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Subclass for Higher-Order Derivatives of Multivalent Analytic Functions on Complex Hilbert Space with Some Applications in Fractional Calculus

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Abstract: In the present investigation, we introduce and study a certain subclass for higher-order derivatives of multivalent analytic functions defined on complex Hilbert space. We determine some properties of this class, like, coefficient estimates, radii of starlikeness and convexity and convex combination. Also, we give an applications of the fractional calculus techniques.

Key words: Multivalent functions, higher-order derivatives, Hilbert space, radii of starlikeness and convexity, fractional calculus, coefficient

INTRODUCTION

Let A_p^m indicate the family of all functions f of the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \quad (p<m,\ p,\ m \in N = \{1, 2, ...\}) \tag{1}$$

Which are analytic and multivalent in the open unit disk $U = \{z \in C: |z|<1\}$. Let K_p^m denote the subclass of A_p^m consisting of functions of the form:

$$f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n \quad (a_n \geq 0,\ p<m,\ p,\ m \in N = \{1, 2, ...\}) \tag{2}$$

Upon differentiating both sides of (1.2) α times with respect to z, we obtain (Chen *et al*., 1995):

$$f^{(\alpha)}(z) = \varphi(p, \alpha) z^{p-\alpha} + \sum_{n=m}^{\infty} \varphi(n, \alpha) a_n z^{n-\alpha}$$
$$(p, m \in N; \alpha \in N_0 = N \cup \{0\}; p>\alpha)$$

Where:

$$\varphi(p, \alpha) = \frac{p!}{(p-\alpha)!} = \begin{cases} 1 & (\alpha = 0) \\ p(p-1)\cdots(p-\alpha+1) & (\alpha \neq 0) \end{cases}$$

Several researchers have investigated higher-order derivatives of multivalent functions, see for example (Altıntas, 2007; Irmak and Cho, 2007; Wanas, 2015; Wanas, 2017 and Wanas and Majeed, 2018). A function $f \in A_p^m$ is said to be multivalent starlike of order γ ($0 \leq \gamma < p$) ifit satisfies the condition:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in U)$$

And is said to be multivalent convex of order γ ($0 \leq \gamma < p$) if it satisfies the condition:

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in U)$$

Denote by $S_m^*(p, \gamma)$ and $C_m(p, \gamma)$ the classes of multivalent starlike and multivalent convex functions of γ order, respectively which were introduced by Owa (1992). It is known that (Goodman, 1983) and (Owa, 1985):

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$$f \in C_m\,(p,\gamma) \text{ if and only if } \frac{zf'(z)}{p} \in S_m^*\,(p,\gamma)$$

The classes $S_m^* = S^*$ (p, γ) and C_1 (p, γ) = C (p, γ) were studied by Owa (1985). Let H be a complex Hilbert space and T be a bounded linear operator on H. For a complex analytic function f on the Unit disk U, we denoted f (T), the operator on H defined by the usual Riesz-Dunford integral (Dunford and Schwarz, 1988):

$$f(T) = \frac{1}{2\pi i}\int_C f(z)(zI-T)^{-1}dz$$

Where:

I = The Identity operator on H, C is a positively oriented simple closed rectifiable contour lying in
U = Containing the spectrum σ (T)
T = Interior domain (Fan, 1978)

Also f (T) can be defined by the following series:

$$f(T) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}T^n$$

Which converges in the norm topology (Fan, 1979).

Definition 1.1 (Selvaraj *et al*., 2009): The fractional integral operator of order λ (λ>0) is defined by:

$$D_T^{-\lambda} f(T) = \frac{1}{\Gamma(\lambda)}\int_0^1 \frac{T^{\lambda} f(tT)}{(1+t)^{1-\lambda}}dt$$

where, f is analytic function in a simple connected region of z-plane containing the origin.

Definition 1.2 (Selvaraj *et al*., 2009): The fractional derivative for operator of order λ (0≤λ<1) is defined by:

$$D_T^{\lambda} f(T) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dT}\int_0^1 \frac{T^{1-\lambda} f(tT)}{(1-t)^{\lambda}}dt$$

where, f is analytic in a simply connected region of the z-plane containing the origin. For $f \in K_p^m$, from definitions 1.1 and 1.2 by applying a simple calculation, we get:

$$D_T^{-\lambda} f(T) = \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)}T^{p+\lambda} - \sum_{n=m}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\lambda+1)}a_n T^{n+\lambda} \quad (3)$$

And:

$$D_T^{\lambda} f(T) = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}T^{p-\lambda} - \sum_{n=m}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n T^{n-\lambda} \quad (4)$$

Definition 1.3: A function $f \in K_p^m$ is said to be in the class AK_p^m (α, β, δ, T) if satisfies the inequality:

$$\left\| T^{\alpha} f^{(\alpha+\beta)}(T) - \frac{(p-\beta)!}{(p-\alpha-\beta)!} f^{(\beta)}(T) \right\| < \left\| \delta f^{(\beta)}(T) \right\| \quad (5)$$

where, p ∈ N, α, β ∈ N_0 = N ∪ {0}, α+β<p, δ ∈ C\{0}\and for all operator T with ||T||<1, T≠Ø (Ø denote the zero operator on H).

MATERIALS AND METHODS

Coefficient estimates: In this study, we derive coefficient estimates for the function f to be in the class $AK_p^m(\alpha, \beta, \delta, T)$.

Theorem 2.1: Let $f \in K_p^m$ be defined by (1.2). Then $f \in AK_p^m$ (α, β, δ, T) for all T≠Ø if and only if:

$$\frac{\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!} - \frac{(p-\beta)!}{(p-\alpha-\beta)!} + |\delta|\right]a_n \le |\delta|}{\frac{p!}{(p-\beta)!}} \quad (6)$$

where, p ∈ N, α, β ∈ N_0 = N∪{0}, α+β<p, δ ∈ C\{0}. The result is sharp for the function f given by:

$$f(z) = z^p - \frac{|\delta|\, p!\,(n-\beta)!}{n!\,(p-\beta)!\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!} - \frac{(p-\beta)!}{(p-\alpha-\beta)!} + |\delta|\right]} z^n, \; (n \ge m) \quad (7)$$

Proof: Assume that the inequality (2.1) holds. Then, we get:

$$\left\|T^{\alpha}f^{(\alpha+\beta)}(T)-\frac{(p-\beta)!}{(p-\alpha-\beta)!}f^{(\beta)}(T)\right\|-\|\delta f^{(\beta)}(T)\|=$$

$$\left\|\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}\right]a_nT^{n-\beta}\right\|-\left\|\delta\frac{p!}{(p-\beta)!}T^{p-\beta}-\sum_{n=m}^{\infty}\delta\frac{n!}{(n-\beta)!}a_nT^{n-\beta}\right\|$$

$$\leq\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}\right]a_n\|T\|^{n-\beta}-|\delta|\frac{p!}{(p-\beta)!}\|T\|^{p-\beta}+|\delta|\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}a_n\|T\|^{n-\beta}$$

$$\leq\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]a_n-|\delta|\frac{p!}{(p-\beta)!}\leq 0$$

Therefore, $f\in AK_p^m(\alpha,\beta,\delta,T)$. To show the converse, let $f\in AK_p^m(\alpha,\beta,\delta,T)$. Then:

$$\left\|T^{\alpha}f^{(\alpha+\beta)}(T)-\frac{(p-\beta)!}{(p-\alpha-\beta)!}f^{(\beta)}(T)\right\|<\left\|\delta f^{(\beta)}(T)\right\|$$

Simple calculations gives:

$$\left\|\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}\right]a_nT^{n-\beta}\right\|<$$

$$\left\|\delta\frac{p!}{(p-\beta)!}T^{p-\beta}-\sum_{n=m}^{\infty}\delta\frac{n!}{(n-\beta)!}a_nT^{n-\beta}\right\|$$

Taking T = γl ($0<\gamma<1$) in the above inequality, we have:

$$\frac{\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}\right]a_n\gamma^{n-\beta}}{|\delta|\frac{p!}{(p-\beta)!}\gamma^{p-\beta}-\sum_{n=m}^{\infty}|\delta|\frac{n!}{(n-\beta)!}a_n\gamma^{n-\beta}}<1 \qquad (8)$$

Upon clearing denominator in (2.3) and letting $\gamma\to 1$, we obtain:

$$\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}\right]$$

$$a_n<|\delta|\frac{p!}{(p-\beta)!}-\sum_{n=m}^{\infty}|\delta|\frac{n!}{(n-\beta)!}a_n$$

Or:

$$\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]$$

$$a_n\leq|\delta|\frac{p!}{(p-\beta)!}$$

This completes the proof of the theorem.

Corollary 2.1: If $f\in AK_p^m(\alpha,\beta,\delta,T)$ then:

$$a_n\leq\frac{|\delta|p!(n-\beta)!}{n!(p-\beta)!\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]},(n\geq m) \qquad (9)$$

Radii of starlikeness and convexity

Theorem 3.1: If $f\in AK_p^m(\alpha,\beta,\delta,T)$ then f will be p-valently starlike of order γ ($0\leq\gamma<p$) in the disk $|z|<\gamma_1$ where:

$$\gamma_1=\inf_n\left\{\frac{n!(p-\beta)!(p-\gamma)\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}{|\delta|p!(n-\beta)!(n-\gamma)}\right\}^{\frac{1}{n-p}},$$

$$(n\geq m)$$

The result is sharp for the function f given by (2.2).

Proof: It is enough to show that:

$$\left\|\frac{Tf'(T)}{f(T)}-p\right\|\leq p-\gamma \qquad (10)$$

We have:

$$\left\|\frac{Tf'(T)}{f(T)}-p\right\| \leq \frac{\sum_{n=m}^{\infty}(n-p)a_n\|T\|^{n-p}}{1-\sum_{n=m}^{\infty}a_n\|T\|^{n-p}}$$

Hence, (3.1) will be satisfied if:

$$\sum_{n=m}^{\infty}\left(\frac{n-\gamma}{p-\gamma}\right)a_n\|T\|^{n-p} \leq 1 \quad (11)$$

In view of theorem 2.1, if $f \in AK_p^m(\alpha, \beta, \delta, T)$ then:

$$\sum_{n=m}^{\infty}\frac{n!(p-\beta)!\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}{|\delta|\,p!(n-\beta)!}a_n \leq 1 \quad (12)$$

By making use of (3.3) we observe that (3.2) holds true if:

$$\frac{n-\gamma}{p-\gamma}\|T\|^{n-p} \leq \frac{n!(p-\beta)!\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}{|\delta|\,p!(n-\beta)!}$$

Or equivalently:

$$\|T\| \leq \left\{\frac{n!(p-\beta)!(p-\gamma)\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}{|\delta|\,p!(n-\beta)!(n-\gamma)}\right\}^{\frac{1}{n-p}}$$

This gives the desired result.

Theorem 3.2: If $f \in AK_p^m(\alpha, \beta, \delta, T)$ then f will be p-valently convex of order γ $(0 \leq \gamma < p)$ in the disk $|z| < \gamma_2$ where:

$$\gamma_2 = \inf_n\left\{\frac{pn!(p-\beta)!(p-\gamma)\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}{n|\delta|p!(n-\beta)!(n-\gamma)}\right\}^{\frac{1}{n-p}},$$

$$(n \geq m)$$

The result is sharp for the function f given by (2.2).

Proof: It is enough to show that:

$$\left\|\frac{Tf''(T)}{f'(T)}+1-p\right\| \leq p-\gamma$$

The result follows by application of arguments similar to the proof of theorem 3.1.

RESULTS AND DISCUSSION

Convex combination

Theorem 4.1: The class $AK_p^m(\alpha, \beta, \delta, T)$ is closed under convex combinations.

Proof: For j = 1, 2, ..., let $f_j \in AK_p^m(\alpha, \beta, \delta, T)$ where f_j is given by:

$$f_j(T) = T^p - \sum_{n=m}^{\infty}a_{n,j}T^n$$

Then by (2.1), we obtain:

$$\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]a_{n,j} \leq |\delta|\frac{p!}{(p-\beta)!} \quad (13)$$

For $\sum_{j=1}^{\infty}\mu_j = 1$, $0 \leq \mu_j \leq 1$, the convex combination of f_j may be written as:

$$\sum_{j=1}^{\infty}\mu_j f_j(T) = T^p - \sum_{n=m}^{\infty}\left(\sum_{j=1}^{\infty}\mu_j\, a_{n,j}\right)T^n$$

It follows from (4.1) that:

$$\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\left(\sum_{j=1}^{\infty}\mu_j\, a_{n,j}\right) =$$
$$\sum_{j=1}^{\infty}\mu_j\left(\sum_{n=m}^{\infty}\frac{n!}{(n-\beta)!}\left[\frac{(n-\beta)!}{(n-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]a_{n,j}\right) \leq$$
$$\sum_{j=1}^{\infty}\mu_j|\delta|\frac{p!}{(p-\beta)!} = |\delta|\frac{p!}{(p-\beta)!}$$

Thus:

$$\sum_{j=1}^{\infty}\mu_j f_j(T) \in AK_p^m(\alpha, \beta, \delta, T)$$

Corollary 4.1: The class $AK_p^m(\alpha, \beta, \delta, T)$ is a convex set.

Applications of the fractional calculus

Theorem 5.1: If $f \in AK_p^m(\alpha, \beta, \delta, T)$ then:

$$\left\|D_T^{-\lambda} f(T)\right\| \le \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)}\|T\|^{p+\lambda} \times \left[1+\frac{|\delta| p!(m-\beta)!\Gamma(m+1)\Gamma(p+\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m+\lambda+1)}\|T\|^{m-p}\right] \quad (14)$$

And:

$$\left\|D_T^{-\lambda} f(T)\right\| \ge \frac{\Gamma(p+1)}{\Gamma(p+\lambda+1)}\|T\|^{p+\lambda} \times \left[1-\frac{|\delta| p!(m-\beta)!\Gamma(m+1)\Gamma(p+\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m+\lambda+1)}\|T\|^{m-p}\right] \quad (15)$$

The result is sharp for the function f given by:

$$f(z) = z^p - \frac{|\delta|\, p!\,(m-\beta)!}{m!\,(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]} z^m, \ (p, m \in N) \quad (16)$$

Proof: Let $f \in AK_p^m(\alpha, \beta, \delta, T)$. By (1.3), we deduce that:

$$\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T) = T^p - \sum_{n=m}^{\infty} \frac{\Gamma(n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(n+\lambda+1)} a_n T^n$$

Putting:

$$\psi(n, \lambda) = \frac{\Gamma(n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(n+\lambda+1)} \ (n \ge m, p, m \in N)$$

Then, we obtain:

$$\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T) = T^p - \sum_{n=m}^{\infty} \psi(n, \lambda)\, a_n T^n$$

Since, for $n \ge m$, ψ is a decreasing function of n then we get:

$$0 < \psi(n, \lambda) \le \psi(m, \lambda) = \frac{\Gamma(m+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(m+\lambda+1)} \quad (17)$$

Now, by the application of theorem 2.1 and using (5.4) we find that:

$$\left\|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T)\right\| \le \|T\|^p + \sum_{n=m}^{\infty} \psi(n, \lambda)\, a_n \|T\|^n \le \|T\|^p + \psi(m, \lambda)\|T\|^m \sum_{n=m}^{\infty} a_n \le \|T\|^p + \frac{|\delta| p!(m-\beta)!\Gamma(m+1)\Gamma(p+\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m+\lambda+1)}\|T\|^m$$

Which gives (5.1), we also have:

$$\left\|\frac{\Gamma(p+\lambda+1)}{\Gamma(p+1)} T^{-\lambda} D_T^{-\lambda} f(T)\right\| \ge \|T\|^p - \sum_{n=m}^{\infty} \psi(n, \lambda)\, a_n \|T\|^n \ge \|T\|^p - \psi(m, \lambda)\|T\|^m \sum_{n=m}^{\infty} a_n \ge \|T\|^p - \frac{|\delta| p!(m-\beta)!\Gamma(m+1)\Gamma(p+\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m+\lambda+1)}\|T\|^m$$

Which gives (5.2). By taking $\lambda = 1$ in theorem 5.1, we conclude the following corollary:

Corollary 5.1: If $f \in f \in AK_p^m(\alpha, \beta, \delta, T)$ then:

$$\left\|\int_0^1 Tf(tT)dt\right\| \le \frac{||T||^{p+1}}{p+1}\left[1+\frac{(p+1)|\delta|p!(m-\beta)!}{(m+1)m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}||T||^{m-p}\right]$$

And:

$$\left\|\int_0^1 Tf(tT)dt\right\| \ge \frac{||T||^{p+1}}{p+1}\left[1-\frac{(p+1)|\delta|p!(m-\beta)!}{(m+1)m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}||T||^{m-p}\right]$$

Proof: By definition 1.1 and theorem 5.1 for $\lambda = 1$, we have $D_T^{-\lambda} f(T) = \int_0^1 f(tT)\,dt$, the result is true.

Theorem 5.2: If $f \in AK_p^m(\alpha, \beta, \delta, T)$ then:

$$\left\|D_T^{\lambda} f(T)\right\| \le \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}||T||^{p-\lambda} \times \left[1+\frac{|\delta|p!(m-\beta)!\Gamma(m+1)\Gamma(p-\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m-\lambda+1)}||T||^{m-p}\right] \quad (18)$$

And

$$\left\|D_T^{\lambda} f(T)\right\| \ge \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}||T||^{p-\lambda} \times : \left[1-\frac{|\delta|p!(m-\beta)!\Gamma(m+1)\Gamma(p-\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m-\lambda+1)}||T||^{m-p}\right] \quad (19)$$

The result is sharp for the function f given by (5.3).

Proof: Let $f \in AK_p^m(\alpha, \beta, \delta, T)$. By (1.4), we deduce:

$$\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)}T^{\lambda}D_T^{\lambda}f(T) = T^p - \sum_{n=m}^{\infty}\frac{\Gamma(n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n-\lambda+1)}a_nT^n = T^p - \sum_{n=m}^{\infty}\phi(n,\lambda)a_nT^n$$

Where:

$$\phi(n,\lambda) = \frac{\Gamma(n+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n-\lambda+1)}(n \ge m, p, m \in N)$$

Since, for $n \ge m$, ϕ is a decreasing function of n, thus we have:

$$0<\phi(n,\lambda) \le \phi(m,\lambda) = \frac{\Gamma(m+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(m-\lambda+1)}$$

Also, by using theorem 2.1,we obtain:

$$\sum_{n=m}^{\infty} a_n \le \frac{|\delta|p!(m-\beta)!}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]}$$

Thus:

$$\left\|\frac{\Gamma(p-\lambda+1)}{\Gamma(p+1)}T^{\lambda}D_T^{\lambda}f(T)\right\| \le ||T||^p + \phi(m,\lambda)||T||^m\sum_{n=m}^{\infty}a_n \le ||T||^p + \frac{|\delta|p!(m-\beta)!\Gamma(m+1)\Gamma(p-\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma(m-\lambda+1)}||T||^m$$

Then:

$$\left\|D_T^{\lambda} f(T)\right\| \le \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\|T\|^{p-\lambda} \times \left[1+\frac{|\delta| p!(m-\beta)!\Gamma(m+1)\Gamma(p-\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma\binom{m-\lambda}{+1}}\|T\|^{m-p}\right]$$

And by the same way, we conclude that:

$$\left\|D_T^{\lambda} f(T)\right\| \ge \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\|T\|^{p-\lambda} \times \left[1-\frac{|\delta| p!(m-\beta)!\Gamma(m+1)\Gamma(p-\lambda+1)}{m!(p-\beta)!\left[\frac{(m-\beta)!}{(m-\alpha-\beta)!}-\frac{(p-\beta)!}{(p-\alpha-\beta)!}+|\delta|\right]\Gamma(p+1)\Gamma\binom{m-\lambda}{+1}}\|T\|^{m-p}\right]$$

CONCLUSION

The operators on Hilbert space were considered recently by Xiaopei (1994), Joshi (1998), Chrakim *et al*. (1998), Ghanim and Darus (2008), Selvaraj *et al*. (2009) and Wanas and Jebur (2018).

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