	1.79049e-06	5.04412e-04
	I2BBDF(5)	50,000	1.66189e-06	4.34397e-04
10^{-7}	2P4BBDF	5,000,000	6.90456e-06	-
	BBDF(5)	5,000,000	2.26481e-10	7.31401e-02
	I2BBDF(5)	5,000,000	1.79400e-10	3.10261e-02

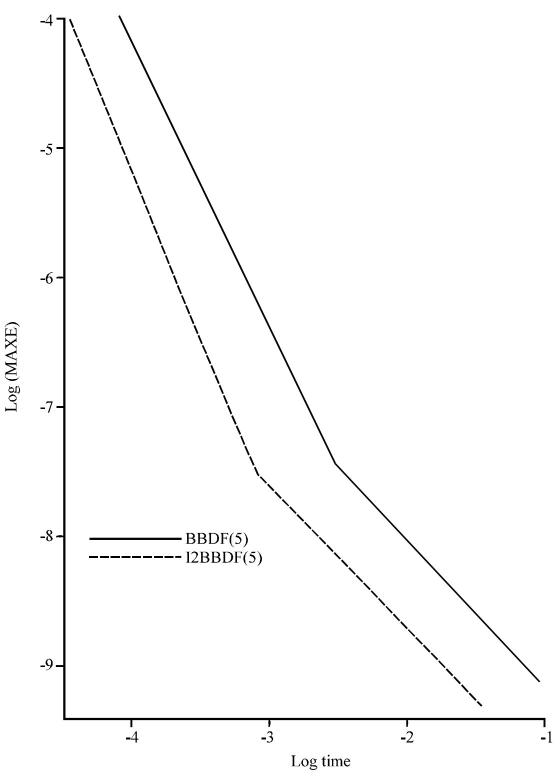

Fig. 2: The efficiency curves for problem 1

Therefore, the selected tested problems can be classified as stiff problems, since, the eigenvalues for all the problems are negative number. Below are the abbreviations that will be used in Table 1:

- h: Step size used
- 2P4BBDF: 2-point implicit block method with an off-stage function
- BBDF(5): Fifth order of Block Backward Differentiation Formula
- I2BBDF(5): New 2-point Block Backward Differentiation Formula of fifth order derived in this study
- TS: Total Steps taken
- MAXE : Maximum Error
- Time: Execution time in seconds

All the tested problems are solved using C code programming with h = 10^{-3}, 10^{-5} and 10^{-7}. The results are tabulated in Table 1.

In addition, the efficiency curves are plotted to show the effectiveness of the method when compare with the existing methods. The graphs of log MAXE against log time are given in Fig. 2-4. The numerical results presented in the previous section are discussed. Referring to Table 1, we observed that the I2BBDF(5) method give a better accuracy, MAXE when compared with the 2P4BBDF method. It is because the value of ρ = -7/8 considered in the derived method compute the solutions accurately as compared to the value of ζ = -1/4 in the 2P4BBDF method. For the BBDF(5) method, the MAXE produced are comparable but still the I2BBDF(5) method outperformed the results in terms of execution time. This is due to the extra future point considered in the new method. Based on Fig. 2-4, the figures demonstrate the efficiency curves for the I2BBDF(5) and the BBDF(5)

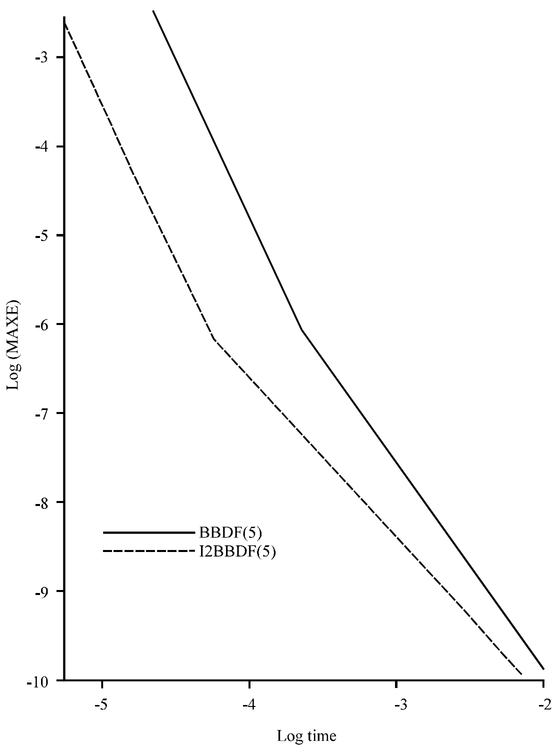

Fig. 3: The efficiency curves for problem 2

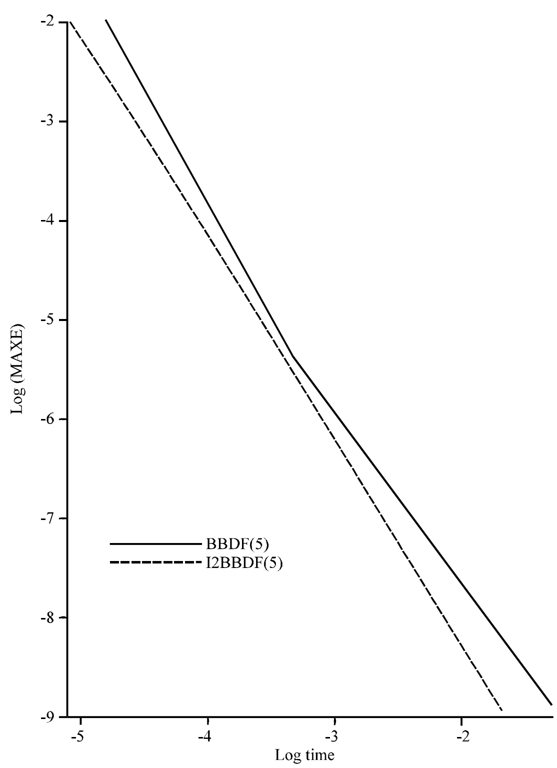

Fig. 4: The efficiency curves for problem 3

methods only. This is because the execution time for the 2P4BBDF method cannot be computed, since, the code is not available. From the graphs it is clear that the I2BBDF(5) method converge faster than the BBDF(5) method for all tested problems.

CONCLUSION

In this study, we have proposed an implicit 2-point block method that can approximate the solution at two points a time called improved 2-point Block Backward Differentiation Formula of fifth order (I2BBDF(5)) method to solve stiff ODEs. Stability of the method is proved to be A-stable indicate that the method is suitable for solving first order stiff ODEs. One of the most important benefits of using the new method over the methods of comparison is that its effectiveness in increasing the accuracy of the approximation and reducing the number of integration steps. Numerical results suggest that the method I2BBDF(5) is recommended as an alternative solver for solving stiff ODEs.

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