Journal of Engineering and Applied Sciences 14 (23): 8616-8622, 2019
ISSN: 1816-949X

Stability and Convergence for Solving Two-Sided Fractional Percolation Equation

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Abstract: This study suggested a new Two-Sided Fractional Percolation Equation (T-SFPE) with coefficients. The algorithm for the numerical solution for this equation is based on Finite Difference Method (FDM). Stability, consistency and convergence of the fractional order numerical method are discussed. The numerical method has been applied to solve practical numerical examples and comparing results with the exact solution. Numerical experiments illustrating the effectiveness of the theoretical analysis are provided.

Key words: Fractional derivative, Explicit Finite Difference Method (EFDM), Fractional Percolation Equation (FPE), convergence of numerical method, stability, theoretical

INTRODUCTION

Recently, many engineering, mathematical and physical phenomena described successfully by utilizing fractional calculus, integrals theory and non-integer orders derivatives (Kilbas *et al*., 2006; Wei and He, 2014; Abbas, 2015; Yu *et al*., 2013; Hajji *et al*., 2014; Zhao *et al*., 2015; Kirchner *et al*., 2000a, b; Magin, 2006; Podlubny, 1998).

In various fields of physics, geology, biology, chemistry, finance (Sokolov *et al*., 2002; Benson *et al*., 2000a, b; Magin, 2006; Kirchnern *et al*., 2000; Raberto *et al*., 2002) and are frequently modeled through fractional differential equations.

To obtain numerical solutions of Fractional Differential Equations (FDEs), finite element methods introduced by Ma *et al*. (2014), Jiang and Ma (2011) and Liu *et al*. (2014) because most FDEs do not have exact analytic solutions. So that, numerical and approximation techniques must be offered and developed.

Also, based on FDMs by Chen *et al*. (2014), Vong and Wang (2015) and Wang and Du (2014), the numerical treatments have been developed. FPE is a mathematical model is a mathematical model in porous media for fluid dynamics and in groundwater for the seepage flow problems (Wang and Du, 2014; Thusyanthan and Madabhushi, 2003). FPE is a partial differential equation acquired from the equation of conventional percolation (Bear, 1972) by replacing the space integer derivatives by the space fractional derivatives. In addition, a 3D FPE is suggested by He (1998).

Recently, by many researchers, some numerical methods considered. Chen *et al*. (2011) suggested Implicit Finite Difference Method (IFDM) for the one-dimensional (1D) FPE.

Also, Guo *et al*. (2016a, b) suggested an implicit FDM for 1D FPE with Dirichlet and fractional boundary conditions. Chen *et al*. (2013) proposed an alternating direction implicit difference method for 2D case.

Moreover, Chen *et al*. proposed 2D variable-order FPE. Liu *et al*. (2014) suggested two FDMs for the 3D non-continued seepage flow problem. Guo *et al*. (2016a, b) and Liu *et al*. (2009) discussed a second order FDM for the 2D FPE. In this study, we investigate the FPE numerical solutions.

Fractional percolation model: We consider the following (T-SFPM):

$$\begin{aligned}\frac{\partial K}{\partial t} = {} & \frac{\partial^{\beta_1}}{\partial +x^{\beta_1}}\left(M_x\frac{\partial^{\alpha_1}K}{\partial +x^{\alpha_1}}\right)+\frac{\partial^{\beta_1}}{\partial -x^{\beta_1}}\left(M_x\frac{\partial^{\alpha_1}K}{\partial -x^{\alpha_1}}\right)+\frac{\partial^{\beta_2}}{\partial +y^{\beta_2}}\left(M_y\frac{\partial^{\alpha_2}K}{\partial +y^{\alpha_2}}\right)+\frac{\partial^{\beta_2}}{\partial -y^{\beta_2}}\left(M_y\frac{\partial^{\alpha_2}K}{\partial -y^{\alpha_2}}\right)+ \\ & \frac{\partial^{\beta_3}}{\partial +z^{\beta_3}}\left(M_z\frac{\partial^{\alpha_3}K}{\partial +z^{\alpha_3}}\right)+\frac{\partial^{\beta_3}}{\partial -z^{\beta_3}}\left(M_z\frac{\partial^{\alpha_3}K}{\partial -z^{\alpha_3}}\right)+f(x,y,z,t)\end{aligned} \quad (1)$$

Subject to the Initial Condition (IC):

$$K(x,y,z,0)=\varphi(x,y,z)$$

And the Boundary Conditions (BC):

$$\begin{aligned}&K(x_0,y,z,t)=K(x,y_0,z,t)=K(x,y,z_0,t)=0\\&K(x_R,y,z,t)=\Psi_1(y,z,t),(x,y_R,z,t)=\Psi_2(x,z,t),\\&K(x,y,z_R,t)=\Psi_3(x,y,t)\end{aligned}$$

Where (x, y, z)• • , $x_0<x<x_R$, $y_0<y<y_R$, $z_0<z<z_R$, 0• t• T and 0<• $_i$, • $_i$• 1, i = 1, 2, 3. K = (x, ,y, z, t) is the pressure. M_x, M_y and M_z are the percolation coefficients along the x- to z- directions, respectively. The • $_1$ is a known function of y, z and t. • $_2$ is a known function of x, z and t. • $_3$ is a known function of x, y and t. f(x, y, z, t) is the source term. • is the percolation domain. The time fractional derivative is the • $_i$, i = 1, 2, 3 order shifted Grunwald estimate are defined as:

$$\begin{aligned}&\frac{\partial^{\alpha_1}K(x,y,z,t)}{\partial+x^{\alpha_1}}=\frac{1}{(\Delta x)^{\alpha_1}}\sum_{p=0}^{i+1}g_{\alpha_1,p}K^s_{i-p+1,j,f}+0(\Delta x),\ \frac{\partial^{\alpha_2}K(x,y,z,t)}{\partial+y^{\alpha_2}}=\frac{1}{(\Delta y)^{\alpha_2}}\sum_{p=0}^{j+1}g_{\alpha_2,p}K^s_{i,j-p+1,f}+0(\Delta y)\\&\frac{\partial^{\alpha_3}K(x,y,z,t)}{\partial+z^{\alpha_3}}=\frac{1}{\Delta z^{\alpha_3}}\sum_{p=0}^{f+1}g_{\alpha_3,p}K^s_{i,j,f-p+1}+0(\Delta z),\ \frac{\partial^{\alpha_1}K(x,y,z,t)}{\partial\text{-}x^{\alpha_1}}=\frac{1}{(\Delta x)^{\alpha_1}}\sum_{p=0}^{n-i+1}g_{\alpha_1,p}K^s_{i-p+1,j,f}+0(\Delta x)\\&\frac{\partial^{\alpha_2}K(x,y,z,t)}{\partial\text{-}y^{\alpha_2}}=\frac{1}{(\Delta y)^{\alpha_2}}\sum_{p=0}^{m-j+1}g_{\alpha_2,p}K^s_{i,j-p+1,f}+0(\Delta y),\ \frac{\partial^{\alpha_3}K(x,y,z,t)}{\partial\text{-}z^{\alpha_3}}=\frac{1}{\Delta z^{\alpha_3}}\sum_{p=0}^{p-f+1}g_{\alpha_3,p}K^s_{i,j,f-p+1}+0(\Delta z)\end{aligned}\tag{2}$$

MATERIALS AND METHODS

The propose method and its consistency: We discuss proposed EDM and consistency for solving T-SFPE, Eq. (1) can be rewritten as the following form:

$$\begin{aligned}\frac{\partial K}{\partial t}=&\frac{\partial^{\beta_1}M_x}{\partial+x^{\beta_1}}\left(\frac{\partial^{\alpha_1}K}{\partial+x^{\alpha_1}}\right)+M_x\left(\frac{\partial^{\alpha_1+\beta_1}K}{\partial+x^{\alpha_1+\beta_1}}\right)+\frac{\partial^{\beta_1}M_x}{\partial\text{-}x^{\beta_1}}\left(\frac{\partial^{\alpha_1}K}{\partial\text{-}x^{\alpha_1}}\right)+M_x\left(\frac{\partial^{\alpha_1+\beta_1}K}{\partial\text{-}x^{\alpha_1+\beta_1}}\right)+\frac{\partial^{\beta_2}M_y}{\partial+y^{\beta_2}}\left(\frac{\partial^{\alpha_2}K}{\partial+y^{\alpha_2}}\right)+M_y\left(\frac{\partial^{\alpha_2+\beta_2}K}{\partial+y^{\alpha_2+\beta_2}}\right)+\\&\frac{\partial^{\beta_2}M_y}{\partial\text{-}y^{\beta_2}}\left(\frac{\partial^{\alpha_2}K}{\partial\text{-}y^{\alpha_2}}\right)+M_y\left(\frac{\partial^{\alpha_2+\beta_2}K}{\partial\text{-}y^{\alpha_2+\beta_2}}\right)+\frac{\partial^{\beta_3}M_z}{\partial+z^{\beta_3}}\left(\frac{\partial^{\alpha_3}K}{\partial+z^{\alpha_3}}\right)+M_z\left(\frac{\partial^{\alpha_3+\beta_3}K}{\partial+z^{\alpha_3+\beta_3}}\right)+\frac{\partial^{\beta_3}M_z}{\partial\text{-}z^{\beta_3}}\left(\frac{\partial^{\alpha_3}K}{\partial\text{-}z^{\alpha_3}}\right)+M_z\left(\frac{\partial^{\alpha_3+\beta_3}K}{\partial\text{-}z^{\alpha_3+\beta_3}}\right)+f(x,y,z,t)\end{aligned}\tag{3}$$

For the derivation of the EDM for T-SV-OFOE with variable coefficients, first, we construct a computational uniform grid by $x_i = x_0$+i• x where, • x = $(x_R$-$x_0)/n$ for i = 0, 1, …, n and the grid $y_j = y_0$+j• y where for j = 0, 1, …, m also the grid points $z_g = z_0$+f• z where, • z$(z_R$-$z_0)/p$ for f = 0, 1, …, p.

And the grid points t_s = s• t where • t = T/M for, s = 0, ..., M. Similarly, we define $K_{i,j,f}{}^s$ = K (x_i, y_j, z_f, t_s), $q_{i,j,f}{}^s$ = q (x_i, y_j, z_f, t_s), • $_{i,j,f}{}^s$ = • (x_i, y_j, z_f), • $_{j,f}{}^s$ = • (y_j, z_f, t_s), • $_{i,f}{}^s$ = • (x_i, z_f, t_s) and • $_{i,j}{}^s$ = • (x_i, y_j, t_s). Also from the IC and BC one can get:

$$K^0_{i,j,f}=\varphi_{i,j,f},\ K^s_{0,j,f}=K^s_{i,0,f}=K^s_{i,j,0}=0$$

$$K^s_{R,j,f}=\psi^s_{j,f},K^s_{i,R,f}=\psi^s_{i,f},K^s_{i,j,R}=\psi^s_{i,j}$$

Where 0• i• n,0• j• m, 0• f• v and s>0. Now, we apply first order time derivative given by:

$$\left.\frac{\partial K}{\partial t}\right|_{(x_i,y_j,z_f,t_n)}\sim\frac{K(x_i,y_j,z_f,t_{s+1})-K(x_i,y_j,z_f,t_s)}{\Delta t}+0(\Delta t)$$

to approximate Eq. 3. Moreover, the mixed fractional derivatives in Eq. 3 using the Dirichlet boundary condition and Theorem 1 by Chen *et al*. (2011) can be described as:

$$\begin{aligned}&\left.\frac{\partial^{\beta_1}M_x}{\partial+x^{\beta_1}}\left(\frac{\partial^{\alpha_1}K}{\partial+x^{\alpha_1}}\right)\right|_{(x_i,y_j,z_f,t_n)}+\left.M_x\left(\frac{\partial^{\alpha_1+\beta_1}K}{\partial+x^{\alpha_1+\beta_1}}\right)\right|_{(x_i,y_j,z_f,t_n)}=\dot{M}_x\left(\frac{1}{\Delta x^{\alpha_1}}\sum_{p=0}^{i+1}g_{\alpha_1,p}K^s_{i-p+1,j,f}\right)+\frac{M_x}{\Delta x^{\beta_1+\alpha_1}}\sum_{p=0}^{i+1}g_{\beta_1+\alpha_1,p}K^s_{i-p+1,j,f}\\&\left.\frac{\partial^{\beta_1}M_x}{\partial\text{-}x^{\beta_1}}\left(\frac{\partial^{\alpha_1}K}{\partial\text{-}x^{\alpha_1}}\right)\right|_{(x_i,y_j,z_f,t_n)}+\left.M_x\left(\frac{\partial^{\alpha_1+\beta_1}K}{\partial\text{-}x^{\alpha_1+\beta_1}}\right)\right|_{(x_i,y_j,z_f,t_n)}=\dot{M}_x\left(\frac{1}{\Delta x^{\alpha_1}}\sum_{p=0}^{n-i+1}g_{\alpha_1,p}K^s_{i-p+1,j,f}\right)+\frac{M_x}{\Delta x^{\beta_1+\alpha_1}}\sum_{p=0}^{n-i+1}g_{\beta_1+\alpha_1,p}K^s_{i-p+1,j,f}\end{aligned}$$

$$\frac{\partial^{\beta_2} M_y}{\partial + y^{\beta_2}}\left(\frac{\partial^{\alpha_2} K}{\partial + y^{\alpha_2}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} + M_y\left(\frac{\partial^{\alpha_2+\beta_2} K}{\partial + y^{\alpha_2+\beta_2}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} = \dot{M}_y\left(\frac{1}{\Delta y^{\alpha_2}}\sum_{p=0}^{j+1} g_{\alpha_2,p} K^s_{i,j-p+1,f}\right) + \frac{M_y}{\Delta y^{\beta_1+\alpha_1}}\sum_{p=0}^{j+1} g_{\beta_2+\alpha_2,p} K^s_{i,j-p+1,f}$$

$$\frac{\partial^{\beta_2} M_y}{\partial - y^{\beta_2}}\left(\frac{\partial^{\alpha_2} K}{\partial - y^{\alpha_2}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} + M_y\left(\frac{\partial^{\alpha_2+\beta_2} K}{\partial - y^{\alpha_2+\beta_2}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} = \dot{M}_y\left(\frac{1}{\Delta y^{\alpha_2}}\sum_{p=0}^{m-j+1} g_{\alpha_2,p} K^s_{i,j-p+1,f}\right) + \frac{M_y}{\Delta y^{\beta_1+\alpha_1}}\sum_{p=0}^{m-j+1} g_{\beta_2+\alpha_2,p} K^s_{i,j-p+1,f}$$

$$\frac{\partial^{\beta_3} M_z}{\partial + z^{\beta_3}}\left(\frac{\partial^{\alpha_3} K}{\partial + z^{\alpha_3}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} + M_z\left(\frac{\partial^{\alpha_3+\beta_3} K}{\partial + z^{\alpha_3+\beta_3}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} = \dot{M}_z\left(\frac{1}{\Delta z^{\alpha_3}}\sum_{p=0}^{f+1} g_{\alpha_3,p} K^s_{i,j,f-p+1}\right) + \frac{M_z}{\Delta z^{\beta_1+\alpha_1}}\sum_{p=0}^{f+1} g_{\beta_3+\alpha_3,p} K^s_{i,j,f-p+1}$$

$$\frac{\partial^{\beta_3} M_z}{\partial - z^{\beta_3}}\left(\frac{\partial^{\alpha_3} K}{\partial - z^{\alpha_3}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} + M_z\left(\frac{\partial^{\alpha_3+\beta_3} K}{\partial - z^{\alpha_3+\beta_3}}\right)\Bigg|_{(x_i, y_j, z_f, t_n)} = \dot{M}_z\left(\frac{1}{\Delta z^{\alpha_3}}\sum_{p=0}^{v-j+1} g_{\alpha_3,p} K^s_{i,j,f-p+1}\right) + \frac{M_z}{\Delta z^{\beta_1+\alpha_1}}\sum_{p=0}^{v-j+1} g_{\beta_3+\alpha_3,p} K^s_{i,j,f-p+1}$$

Now, we evaluate Eq. 3 at (x_i, y_j, z_f, t_s) and use EFDM to get $K^{s+1}_{i,j,f}$ to give:

$$\begin{aligned}K^{s+1}_{i,j,f} = {} & \dot{M}_x\left(\frac{\Delta t}{\Delta x^{\alpha_1}}\sum_{p=0}^{i+1} g_{\alpha_1,p} K^s_{i-p+1,j,f}\right) + M_x\left(\frac{\Delta t}{\Delta x^{\beta_1+\alpha_1}}\sum_{p=0}^{i+1} g_{\beta_1,\alpha_z,p} K^s_{i-p+1,j,f}\right) + \dot{M}_x\left(\frac{\Delta t}{\Delta x^{\alpha_1}}\sum_{p=0}^{n-i+1} g_{\alpha_1,p} K^s_{i-p+1,j,f}\right) + \\ & M_x\left(\frac{\Delta t}{\Delta x^{\beta_1+\alpha_1}}\sum_{p=0}^{n-i+1} g_{\beta_1,\alpha_1,p} K^s_{i-p+1,j,f}\right) + \dot{M}_y\left(\frac{\Delta t}{\Delta y^{\alpha_2}}\sum_{p=0}^{j+1} g_{\alpha_2,p} K^s_{i,j-p+1,f}\right) + M_y\left(\frac{\Delta t1}{\Delta y^{\beta_1+\alpha_1}}\sum_{p=0}^{j+1} g_{\beta_2,\alpha_2,p} K^s_{i,j-p+1,f}\right) + \\ & \dot{M}_y\left(\frac{\Delta t}{\Delta y^{\alpha_2}}\sum_{p=0}^{m-j+1} g_{\alpha_2,p} K^s_{i,j-p+1,f}\right) + M_y\left(\frac{\Delta t}{\Delta y^{\beta_1+\alpha_1}}\sum_{p=0}^{m-j+1} g_{\beta_2,\alpha_2,p} K^s_{i,j-p+1,f}\right) + \dot{M}_z\left(\frac{\Delta t}{\Delta z^{\alpha_3}}\sum_{p=0}^{f+1} g_{\alpha_3,p} K^s_{i,j,f-p+1}\right) + \\ & M_z\left(\frac{\Delta t}{\Delta z^{\beta_1+\alpha_1}}\sum_{p=0}^{f+1} g_{\beta_3,\alpha_3,p} K^s_{i,j,f-p+1}\right) + \dot{M}_z\left(\frac{\Delta t}{\Delta z^{\alpha_3}}\sum_{p=0}^{v-j+1} g_{\alpha_3,p} K^s_{i,j,f-p+1}\right) + M_z\left(\frac{\Delta t}{\Delta z^{\beta_1+\alpha_1}}\sum_{p=0}^{v-j+1} g_{\beta_3,\alpha_3,p} K^s_{i,j,f-p+1}\right) + K^S_{i,j,f} + \Delta t q^s_{i,j,f}\end{aligned} \tag{4}$$

$$g\beta_d + \alpha_{d,r} = (-1)^r \frac{(\beta_d+\alpha_d)(\beta_d+\alpha_d-1), ..., (\beta_d+\alpha_d-r+1)}{r!},\ d = 1,2,3,\ r = 0,1,2,... \tag{5}$$

Equation 4 is consistent with order O(• t)+O(• x)+O(• z) proposed method. We next define the following difference operators:

$$\bar{\varpi}_{\alpha_1,x} K^s_{i,j,f} = \frac{\dot{M}_x}{\Delta x^{\alpha_1}}\left(\sum_{p=0}^{i+1} g_{\alpha_1,p} K^s_{i-p+1,j,f} + \sum_{p=0}^{n-i+1} g_{\alpha_1,p} K^s_{i+p-1,j,f}\right)$$

$$\varpi_{\beta_{1+}+\alpha_1,x} K^s_{i,j,f} = \frac{M_x}{\Delta x^{\beta_1+\alpha_1}}\left(\sum_{p=0}^{i+1} g_{\beta_1+\alpha_1,p} K^s_{i-p+1,j,f} + \sum_{p=0}^{n-i+1} g_{\beta_1+\alpha_1,p} K^s_{i+p-1,j,f}\right)$$

which is of O(• x) approximation to the • th fractiodznal derivative. Also:

$$\bar{\varpi}_{\alpha_2,y} K^s_{i,j,f} = \frac{\dot{M}_y}{\Delta y^{\alpha_2}}\left(\sum_{p=0}^{i+1} g_{\alpha_2,p} K^s_{i,j-p+1,f} + \sum_{p=0}^{m-j+1} g_{\alpha_2,p} K^s_{i,j+p-1,f}\right)$$

$$\varpi_{\beta_2+\alpha_2,y} K^s_{i,j,f} = \frac{M_y}{\Delta y^{\beta_2+\alpha_2}}\left(\sum_{p=0}^{j+1} g_{\beta_2+\alpha_2,p} K^s_{i,j-p+1,f} + \sum_{p=0}^{m-j+1} g_{\beta_2+\alpha_2,p} K^s_{i,j+p-1,f}\right)$$

$$\bar{\varpi}_{\alpha_3,z} K^s_{i,j,f} = \frac{\dot{M}_z}{\Delta z^{\alpha_3}}\left(\sum_{p=0}^{f+1} g_{\alpha_3,p} K^s_{i,j,f-p+1} + \sum_{p=0}^{v-j+1} g_{\alpha_3,p} K^s_{i,j,f+p-1}\right)$$

$$\varpi_{\beta_3+\alpha_3,z} K^s_{i,j,f} = \frac{M_z}{\Delta z^{\beta_3+\alpha_3}}\left(\sum_{p=0}^{f+1} g_{\beta_3+\alpha_3,p} K^s_{i,j,f-p+1} + \sum_{p=0}^{v-j+1} g_{\beta_3+\alpha_3,p} K^s_{i,j,f+p-1}\right)$$

are of O(• y) and O(• z) approximation of the • and •-order Grunwald shifted fractional derivatives term, respectively. The operator form of the explicit method can be written by:

$$K_{i,j,f}^{s+1} = \left(1+\Delta t\tilde{\varpi}_{\alpha_1,x}+\Delta t\varpi_{\beta_1+\alpha_1,x}+\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}+\Delta t\tilde{\varpi}_{\alpha_3,z}+\Delta t\varpi_{\beta_3+\alpha_3,z}\right)K_{i,j,f}^{s}+\Delta t q_{i,j,f}^{s} \quad (6)$$

Equation 6 may be written in form:

$$K_{i,j,f}^{s+1} = \left(1+\Delta t\tilde{\varpi}_{\alpha_1,x}+\Delta t\varpi_{\beta_1+\alpha_1,x}\right)\left(1+\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}\right)\left(1+\Delta t\tilde{\varpi}_{\alpha_3,z}+\Delta t\varpi_{\beta_3+\alpha_3,z}\right)K_{i,j,f}^{s}+\Delta t q_{i,j,f}^{s} \quad (7)$$

which introduces an additional perturbation error equal to:

$$\left[\Delta t\tilde{\varpi}_{\alpha_1,x}\left(\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}\right)+\Delta t\varpi_{\beta_1+\alpha_1,x}\left(\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}\right)\right]K_{i,j,f}^{s}+$$
$$\begin{bmatrix}\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}+\Delta t\tilde{\varpi}_{\alpha_1,x}+\Delta t\tilde{\varpi}_{\alpha_1,x}\left(\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}\right)+\\ \Delta t\varpi_{\beta_1+\alpha_1,x}+\Delta t\varpi_{\beta_1+\alpha_1,x}\left(\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}\right)\end{bmatrix}\left(\Delta t\tilde{\varpi}_{\alpha_3,z}+\Delta t\varpi_{\beta_3+\alpha_3,z}\right)K_{i,j,f}^{s}$$

Hence, we obtain the following fractional explicit scheme at time t_s:

$$K_{i,j,f}^{s+1} = \left(1+\Delta t\tilde{\varpi}_{\alpha_1,x}+\Delta t\varpi_{\beta_1+\alpha_1,x}\right)K_{i,j,f}^{s/3}+\Delta t q_{i,j,f}^{s} \quad (8)$$

$$K_{i,j,f}^{s/3} = \left(1+\Delta t\tilde{\varpi}_{\alpha_2,y}+\Delta t\varpi_{\beta_2+\alpha_2,y}\right)K_{i,j,f}^{2s/3} \quad (9)$$

and:

$$K_{i,j,f}^{2s/3} = \left(1+\Delta t\tilde{\varpi}_{\alpha_3,z}+\Delta t\varpi_{\beta_3+\alpha_3,z}\right)K_{i,j,f}^{s} \quad (10)$$

Computationally, the explicit method defined for Eq. 7 can be solved using the following steps. At time tn:

- If (y_j, z_f) is fixed, we will obtain an intermediate solution $K_{i,j,f}^{s/3}$ from Eq. 8
- If (x_i, z_f) is fixed, we will obtain an intermediate solution $K_{i,j,f}^{2s/3}$ from Eq. 9
- If (x_i, y_j) is fixed we will obtain an intermediate solution from Eq. 10 using information compiled during the previous step

RESULTS AND DISCUSSION

Stability and convergence of the EFDM: In this study, we discuss the stability and convergence of the EFDM.

Theorem: The explicit system defined by the linear difference Eq. (8-10) with 1<•$_1$+•$_1$, •$_2$+•$_2$, •$_3$+•$_3$• 2 is conditionally stable, if:

$$\alpha_1\left(\frac{M'_{x_{max}}\Delta t}{\Delta x^{\alpha_1}}\right)\le 1-(\beta_1+\alpha_1)\left(\frac{M_{x_{max}}\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right)$$

$$\alpha_2\left(\frac{M'_{y_{max}}\Delta t}{\Delta y^{\alpha_2}}\right)\le 1-(\beta_2+\alpha_2)\left(\frac{M_{y_{max}}\Delta t}{\Delta y^{\beta_2+\alpha_2}}\right)$$

and:

$$\alpha_3\left(\frac{M'_{z_{max}}\Delta t}{\Delta z^{\alpha_3}}\right)\le 1-(\beta_3+\alpha_3)\left(\frac{M_{z_{max}}\Delta t}{\Delta z^{\beta_3+\alpha_3}}\right)$$

Proof: Can be written Eq. 8 in the following matrix form:

$$\underline{K_{p_1,p_2}^{s+1}} = C_{p_1,p_2}\underline{K_{p_1,p_2}^{s/3}}+\Delta t\underline{Q_{p_1,p_2}^{s}}$$

Where:

$$\underline{K_{p_1,p_2}^{s+1}} = \left[K_{1,p_1,p_2}^{s+1},K_{2,p_1,p_2}^{s+1},\ldots,K_{n-1,p_1,p_2}^{s+1}\right]^T$$
$$\underline{K_{p_1,p_2}^{s/3}} = \left[K_{1,p_1,p_2}^{s/3},K_{2,p_1,p_2}^{s/3},\ldots,K_{n-1,p_1,p_2}^{s/3}\right]^T$$
$$\Delta t Q_{p_1,p_2}^{s} = \left[\Delta t q_{1,p_1,p_2}^{s},\Delta t q_{2,p_1,p_2}^{s},\ldots,\Delta t q_{n-1,p_1,p_2}^{s}\right]^T$$

$C_{i,j}$ for i = 1, ..., n-1 and j = 1, ..., n-1 are defined by:

$$C_{i,j}=\begin{cases}1+\delta_{i,p_1,p_2}g_{\alpha_1,1}+\xi_{i,p_1,p_2}g_{\beta_1+\alpha_1,1} & \text{for } j=1\\ \delta_{i,p_1,p_2}g_{\alpha_1,0}+\xi_{i,p_1,p_2}g_{\beta_1+\alpha_1,0} & \text{for } j=i-1\\ \delta_{i,p_1,p_2}g_{\alpha_1,2}+\xi_{i,p_1,p_2}g_{\beta_1+\alpha_1,2} & \text{for } j=i+1\\ \delta_{i,p_1,p_2}g_{\alpha_1,j+1}+\xi_{i,p_1,p_2}g_{\beta_1+\alpha_1,j+1} & \text{for } j<i+1\end{cases}$$

Where the coefficients:

$$\delta_{i,p_1,p_2} = \frac{M'_x\Delta t}{\Delta x^{\alpha_1}}$$

$$\xi_{i,p_1,p_2} = \frac{M_x\Delta t}{\Delta x^{\beta_2+\alpha_1}}$$

To explain this matrix pattern:

$$K^{s+1}_{1,p_1,p_2} = \left(\delta_{1,p_1,p_2} g_{\alpha_1,0} + \xi_{1,p_1,p_2} g_{\beta_1+\alpha_1,0}\right) K^{s/3}_{0,p_1,p_2} + \left(1+\delta_{1,p_1,p_2} g_{\alpha_1,1} + \xi_{1,p_1,p_2} g_{\beta_1+\alpha_1,1}\right) K^{s/3}_{1,p_1,p_2} +$$
$$\left(\delta_{1,p_1,p_2} g_{\alpha_1,2} + \xi_{1,p_1,p_2} g_{\beta_1+\alpha_1,2}\right) K^{s/3}_{2,p_1,p_2} + \ldots + \left(\delta_{1,p_1,p_2} g_{\alpha_1,n} + \xi_{1,p_1,p_2} g_{\beta_1+\alpha_1,n}\right) K^{s/3}_{n,p_1,p_2} + \Delta tq^{s}_{1,p_1,p_2}$$

$$K^{s+1}_{2,p_1,p_2} = \left(\delta_{2,p_1,p_2} g_{\alpha_1,0} + \xi_{2,p_1,p_2} g_{\beta_1+\alpha_1,0}\right) K^{s/3}_{0,p_1,p_2} + \left(1+\delta_{2,p_1,p_2} g_{\alpha_1,1} + \xi_{2,p_1,p_2} g_{\beta_1+\alpha_1,1}\right) K^{s/3}_{1,p_1,p_2} +$$
$$\left(\delta_{2,p_1,p_2} g_{\alpha_1,2} + \xi_{2,p_1,p_2} g_{\beta_1+\alpha_1,2}\right) K^{s/3}_{2,p_1,p_2} + \left(\delta_{2,p_1,p_2} g_{\alpha_1,3} + \xi_{2,p_1,p_2} g_{\beta_1+\alpha_1,3}\right) K^{s/3}_{3,p_1,p_2} + \ldots +$$
$$\left(\delta_{2,p_1,p_2} g_{\alpha_1,n} + \xi_{2,p_1,p_2} g_{\beta_1+\alpha_1,n}\right) K^{s/3}_{n,p_1,p_2} + \Delta tq^{s}_{2,p_1,p_2}$$
$$\vdots$$
$$K^{s+1}_{n-1,p_1,p_2} = \left(\delta_{n-1,p_1,p_2} g_{\alpha_1,0} + \xi_{n-1,p_1,p_2} g_{\beta_1+\alpha_1,0}\right) K^{s/3}_{0,p_1,p_2} + \left(1+\delta_{n-1,p_1,p_2} g_{\alpha_1,1} + \xi_{n-1,p_1,p_2} g_{\beta_1+\alpha_1,1}\right) K^{s/3}_{1,p_1,p_2} +$$
$$\left(\delta_{n-1,p_1,p_2} g_{\alpha_1,2} + \xi_{n-1,p_1,p_2} g_{\beta_1+\alpha_1,2}\right) K^{s/3}_{1,p_1,p_2} + \ldots + \left(\delta_{n-1,p_1,p_2} g_{\alpha_1,n} + \xi_{n-1,p_1,p_2} g_{\beta_1+\alpha_1,n}\right) K^{s/3}_{n,p_1,p_2} + \Delta tq^{s}_{n-1,p_1,p_2}$$

According to theorem by Isaacson and Keller (1966), the union of the circles centered at $C_{i,j}$ with radius $r_i = \bullet_{1=0}^{n}$ $c_{i,1}$. Here, we have:

$$c_{i,j} = 1+\left(\delta_{1,p_1,p_2} g_{\alpha_1,1} + \xi_{1,p_1,p_2} g_{\beta_1+\alpha_1,1}\right) = 1-\alpha_1\delta_{1,p_1,p_2} - (\beta_1+\alpha_1)\xi_{i,p_1,p_2}$$

And:

$$r_i = \sum_{1=0}^{n} c_{i,1} = \delta_{i,p_1,p_2} \sum_{1=0}^{n} g_{\alpha_1,i-j+1} + \xi_{i,p_1,p_2} \sum_{1=0}^{n} g_{\beta_1+\alpha_1,i-j+1} \leq \alpha_1\delta_{1,p_1,p_2} + (\beta_1+\alpha_1)\xi_{i,p_1,p_2}$$

and therefore, $c_{i,j}+r_i \bullet 1$. We also have:

$$c_{i,j}-r_i \geq 1-\alpha_1\delta_{1,p_1,p_2} - (\beta_1+\alpha_1)\xi_{i,p_1,p_2} = 1-\alpha_{1,p_1,p_2} - (\beta_1+\alpha_1)\xi_{i,p_1,p_2} = 1-\alpha_1\left(\frac{\dot{M}_x\Delta t}{\Delta x^{\alpha_1}}\right) - (\beta_1+\alpha_1)\left(\frac{M_x\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right) \geq 1-\alpha_1\left(\frac{M'_{x_{max}}\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right) - (\beta_1+\alpha_1)\left(\frac{M_{x_{max}}\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right)$$

Therefore, at most suffices to have the spectral radius of the matrix C to be one:

$$1-\alpha_1\left(\frac{M'_{x_{max}}\Delta t}{\Delta x^{\alpha_1}}\right) - (\beta_1+\alpha_1)\left(\frac{M_{x_{max}}\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right) \geq -1 \rightarrow \alpha_1\left(\frac{M'_{x_{max}}\Delta t}{\Delta x^{\alpha_1}}\right) + (\beta_1+\alpha_1)\left(\frac{M_{x_{max}}\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right) \leq 1 \rightarrow \alpha_1\left(\frac{M'_{x_{max}}\Delta t}{\Delta x^{\alpha_1}}\right) \leq 1-(\beta_1+\alpha_1)\left(\frac{M_{x_{max}}\Delta t}{\Delta x^{\beta_1+\alpha_1}}\right)$$

Similarly method above, we have the matrix form of Eq. 9:

$$\underline{K^{s/3}_{p_1,p_2}} = \underline{S_{p_1,p_2} K^{2s/3}_{p_1,p_2}}$$

Where:

$$\underline{K^{s/3}_{p_1,p_2}} = \left[K^{s/3}_{p_1,1,p_2}, K^{s/3}_{p_1,2,p_2}, \ldots, K^{s/3}_{p_1,m-1,p_2}\right]^T$$

and:

$$\left[K^{2s/3}_{p_1,1,p_2}, K^{2s/3}_{p_1,2,p_2}, \ldots, K^{2s/3}_{p_1,m-1p_2}\right]^T$$

and $\underline{s_{i,j}}$ is the matrix of coefficients for $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-1$ are defined by:

$$S_{i,j} = \begin{cases} 1+\Omega_{p_1,j,p_2} g_{\alpha_2,1} + \psi_{p_1,j,p_2} g_{\beta_2+\alpha_2,1} & \text{for } j = i \\ \Omega_{p_1,j,p_2} g_{\alpha_2,0} + \psi_{p_1,j,p_2} g_{\beta_2+\alpha_2,0} & \text{for } j = i-1 \\ \Omega_{p_1,j,p_2} g_{\alpha_2,2} + \psi_{p_1,j,p_2} g_{\beta_2+\alpha_2,2} & \text{for } j = i+1 \\ \Omega_{p_1,j,p_2} g_{\alpha_2,j+1} + \psi_{p_1,j,p_2} g_{\beta_2+\alpha_2,j+1} & \text{for } j<i+1 \end{cases}$$

Where the coefficients:

$$\Omega_{p_1,j,p_2} = \frac{M'_y\Delta t}{\Delta y^{\alpha_2}}$$
$$\psi_{p_1,j,p_2} = \frac{M_y\Delta t}{\Delta y^{\beta_2+\alpha_2}}$$

So and in the same way, according to theorem by Isaacson and Keller (1966), we get:

$$\alpha_2\left(\frac{M'_{y_{max}}\Delta t}{\Delta y^{\alpha_2}}\right) \leq 1-(\beta_2+\alpha_2)\left(\frac{M_{y_{max}}\Delta t}{\Delta y^{\beta_2+\alpha_2}}\right)$$

Now, resulting the system of equations defined by Eq. 10 is then defined by:

$$\underline{K^{2s/3}_{p_1,p_2}} = A_{p_1,p_2} K^{s}_{p_1,p_2}$$

where:

$$\underline{K^{2s/3}_{p_1,p_2}} = \left[K^{2s/3}_{p_1,p_2,1}, K^{2s/3}_{p_1,p_2,2}, \ldots, K^{2s/3}_{p_1,p_2,m-1}\right]^T$$

and:

$$\left[K^{s}_{p_1,p_2,1}, K^{s}_{p_1,p_2,2}, \ldots, K^{s}_{p_1,p_2,m-1}\right]^T$$

since, $A_{i,j}$ for i = 1, 2, ..., m-1 and j = 1, ..., m-1 are defined by:

$$A_{i,j} = \begin{cases} 1+\mu_{p_3,p_3,f} g_{\alpha_3,1} + \sigma_{p_3,p_3,f} g_{\beta_3+\alpha_3,1} & \text{for } j=i \\ \mu_{p_3,p_3,f} g_{\alpha_3,0} + \sigma_{p_3,p_3,f} g_{\beta_3+\alpha_3,0} & \text{for } j=i-1 \\ \mu_{p_3,p_3,f} g_{\alpha_3,2} + \sigma_{p_3,p_3,f} g_{\beta_3+\alpha_3,2} & \text{for } j=i+1 \\ \mu_{p_3,p_3,f} g_{\alpha_3,j+1} + \sigma_{p_3,p_3,f} g_{\beta_3+\alpha_3,j+1} & \text{for } j<i+1 \end{cases}$$

where the coefficients:

$$\mu_{p_3,p_3,f} = \frac{\dot{M}_z}{\Delta z^{\alpha_3}}$$

$$\psi_{p_1,p_2,f} = \frac{M_z \Delta t}{\Delta z^{\beta_3+\alpha_3}}$$

So, according to the Greshgorin Theorem (Smith, 1985), we get:

$$\alpha_3\left(\frac{M'_{z_{max}}\Delta t}{\Delta z^{\alpha_3}}\right) \leq 1-(\beta_3+\alpha_3)\left(\frac{M_{z_{max}}\Delta t}{\Delta z^{\beta_3+\alpha_3}}\right)$$

on show above that EFDM is consistent and conditionally stable, then, it converges at the rate O(• x+• y+• z+• t) by Theorem (Smith, 1985).

Numerical simulation and comparison: We present numerical example for solving the T-SFPE by the EFDM given in Section 3.

Example: Consider Eq. 1 with the following BC and IC:

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial + x}\left(M_x \frac{\partial^{0.8} K}{\partial + x^{0.8}}\right) + \frac{\partial}{\partial - x}\left(M_x \frac{\partial^{0.8} K}{\partial - x^{0.8}}\right) + \frac{\partial}{\partial + y}\left(M_y \frac{\partial^{0.8} K}{\partial + y^{0.8}}\right) + \frac{\partial}{\partial - y}\left(M_y \frac{\partial^{0.8} K}{\partial - y^{0.8}}\right) + \frac{\partial}{\partial + z}\left(M_z \frac{\partial^{0.6} K}{\partial + z^{0.6}}\right) + \frac{\partial}{\partial - z}\left(M_z \frac{\partial^{0.6} K}{\partial - z^{0.6}}\right) + f(x,y,z)$$

Where $f(x, y, z) = -e^{-t}([-2\bullet (3.2)/\bullet (2.4)x^{2.4}y^3z^3]+[-2\bullet (3.2)/\bullet (2.4)xy^3z^3(1-x)^{1.4}+[\bullet (3.2)/\bullet (1.4)x^{0.4}y^3z^3(2-x^2)]+[\bullet (3.2)/\bullet (1.4) y^3z^3(1-x)^{0.4}(2-x^2)]+[-2\bullet (3)/\bullet (2.2)x^{3.2}y^{2.2}z^3]+[-2\bullet (3)/\bullet (2.2)x^{3.2}y (1-y)^{1.2}z^3]+[\bullet (3)/\bullet (1.2)x^{3.2}y^{0.2}z^3(2-y^2)]+[\bullet (3)/\bullet (1.2)x^{3.2}(1-y)^{0.2}z^3(2-y^2)]+[-2\bullet (3)/\bullet (2.4)x^{3.2}y^3z^{1.4}]+[-2\bullet (3)/\bullet (2.4)x^{3.2}y^3z(1-z)^{0.4}] +[\bullet (3)/\bullet (1.4)x^{3.2}y^3z^{0.4}(2-z^2)]+\bullet (3)/\bullet (1.4)x^{3.2}y^3(1-z)^{0.1}(2-z^2)+[-2\bullet (3)/\bullet (2.4)x^{3.2}y^3z^{1.4}]+[-2\bullet (3)/\bullet (2.4)x^{3.2}y^3z(1-z)^{0.4}]+[\bullet (3)/\bullet (1.4)x^{3.2}y^3z^{0.4}(2-z^2)]+\bullet (3)/\bullet (1.4)x^{3.2}y^3(1-z)^{0.4}(2-z^2)+[-2 \bullet (3)/\bullet (2.4)x^{3.2}y^3z^{2.4}]+[\bullet (3)/\bullet (1.4)x^{3.2}y^3z^{0.4}(2-z^2)])$.

Subjects to the IC:

$$K(x,y,z,0) = x^{2.2}y^2z^2$$

And Dirichlet BC:

$$K(x,y,z,t) = K(x,0,z,t) = K(x,y,0,t) = 0$$
$$K(1,y,z,t) = e^{-t}y^2z^2, K(x,1,z,t) = e^{-t}y^{2.2}z^2,$$
$$K(x,y,1,t) = e^{-t}x^{2.2}y^2z^2$$

That the exact solution to this problem is:

$$K(x,y,1,t) = e^{-t}x^{2.2}y^2z^2$$

Table 1 shows the numerical solution obtained from the EFDM. This method compares well with the exact analytic solution. The numerical result shown is with time • t = 0.0125, • x = 0.02.

Table 1: The numerical solution of example by using the EFDM for • x = 0.2 and • t = 0.0125

x = y = z	t-test	Numerical solutions	Exact solutions	Errors
0.2	0.0125	8.52 E-5	4.581 E-5	3.939 E-5
0.4	0.0125	4.549 E-3	3.368 E-3	1.181 E-3
0.6	0.0125	0.052	0.042	0.01
0.8	0.0125	0.446	0.252	0.194
0.2	0.0250	4.083 E-4	4.524 E-5	3.631E-4
0.4	0.0250	0.012	3.326 E-3	8.674E-3
0.6	0.0250	0.157	0.041	0.116
0.8	0.0250	0.479	0.245	0.234

CONCLUSION

We have presented EFDM for the T-SFPE. The consistency, stability and convergence of the method has been described and demonstrated. The effectiveness of the method is apparent in the numerical example provided.

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