Journal of Engineering and Applied Sciences 15 (5): 1148-1157, 2020
ISSN: 1816-949X
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Stability of Generalized AQCQ Functional Equation in Modular Space

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Abstract: Mixed type functional equation is a further step of development in the broad area of functional equations. Many researchers introduced, various mixed type functional equations like additive-quadratic, quadratic-cubic, quadratic-quartic, additive-quadratic-cubic and so on. But even today notably, we have only one famous mixed type of additive-quadratic-cubic-quartic functional equation. In this study, the researchers made an attempt to introduce such new mixed type of additive-quadratic-cubic-quartic functional equation with its general solution and various stabilities related to Ulam problem in modular space.

Key words: Modular space, functional equations, stabilities related to Ulam problem, Δ_α-condition, Fatou property

INTRODUCTION

For the detailed study on Ulam problem and its recent developments called generalized Hyers-Ulam-Rassias stability, one can refer (Aoki, 1950; Gavruta, 1994; Hyers, 1941; Rassias, 1982; Ravi *et al*., 2008, 2009; Rassias, 1978). In 1950, Nakano (1950) established the modular linear spaces and further developed by many researchers, one can refer (Amemiya, 1957; Koshi and Shimogaki, 1961; Luxemburg, 1959; Musielak, 1983; Orlicz, 1988; Turpin, 1978; Yamamuro, 1959). The definitions related to our main theorem related to modular space can be referred by El-Fassi and Kabbaj (2016), Kim and Shin (2017). In 2015, Bodaghi *et al.* (2015) investigated the stabilities of following mixed type equation:

$$h(3y+z)-5h(2y+z)+h(2y-z)+10h(y+z)-5h(y-z)=10h(z)+4h(2y)-8h(y)$$

For all y, $z\in R$. In 2016, Narasimman *et al.* (2016) introduced the equations quintic and sextic, respectively of the form:

$$p\left[h(py-z)+h(py+z)\right]+h(y-pz)+h(y+pz)=\left(p^4+p^2\right)\left[h(y-z)+h(y+z)\right]+2\left(p^6-p^4-p^2+1\right)h(y)$$

$$h(py-z)+h(py+z)+h(y-pz)+h(y+pz)=\left(p^4+p^2\right)\left[h(y-z)+h(y+z)\right]+2\left(p^6-p^4-p^2+1\right)\left[h(y)+h(z)\right]$$

With $p\in R-\{0, \pm 1\}$ also discussed their various stabilities related to Ulam problem. In 2010, researchers Xu *et al.* (2010) introduced a general mixed AQCQ-functional equation and investigated generalized Ulam-Hyers stability in multi-Banach spaces using fixed point method.

In 2017, researchers Kim and Hong (2017) investigated the alternative stability theorem in a modular space using Δ_3-condition of a modified quadratic equation.

In 2019, researchers Rassias *et al.* (2019) investigated Ulam stability problem in non-Archimedean intuitionistic fuzzy normed spaces of the generalized quartic equation:

$$h(py-z)+h(py+z)+h(y-pz)+h(y+pz)=2p^2\{h(y-z)+h(y+z)\}+2\left(p^2-1\right)^2\{h(y)+h(z)\};p\neq 0,\pm 1$$

Motivation from the above literature, the researchers made an attempt to introduce a new mixed type equation satisfied by $f(x) = x+x^2+x^3+x^4$ of the form:

$$\begin{aligned}&f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)=\left(a+a^2\right)\\&\left[f(x+y)+f(x-y)\right]+2f(ax)-\left(a^2+a-1\right)\\&\left[2f(x)+2f(y)+2f(-y)-f(y)-f(-y)\right]+\\&2f(ay)+2f(-ay)-f(ay)-f(-ay)+\frac{a^2-a}{24}\\&\begin{pmatrix}f(2(x+y))+f(-2(x+y))-4f(x+y)-4f(-(x+y))+\\f(2(x-y))+f(-2(x-y))-4f(x-y)-4f(-(x-y))\end{pmatrix}+\frac{a-a^2}{12}\\&\left(f(2y)+f(-2y)-4f(y)-4f(-y)+f(2x)+f(-2x)-4f(x)-4f(-x)\right)\end{aligned} \quad (1)$$

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For all x, y∈R, a≠0, ±1. Mainly, researchers obtain its general solution and investigate various stabilities concerning Ulam problem in modular spaces.

General solution of (1): additive case

Lemma 2.1: Let X and Y are linear spaces, a mapping f: X→Y is additive and odd if f satisfies:

$$f(ax+y)+f(ax-y)+f(x-ay)+f(x+ay)=(a+a^2)[f(x+y)+f(x-y)]-2(a^2-1)f(x) \quad (2)$$

For all x, y∈X, α≠0, ±1.

Proof: Consider f satisfies Eq. 2. Replacing (x, y) by (0, 0) and (x, 0) in Eq. 2, we get f (0) = 0 and:

$$f(ax)=af(x) \quad (3)$$

Respectively, for all x∈X. Therefore, f is additive function. Let (x, y) = (0, x) in Eq. 2 and by Eq. 3 we reached:

$$f(-x)=-f(x);x\in X \quad (4)$$

Thus f is an odd function.

Theorem 2.2: A function f: X→Y is a solution of Eq. 2 iff A(x) is the diagonal of the additive symmetric map A_1: X→Y such that f is of the form f (x) = A (x) for all x∈X.

Proof: Let f satisfies Eq. 2 when f is additive. We can rewrite Eq. 2 as follows:

$$f(x)+\frac{1}{2(a^2-1)}f(ax+y)+\frac{1}{2(a^2-1)}f(ax-y)+\frac{1}{2(a^2-1)}f(x+ay)+\frac{1}{2(a^2-1)}f(x-ay)-\frac{a+a^2}{2(a^2-1)}f(x+y)-\frac{a+a^2}{2(a^2-1)}f(x-y)=0 \quad (5)$$

For all x, y∈X. Theorems 3.5 and 3.6 in (Xu *et al*., 2012) implies that f is of the form:

$$f(x)=A^1(x)+A^0(x) \quad (6)$$

For all x∈X, $A^0(x)=A^0$ and for i = 1, $A^i(x)$ is the diagonal of the i-additive symmetric map A_i: X^i→Y. We get $A^0(x)=A^0=0$ and f is odd by f (0) = 0 and f (−x) = −f (x), respectively. It follows that $f(x)=A^1(x)$.

Conversely, $A^1(x)$ is the diagonal of the additive symmetric map A_1: X^1 → Y such that $f(x)=A^1(x)$ for all x∈X, from:

$$A^1(x+y)=A^1(x)+A^1(y)$$
$$A^1(rx)=r^1A^1(x);x,y\in X,r\in Q$$

We see that f satisfies Eq. 2 and this completes the proof of theorem 2.2.

General solution of (1): quadratic case

Lemma 3.1: Let X and Y are linear spaces, a mapping f: X→Y is quadratic and even if f satisfies:

$$f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)=f(x+y)+f(x-y)+2a^2\{f(x)+f(y)\} \quad (7)$$

For all x, y∈X, a≠0, ±1

Proof: Assume f satisfies the functional Eq. 7. Letting (x, y) by (0. 0) in Eq. 7, we get f (0) = 0. Setting y = 0 in Eq. 7, we obtain:

$$f(ax)=a^2f(x) \quad (8)$$

For all x∈X. Thus, f is quadratic. Replacing (x, y) by (0, x) in Eq. 7 and by Eq. 8, we get f (−x) = f (x) for all x∈X. Thus, f is an even function.

Theorem 3.2: A function f: X→Y is a solution of the functional Eq. 7 if and only if f is of the form $f(x)=E^2(x)$ for all x∈X where $E^2(x)$ is the diagonal of the 2-additive symmetric map E_2: X^2→Y.

Proof: The functional Eq. 7 can rewrite in the form:

$$f(x)-\frac{1}{2a^2}f(ax+y)-\frac{1}{2a^2}f(ax-y)-\frac{1}{2a^2}f(x+ay)+\frac{1}{2a^2}f(x+y)+\frac{1}{2a^2}f(x-y)-\frac{1}{2a^2}f(x-ay)+f(y)=0 \quad (9)$$

For all x, y∈X. By Xu *et al.* (2012), theorems 3.5 and 3.6, f is a generalized polynomial function of degree at most 2 that is f is of the form:

$$f(x)=E^2(x)+E^1(x)+E^0(x) \quad (10)$$

For all x∈X, where $E^0(x)=E^0$ is an arbitrary element of Y and $E^i(x)$ is the diagonal of the i-additive symmetric map E_i: X^i→Y for i = 1, 2. By f (0) = 0 and f (−x) = f (x) for all x∈X, we get $E^0(x)=E^0=0$ and the function f is even. Thus $E^1(x)=0$. It follows that $f(x)=E^2(x)$.

Conversely, assume that f (x) = E^2 (x) for all x∈X where E^2 (x) is the diagonal of 2-additive symmetric map E_2: X^2→Y from:

$$E^2(x+y)=E^2(x)+2E^{2,2}(x,y)+E^2(y)$$
$$E^2(rx)=r^2E^2(x)$$
$$E^{2,2}(x,ry)=r^2E^{2,2}(x,y), E^{2,2}(rx,y)=r^2E^{2,2}(x,y)$$

For all x, y∈X, r∈Q, we see that f satisfies Eq. 7 which completes the proof of theorem 3.2.

General solution of (1): cubic case

Lemma 4.1: Let X and Y are linear spaces, a mapping f: X→Y is cubic and odd if f satisfies:

$$\begin{aligned}&f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)=\\&(a+a^2)[f(x+y)+f(x-y)]+2(a^3-a^2-a+1)f(x)\end{aligned} \quad (11)$$

for all x, y∈X.

Proof: Consider f satisfies Eq. 11. Replacing (x, y) by (0, 0) and (x, 0) in Eq. 11, we get f (0) = 0. And:

$$f(ax)=a^3f(x) \quad (12)$$

respectively, for all x∈X. Therefore, f is cubic function. Let (x, y) by (0, x) in Eq. 11 and using Eq. 12, we obtain:

$$f(-x)=-f(x); x\in X \quad (13)$$

Thus, f is an odd function.

Theorem 4.2: A function f: X→Y is a solution of Eq. 11 iff C^3 (x) is the diagonal of the 3-additive symmetric map C_3: X^3→Y such that f is of the form f (x) = C^3 (x) for all x ∈X.

Proof: Let f satisfies Eq. 11 when f is cubic. We can rewrite Eq. 11 as follows:

$$\begin{aligned}&f(x)+\frac{1}{2(a^2-1)}f(ax+y)+\frac{1}{2(a^2-1)}f(ax-y)+\\&\frac{1}{2(a^2-1)}f(x+ay)+\frac{1}{2(a^2-1)}f(x-ay)-\\&\frac{a+a^2}{2(a^2-1)}f(x+y)-\frac{a+a^2}{2(a^2-1)}f(x-y)=0\end{aligned} \quad (14)$$

ror all x, y∈X. Theorems 3.5 and 3.6 by Xu *et al.* (2012) implies that f is of the form:

$$f(x)=C^3(x)+C^2(x)+C^1(x)+C^0(x) \quad (15)$$

For all x∈ X where C^0 (x) = C^0 and i = 1, 2, 3, C^i (x) is the diagonal of the i-additive symmetric map C_i: X^i→Y. We get C^0 (x) = C^0 = 0 and f is odd by f (0) = 0 and f (−x) = f (x), respectively. Therefore, C^2(x) = 0. It follows that f (x) = C^3 (x)+C^1 (x). By Eq. 12 and C^n (rx) = r^n C^n (x) for all x∈X and r∈Q, we obtain n^1 C^1 (x) = n^3 C^1 (x). Hence, C^1 (x) = 0 for all x∈X. Therefore, f (x) = C^3 (x).

Conversely, C^3 (x) is the diagonal of the 3-additive symmetric map C_3: X^3→Y such that f (x) = C^3 (x) for all x∈X from:

$$C^3(x+y)=C^3(x)+3C^{2,1}(x,y)+3C^{1,2}(x,y)+C^3(y)$$
$$C^3(rx)=r^3C^3(x), C^{2,1}(x,ry)=r^1C^{2,1}(x,y),$$
$$C^{2,1}(rx,y)=r^2C^{2,1}(x,y), C^{1,2}(x,ry)=r^2C^{1,2}(x,y),$$
$$C^{1,2}(rx,y)=r^1C^{1,2}(x,y); x,y\in X, r\in Q$$

We see that f satisfies Eq. 11 and this completes the proof of theorem 4.2.

General solution of (1): quartic case

Lemma 5.1: Let X and Y are linear spaces, a mapping f: X→Y is quartic and even if f satisfies:

$$\begin{aligned}&f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)=\\&2a^2\{f(x+y)+f(x-y)\}+2(a^4-2a^2+1)\{f(x)+f(y)\}\end{aligned} \quad (16)$$

For all x, y∈X.

Proof: Consider, f satisfies Eq. 16. Assuming (x, y) by (0, 0) in Eq. 16 gives f (0) = 0. Setting y = 0 in Eq. 16 to obtain:

$$f(ax)=a^4f(x) \quad (17)$$

∀x∈X. So, f is quartic. By Eq. 17 and x = 0 in Eq. 16, we arrive f (-y) = f (y) for all y∈X. So, f is even.

Theorem 5.2: f: X→Y is a solution of Eq. 16 if and only if E^4 (x) is the diagonal of symmetric 4-additive map, f (x) = E^4 (x), ∀x ∈ X.

Proof: Rewrite Eq. 16 as:

$$\begin{aligned}&f(x)-\frac{1}{2(a^4-2a^2+1)}f(ax+y)-\frac{1}{2(a^4-2a^2+1)}f(ax-y)-\\&\frac{1}{2(a^4-2a^2+1)}f(x+ay)+\frac{a^2}{a^4-2a^2+1}f(x+y)+\\&\frac{a^2}{a^4-2a^2+1}f(x-y)-\frac{1}{2(a^4-2a^2+1)}f(x-ay)+f(y)=0\end{aligned} \quad (18)$$

$\forall$x, y$\in$X. Therefore, f is follows:

$$f(x)=E^4(x)+E^3(x)+E^2(x)+E^1(x)+E^0(x) \quad (19)$$

for all x$\in$X. As same as theorem 4.2, prove the remaining part of this proof.

Stability of functional Eq. 1: additive case: Assume that the linear space X, μ-complete convex modular space X_μ in the following theorems and corollaries. Now, we obtain the stability of Eq. 1 called generalized Hyers-Ulam-Rassias in modular spaces without Δ_α-condition and the Fatou property. Hereafter, we use the following notation:

$$D_A f(x,y)=f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)-(a+a^2)\left[f(x+y)+f(x-y)\right]+2(a^2-1)f(x), \forall x,y\in X$$

Theorem 6.1: Let a mapping f: X→X_μ satisfies:

$$\mu(D_A f(x,y))\leq \nu(x,y) \quad (20)$$

And a mapping ν : X^2→ [0, ∞) such that:

$$\zeta(x,y)=\sum_{j=0}^{\infty}\frac{\nu(p^j x,p^j y)}{p^j}<\infty, x,y\in X \quad (21)$$

Then there exists A_1: X→X_μ a unique additive mapping defined by $A_1(x) = \lim_{n\to\infty} f(a^n x)/a^n$, x$\in$X which satisfies Eq. 2 and:

$$\mu(f(x)-A_1(x))\leq \frac{1}{2a}\zeta(x,0), \forall x\in X \quad (22)$$

Proof: Substituting y = 0 in Eq. 20, we obtain:

$$\mu(f(ax)-af(x))\leq \frac{1}{2}\nu(x,0) \quad (23)$$

And so:

$$\mu\left(f(x)-\frac{f(ax)}{a}\right)\leq \frac{1}{2a}\nu(x,0), \forall x\in X \quad (24)$$

By induction on n, we arrive:

$$\mu\left(f(x)-\frac{f(a^n y)}{a^n}\right)\leq \frac{1}{2}\sum_{j=0}^{n-1}\frac{\nu(a^j y,0)}{a^{j+1}}, \forall x\in X \quad (25)$$

Substituting x by a^mx in Eq. 25, we obtain:

$$\mu\left(\frac{f(a^m x)}{a^m}-\frac{f(a^{n+m}x)}{a^{n+m}}\right)\leq \frac{1}{2a}\sum_{j=m}^{n+m-1}\frac{\nu(a^j x,0)}{a^j} \quad (26)$$

By assumption Eq. 21 it converges to zero as m→∞. Hence, by inequality Eq. 26 the sequence:

$$\left\{\frac{f(a^n x)}{a^n}\right\}, \forall x\in X$$

is μ-Cauchy and hence, it is convergent in X_μ, since, X_μ is μ-complete. Thus, a mapping A_1: X→X_μ is defined by:

$$A_1(y)=\mu\text{-}\lim_{n\to\infty}\left\{\frac{f(a^n x)}{a^n}\right\}$$

For all x$\in$X which implies:

$$\lim_{n\to\infty}\mu\left(\frac{f(a^n x)}{a^n}-A_1(x)\right)=0, \forall x\in X$$

Next, we claim the mapping A_1 satisfies Eq. 2. Setting (x, y) = (a^n x, a^n y) in Eq. 20 and dividing the resultant by a^n, we arrive:

$$\frac{\mu(D_A f(a^n x,a^n y))}{a^n}\leq\frac{\nu(a^n x,a^n y)}{a^n}, \forall x,y\in X$$

Hence, by property $\mu(\alpha u)\leq\alpha\mu(u)$, $0<\alpha\leq 1$, u$\in X_\mu$, we get:

$$\mu\left(\frac{1}{4a^2+2a+3}DA_1(x,y)\right)\leq\mu\left(\frac{1}{4a^2+2a+3}DA_1(x,y)-\frac{Df(a^n x,a^n y)}{(4a^2+2a+3)a^n}+\frac{Df(a^n x,a^n y)}{(4a^2+2a+3)a^n}\right)\leq$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(ax+y)-\frac{f(a^n(ax+y))}{a^n}\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(ax-y)-\frac{f(a^n(ax-y))}{a^n}\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(x+ay)-\frac{f(a^n(x+ay))}{a^n}\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(x-ay)-\frac{f(a^n(x-ay))}{a^n}\right)+$$

$$\frac{a+a^2}{4a^2+2a+3}\mu\left(A_1(x+y)-\frac{f(a^n(x+y))}{a^n}\right)+$$

$$\frac{a+a^2}{4a^2+2a+3}\mu\left(A_1(x-y)-\frac{f(a^n(x-y))}{a^n}\right)+$$

$$\frac{2(a^2-1)}{4a^2+2a+3}\mu\left(A_1(x)-\frac{f(a^n y)}{a^n}\right)+\frac{1}{4a^2+2a+3}\mu\left(\frac{Df(a^n x,a^n y)}{a^n}\right)$$

For all x, y∈X and n is positive integers. We obtain:

$$\mu\left(\frac{1}{4a^2+2a+3}DA_1(x,y)\right)=0$$

if n→∞. Hence, $DA_1(x, y) = 0$, ∀x, y∈X. Thus, A_1 satisfies Eq. 2 and hence, it is additive. Since:

$$\sum_{i=0}^{n}\frac{1}{a^{i+1}}+\frac{1}{a}\leq 1$$

For all n∈N, by the convexity of modular μ and Eq. 23, we arrive:

$$\begin{aligned}&\mu(f(x)\text{-}A_1(x))=\\&\mu\left(f(x)\text{-}\frac{f(a^n x)}{a^n}\right)+\rho\left(\frac{f(a^n y)}{a^n}\text{-}A_1(x)\right)\leq\\&\frac{1}{2}\sum_{i=0}^{n-1}\frac{1}{a^{i+1}}\phi(a^i x,0)+\rho\left(\frac{f(a^n.x)}{a^n}\text{-}A_1(x)\right)\leq\\&\frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{a^{i+1}}\mu(a^i x,0)=\frac{1}{2a}\zeta(x,0)\end{aligned}$$

for all x∈X. Now, to prove the uniqueness of A_1, we consider that there exists as additive mapping D_1: $X\to X_\mu$ satisfying:

$$\mu(f(x)\text{-}D_1(x))\leq\frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{a^{j+1}}\nu(a^j x,0),\forall x\in X$$

But if $A_1(x_0) \neq D_1(x_0)$ for some x_0∈X. Then there exists a constant ∈>0 which is positive such that ∈<ρ $(A_1(x_0)-D_1(x_0))$. By Eq. 21, there is a positive integer n_0∈N such that:

$$\sum_{j=n_0}^{\infty}\frac{1}{a^{j+1}}\nu(a^j y,0)<\frac{\in}{2}$$

Since, A_1 and D_1 are additive mappings, by $A_1(a^{n_0}x_0) = a^{n_0}A_1(x_0)$ and $D_1(a^{n_0}x_0) = a^{n_0}D_1(x_0)$, we arrive:

$$\begin{aligned}&\in<\mu(A_1(X_0)\text{-}D_1(x_0))=\\&\mu\left(\frac{A_1(a^{n_0}x_0)\text{-}f(a^{n_0}x_0)}{a^{n_0}}+\frac{f(a^{n_0}x_0)\text{-}D_1(a^{n_0}x_0)}{a^{n_0}}\right)\\&\leq\frac{1}{a^{n_0}}\mu(A_1(a^{n_0}x_0)\text{-}f(a^{n_0}x_0))+\frac{1}{a^{n_0}}\mu(f(a^{n_0}x_0)\text{-}D_1(a^{n_0}x_0))\\&\leq\frac{1}{a^{n_0}}\sum_{j=0}^{\infty}\frac{\nu(a^{j+n_0}x_0,0)}{a^{j+1}}\\&\leq\sum_{j=n_0}^{\infty}\frac{\nu(a^j x_0,0)}{a^{j+1}}<\in\end{aligned}$$

Which implies a contradiction. Therefore, the mapping A_1 is a unique additive mapping near f satisfying Eq. 22 in X_μ. From the above theorem 6.1, we obtain Hyers-Ulam and generalized Hyers-Ulam stabilities, respectively in the following corollaries.

Corollary 6.2: Let a mapping f: $X\to X_\mu$ satisfying:

$$\mu(D_A f(x,y))\leq\in\ \forall x,y\in X$$

For some ∈>0. Then there exists A_1: $X\to X_\mu$ a unique additive mapping satisfies Eq. 2 and:

$$\mu(f(x)\text{-}A_1(x))\leq\frac{\in}{2(a\text{-}1)}\qquad(27)$$

For all x∈X.

Proof: Letting ν (x, y) = ∈ in theorem 6.1, we arrive:

$$\mu(f(x)\text{-}A_1(x))\leq\frac{1}{2a}\sum_{j=0}^{\infty}\frac{\in}{a^j}=\frac{\in}{2a}\left(1\text{-}\frac{1}{a}\right)^{-1}\leq\frac{\in}{2(a\text{-}1)}\qquad(28)$$

For all x∈X.

Corollary 6.3: If f: $X\to X_\mu$ a mapping satisfies:

$$\mu(D_A f(x,y))\leq\in\left(\|x\|^m+\|y\|^m\right),\forall x,y\in X,m<1$$

A real number ∈>0 then there exists A_1: $X\to X_a$ a unique additive mapping satisfying:

$$\mu(f(x)\text{-}A_1(x))\leq\frac{\in}{2(a\text{-}a^m)}\|x\|^m,\forall x\in X\qquad(29)$$

where, x ≠0 if r<0.

Proof: Assuming ν (x, y) = ∈ $(\|x\|^m+\|y\|^m)$ in theorem 6.1, we arrive:

$$\begin{aligned}&\mu(f(x)\text{-}A_1(x))\leq\frac{1}{2a}\sum_{j=0}^{\infty}\frac{\in\left(\|a^j x\|^m+0\right)}{a^j}\leq\\&\frac{\in}{2a}\sum_{j=0}^{\infty}\left(\frac{a^m}{a}\right)^j\|x\|^m\leq\frac{\in}{2a}\left(1\text{-}\frac{a^m}{a}\right)^{-1}\|x\|^m\leq\\&\frac{\in}{2(a\text{-}a^m)}\|x\|^m\end{aligned}\qquad(30)$$

for all x∈X. Assuming μ satisfies the Δ_a-condition and if there exists β>0 defined by μ(ax)≤βμ(x) for all x∈X_μ.

Theorem 6.4: Letting f: $X \to X_\mu$ and ν: $X^2 \to [0, \infty)$ be the mappings satisfies:

$$\mu(D_A f(x, y)) \leq \nu(x, y) \tag{31}$$

And:

$$\Psi(x, y) = \sum_{j=0}^{\infty} \frac{\beta^{2j}}{a^j} \nu\left(\frac{x}{a^j}, \frac{y}{a^j}\right) < \infty, \forall x, y \in X \tag{32}$$

Then there exists A_2: $X \to X_\mu$ a unique additive mapping such that $A_2(x) = \lim_{n\to\infty} a^n f(x/a^n)$ which satisfies Eq. 2 and:

$$\mu(f(x)\text{-}A_2(x)) \leq \frac{1}{2a}\zeta(x, 0), \forall x \in X \tag{33}$$

Proof: Equation 23, implies that:

$$\mu\left(f(x)\text{-}af\left(\frac{x}{a}\right)\right) \leq \frac{1}{2}\nu\left(\frac{x}{a}, 0\right), x \in X \tag{34}$$

Hence, by the convexity μ, we have:

$$\mu\left(f(x) - a^2 f\left(\frac{x}{a^2}\right)\right) \leq \frac{1}{a}\mu \left(af(x)\text{-}a^2 f\left(\frac{x}{a}\right)\right) + \frac{1}{a}\mu\left(a^2 f\left(\frac{x}{a}\right)\text{-}a^3 f\left(\frac{x}{a^2}\right)\right) \leq \frac{\beta}{2a}\nu\left(\frac{x}{a}, 0\right) + \frac{\beta^2}{2a}\nu\left(\frac{x}{a^2}, 0\right), \forall x \in X$$

Then by induction on n>1, we have:

$$\mu\left(f(x)\text{-}a^n f\left(\frac{x}{a^n}\right)\right) \leq \frac{1}{2}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{1}{2}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{x}{a^n}, 0\right) \tag{35}$$

For all x∈X. Considering Eq. 35 holds true for n and we deduce the following by using the convexity of μ:

$$\mu\left(f(x)\text{-}p^{n+1} f\left(\frac{x}{a^{n+1}}\right)\right) = \frac{1}{a}\mu\left(af(x)\text{-}a^2 f\left(\frac{x}{a}\right)\right) + \frac{1}{2}\mu\left(a^2 f\left(\frac{x}{a}\right)\text{-}a^{n+2} f\left(\frac{x}{a^{n+1}}\right)\right) \leq \frac{\beta}{a}\mu\left(f(x)\text{-}af\left(\frac{x}{a}\right)\right) + \frac{\beta^2}{a}\mu\left(f\left(\frac{x}{a}\right)\text{-}a^n f\left(\frac{x}{a^{n+1}}\right)\right) \leq \frac{\beta}{2a}\nu\left(\frac{x}{a}, 0\right) + \frac{\beta^2}{2a}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^{j+1}}, 0\right) + \frac{\beta^2}{2a}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{x}{a^{n+1}}, 0\right) = \frac{1}{2}\sum_{j=1}^{n}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{1}{2}\frac{\beta^{2n}}{a^n}\nu\left(\frac{x}{a^{n+1}}, 0\right) \tag{36}$$

The above inequality proves Eq. 35 for n+1. Substituting x by x/a^m in Eq. 35, we arrive:

$$\mu\left(a^m f\left(\frac{x}{a^m}\right)\text{-}a^{n+m} f\left(\frac{x}{a^{n+m}}\right)\right) \leq \beta^m \mu\left(f\left(\frac{x}{a^m}\right) - a^n f\left(\frac{x}{a^{n+m}}\right)\right) \leq \beta^m \frac{1}{2}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^{j+m}}, 0\right) + \beta^m \frac{1}{2}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{x}{a^{n+m}}, 0\right) \leq \frac{a^m}{2\beta^m}\sum_{j=m+1}^{n+m-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{a^m}{2\beta^m}\frac{\beta^{2(n+m-1)}}{a^{n+m-1}}\nu\left(\frac{x}{a^{n+m}}, 0\right)$$

By Eq. 32 it converges to zero as $m \to \infty$. Hence, $\{a^n f(x/a^n)\}$ is μ-Cauchy for all x∈X and hence, it is μ-convergent in X_μ, since, X_μ is μ-complete. Hence, we have:

$$A_2(x) = \mu\text{-}\lim_{n\to\infty} a^n f\left(\frac{x}{a^n}\right), \forall x \in X \tag{37}$$

and by Eq. 37, we obtain:

$$\lim_{n\to\infty}\mu\left(a^n f\left(\frac{x}{a^n}\right)\text{-}A_2(x)\right) = 0, \forall x \in X$$

Hence, by the Δ_a-condition, we arrive by taking $n \to \infty$:

$$\mu(f(x)\text{-}A_2(x)) \leq \frac{1}{a}\mu\left(af(x)\text{-}a^{n+1} f\left(\frac{x}{a^n}\right)\right) + \frac{1}{a}\mu\left(a^{n+1} f\left(\frac{x}{a^n}\right)\text{-}aA_2(x)\right) \leq \frac{\beta}{a}\mu\left(f(x)\text{-}a^n f\left(\frac{x}{a^n}\right)\right) + \frac{\beta}{a}\mu\left(a^n f\left(\frac{x}{a^n}\right)\text{-}A_2(x)\right) \leq \frac{\beta}{2a}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{\beta}{2a}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{y}{a^n}, 0\right) + \frac{\beta}{a}\nu\left(a^n f\left(\frac{x}{a^n}\right)\text{-}A_2(x)\right) \leq \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^j}, 0\right)$$

Next, we prove A_2 satisfies Eq. 2. Assuming $(x, y) = (x/a^n, y/a^n)$ in Eq. 31 and multiplaying the resultant by a^n, we obtain:

$$\mu\left(a^n Df\left(\frac{x}{a^n}, \frac{y}{a^n}\right)\right) \leq \beta^n \nu\left(\frac{x}{a^n}, \frac{y}{a^n}\right) \leq \frac{\beta^{2n}}{a^n}\nu\left(\frac{x}{a^n}, \frac{y}{a^n}\right)$$

As $n \to \infty$ which tends to zero. Hence, the property $\mu(\gamma u) \leq \gamma\mu(u)$, $0 < \gamma \leq 1$, $u \in X_u$ implies that:

$$\mu\left(\frac{1}{4a^2+2a+3}D_A A_2(x,y)\right) \le \mu\left(\frac{1}{4a^2+2a+3}D_A A_2(x,y) - a^n\frac{D_A f\left(\frac{x}{a^n},\frac{y}{a^n}\right)}{(4a^2+2a+3)} + a^n\frac{D_A f\left(\frac{x}{a^n},\frac{y}{a^n}\right)}{(4a^2+2a+3)}\right) \le$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_2(ax+y)-a^n f\left(\frac{ax+y}{a^n}\right)\right)+\frac{1}{4a^2+2a+3}\mu\left(A_2(ax-y)-a^n f\left(\frac{ax-y}{a^n}\right)\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_2(x+ay)-a^n f\left(\frac{x+ay}{a^n}\right)\right)+\frac{1}{4a^2+2a+3}\mu\left(A_2(x-ay)-a^n f\left(\frac{x-ay}{a^n}\right)\right)+$$

$$\frac{a+a^2}{4a^2+2a+3}\mu\left(A_2(x+y)-a^n f\left(\frac{x+y}{a^n}\right)\right)+\frac{a+a^2}{4a^2+2a+3}\mu\left(A_2(x-y)-a^n f\left(\frac{x-y}{a^n}\right)\right)+$$

$$\frac{2(a^2-1)}{4a^2+2a+3}\mu\left(A_2(x)-a^n f\left(\frac{x}{a^n}\right)\right)+\frac{1}{4a^2+2a+3}\mu\left(a^n D_A f\left(\frac{x}{a^n},\frac{y}{a^n}\right)\right),\ \forall x, y\in X$$

As the limit $n\to\infty$, we obtain:

$$\mu\left(\frac{1}{4a^2+2a+3}D_A A_2(x,y)\right)=0$$

And hence, $D_A A_2(x, y) = 0$, $\forall x, y \in X$ and A_2 satisfies Eq. 2. Hence, it is additive. To prove the uniqueness of A_2, assume that D_2: $X\to X_\mu$, a additive mapping satisfies:

$$\mu(f(x)-D_2(x)) \le \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^j},0\right),\ \forall x\in X$$

Since, A_2 and D_2 are additive mappings and:

$$a^n A_2\left(\frac{x}{a^n}\right)=A_2(x),\ a^n D_2\left(\frac{x}{a^n}\right)=D_2(x)$$

Implies that:

$$\mu(D_2(x)-A_2(x))=\mu\begin{pmatrix}\frac{a^{n+1}}{a}\left(D_2\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)\right)+\\ \frac{a^{n+1}}{a}\left(f\left(\frac{x}{a^n}\right)-A_2\left(\frac{x}{a^n}\right)\right)\end{pmatrix}$$

$$\le\frac{\beta^{n+1}}{a}\mu\left(D_2\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)\right)+\frac{\beta^{n+1}}{a}\mu\begin{pmatrix}f\left(\frac{x}{a^n}\right)-\\ A_2\left(\frac{x}{a^n}\right)\end{pmatrix}$$

$$\le\frac{\beta^{n+1}}{a}\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^{j+n}},0\right)+\frac{\beta^{n+1}}{a}\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^{j+n}},0\right)$$

$$\le\frac{\beta^{n+1}}{a^2}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^{j+n}},0\right)\le\frac{\beta a^n}{a^2\beta^n}\sum_{j=1}^{\infty}\frac{\beta^{2(j+n)}}{a^{j+n}}\nu\left(\frac{x}{a^{j+n}},0\right),\ x\in X$$

As $n\to\infty$ it tends to zero. Therefore, A_2 satisfying Eq. 33 and is a unique additive mapping.

In the following corollaries of theorem 6.4, we obtain Hyers-Ulam and Hyers-Ulam-Rassias stabilities, respectively.

Corollary 6.5: Let a mapping f: $X\to X_\mu$ satisfying:

$$\mu(Df(x,y))\le\epsilon,\ x,y\in X,\epsilon>0$$

for some $\beta^2< a$. Hence, there exists a unique additive mapping A_2: $X\to X_\mu$ which satisfies Eq. 2 and:

$$\mu(f(x)-A_2(x))\le\frac{\epsilon\beta^2}{2a(a-\beta^2)},\ \forall x\in X \quad (38)$$

Proof: Considering $\nu(x, y) = \epsilon$ in theorem 6.4, we arrive:

$$\mu(f(x)-A_2(x))\le\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\epsilon\beta^{2j}}{a^j}\le\frac{\epsilon\beta^2}{a}\left(\frac{a-\beta^2}{a}\right)^{-1}\le\frac{\epsilon\beta^2}{2a(a-\beta^2)},\ \forall x\in X \quad (39)$$

Corollary 6.6: If f: $X\to X_\mu$ a mapping satisfies:

$$\mu(D_A f(x,y))\le\epsilon\left(\|x\|^m+\|y\|^m\right),\ \forall x,y\in X$$

For given real numbers $\beta^2<a^{r+1}$ and $\epsilon>0$ then there exists A_2: $X\to X_\mu$ a unique additive mapping such that:

$$\mu(f(x)-A_2(x))\le\frac{\epsilon\beta^2}{2a(a^{m+1}-\beta^2)}\|x\|^m,\ \forall x\in X \quad (40)$$

where, $x\neq 0$, if $r<0$.

Proof: Considering $\nu(x, y) = \epsilon(\|x\|^m+\|y\|^m)$ in theorem 6.1, we arrive:

$$\begin{aligned}\mu\left(f(x)-A_2(x)\right) &\le \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\left(\in\left\|\frac{x}{a^j}\right\|^m\right)\\ &\le \frac{\in}{2a}\sum_{j=1}^{\infty}\left(\frac{\beta^2}{a.a^m}\right)^j\|x\|^m\\ &\le \frac{\in\beta^2}{2a\left(a.a^m\right)}\left(1-\frac{\beta^2}{a.a^m}\right)^{-1}\|x\|^m\\ &\le \frac{\in\beta^2}{2a\left(a^{m+1}-\beta^2\right)}\|x\|^m, \forall x\in X\end{aligned} \tag{41}$$

Stability of functional Eq. (1): cubic case: We obtain generalized Hyers-Ulam-Rassias stability of Eq. 1 in modular spaces without Δ_p-condition and the Fatou property. Hereafter, we use the following notation:

$$D_f(x,y) = f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)-\left(a+a^2\right)\left[f(x+y)+f(x-y)\right]-2\left(a^3-a^2-a+1\right)f(x)$$

For all x, y∈X.

Theorem 7.1: Considering f: X→X_μ a mapping satisfies:

$$\mu\left(Df(x,y)\right)\le\nu(x,y) \tag{42}$$

And a mapping ν: X^2→[0, ∞) satisfies:

$$\zeta(x,y)=\sum_{j=0}^{\infty}\frac{\nu\left(a^jx,a^jy\right)}{a^{3j}}<\infty, \forall x,y\in X \tag{43}$$

Then there exists C_1: X→X_μ a unique cubic mapping defined by:

$$C_1(x)=\lim_{n\to\infty}\frac{f\left(a^nx\right)}{a^{3n}}, x\in X$$

Which satisfies the Eq. 11 and:

$$\mu\left(f(x)-C_1(x)\right)\le\frac{1}{2a^3}\zeta(x,0), \forall y\in X \tag{44}$$

Proof: Assuming x = 0 in Eq. 42, we obtain:

$$\mu\left(f(ax)-a^3f(x)\right)\le\frac{1}{2}\nu(x,0) \tag{45}$$

And hence:

$$\mu\left(f(x)-\frac{f(ax)}{a^3}\right)\le\frac{1}{2a^3}\nu(x,0), \forall x\in X \tag{46}$$

Generalizing, we arrive:

$$\mu\left(f(x)-\frac{f\left(a^nx\right)}{a^{3n}}\right)\le\frac{1}{2}\sum_{j=0}^{n-1}\frac{\nu\left(a^jx,0\right)}{a^{3(j+1)}}, \forall x\in X \tag{47}$$

Substituting x by a^m x in Eq. 47, we obtain:

$$\mu\left(\frac{f\left(a^mx\right)}{a^{3m}}-\frac{f\left(a^{n+m}x\right)}{a^3(n+m)}\right)\le\frac{1}{2a^3}\sum_{j=m}^{n+m-1}\frac{\phi\left(a^jx,0\right)}{a^{3j}} \tag{48}$$

By the assumption Eq. 43 it converges to zero as m→∞. Hence, Eq. 48 implies that the sequence $\left\{\frac{f\left(a^nx\right)}{a^{3n}}\right\}$ is μ-Cauchy and therefore it is convergent in X_μ, since, the X_μ is μ-complete. Hence, we define C_1: X→X_p as:

$$C_1(x)=\mu-\lim_{n\to\infty}\left\{\frac{f\left(a^nx\right)}{a^{3n}}\right\}, \forall x\in X$$

Which implies:

$$\lim_{n\to\infty}\mu\left(\frac{f\left(a^nx\right)}{a^{3n}}-C_1(x)\right)=0, \forall x\in X$$

Hereafter, we complete this proof by similar way of theorem 6.1. In the following corollaries of theorem 7.1, we obtain stabilities called Hyers-Ulam and Hyers-Ulam-Rassias, respectively.

Corollary 7.2: Let a mapping f: X→X_μ satisfying:

$$\mu\left(Df(x,y)\right)\le\in, \forall x,y\in X$$

for some ∈>0 and a^3>1. Then, there exists C_1: X→X_μ a unique cubic mapping which satisfies Eq. 11 and:

$$\mu\left(f(x)-C_1(x)\right)\le\frac{\in}{2\left(a^3-1\right)} \tag{49}$$

For all x∈X.

Proof: Assuming ν (x, y) = ∈ in theorem 7.1, we arrive:

$$\mu\left(f(x)-C_1(x)\right)\le\frac{1}{2a^3}\sum_{j=0}^{\infty}\frac{\in}{a^{3j}}=\frac{\in}{2a^3}\left(1-\frac{1}{a^3}\right)^{-1}\le\frac{\in}{2\left(a^3-1\right)} \tag{50}$$

For all x∈X.

Corollary 7.3: If f: $X \to X_\mu$ a mapping satisfies:

$$\rho(D_C f(x,y)) \le \in \left(\|x\|^m + \|y\|^m\right), \forall x, y \in X$$

For given real numbers m<3 and $\in>0$ then there exists a unique cubic mapping C_1: $X \to X_\mu$ such that:

$$\mu(f(x)-C_1(x)) \le \frac{\in}{2(a^3-a^m)}\|x\|^m, \forall x \in X \tag{51}$$

where, $a\neq 0$ if m<0.

Proof: Assuming $\nu(x, y) = \in (\|x\|^m + \|y\|^m)$ in theorem 7.1, we obtain:

$$\mu(f(x)-C_1(x)) \le \frac{1}{2a^3}\sum_{j=0}^{\infty}\frac{\in\left(\|a^j x\|^m + 0\right)}{a^{3j}} \le \frac{\in}{2a^3}\sum_{j=0}^{\infty}\left(\frac{a^m}{a^3}\right)^j \|x\|^m \le \frac{\in}{2a^3}\left(1-\frac{a^m}{a^3}\right)^{-1}\|x\|^m \le \frac{\in}{2(a^3-a^m)}\|x\|^m \tag{52}$$

For all $x \in X$.

Assuming a nontrivial convex modular μ satisfies the Δ_a-condition if there exists $\beta>0$ such that $\mu(ax) \le \beta\mu(x)$ for all $x \in X_\mu$ where $\beta \ge a$ and hence, $\mu(a^3 x) \le M\,\rho(x)$.

Theorem 7.4: If a mapping f: $X \to X_\mu$ satisfies:

$$\mu(Df(x,y)) \le \nu(x,y) \tag{53}$$

And ν: $X^2 \to [0, \infty)$ is a mapping such that:

$$\zeta(x,y) = \sum_{j=1}^{\infty}\frac{M^{2j}}{a^{3j}}\nu\left(\frac{x}{a^j}, \frac{y}{a^j}\right) < \infty, \forall x, y \in X \tag{54}$$

Then a unique cubic mapping C_2: $X \to X_\rho$ exists and defined by $C_2(x) = \lim_{n\to\infty} a^{3n} f\left(\frac{x}{a^n}\right)$, $x \in X$ which satisfies Eq. 11 and:

$$\mu(f(x)-C_2(x)) \le \frac{1}{2a}\zeta(x,0), \forall x \in X \tag{55}$$

Proof: Equation 45 implies that:

$$\mu\left(f(x)-af\left(\frac{x}{a}\right)\right) \le \frac{1}{2}\phi\left(\frac{x}{a},0\right), \forall x \in X \tag{56}$$

Hence, by the convexity μ, we arrive:

$$\mu\left(f(x)-(a^3)^2 f\left(\frac{x}{a^2}\right)\right) \le \frac{1}{a^3}\mu\left(a^3 f(x)-(a^3)^2 f\left(\frac{x}{a}\right)\right) + \frac{1}{a^3}\mu\left((a^3)^2 f\left(\frac{x}{a}\right)-(a^3)^3 f\left(\frac{x}{a^2}\right)\right) \le \frac{M}{2a^3}\nu\left(\frac{x}{a},0\right) + \frac{M^2}{2a^3}\phi\left(\frac{x}{a^2},0\right), \forall x \in X$$

Generalizing, we obtain:

$$\mu\left(f(x)-(a^3)^n f\left(\frac{x}{a^n}\right)\right) \le \frac{1}{2}\sum_{j=1}^{n-1}\frac{M^{2j-1}}{a^{3j}}\nu\left(\frac{x}{a^j},0\right) + \frac{1}{2}\frac{M^{2(n-1)}}{a^{3(n-1)}}\nu\left(\frac{x}{a^n},0\right) \tag{57}$$

For all $x \in X$. The rest of proof is similar to that of theorem 6.4. In the following corollaries of theorem 7.4, we obtain the stabilities called Hyers-Ulam and Hyers-Ulam-Rassias, respectively.

Corollary 7.5: If a mapping f: $X \to X_\mu$ satisfying:

$$\mu(D_C f(x,y)) \le \in, \forall x, y \in X$$

For some $\in>0$ and $M^2<a^3$. Then there exists C_2: $X \to X_\mu$ a unique cubic mapping which satisfies Eq. 11 and:

$$\mu(f(x)-C_2(x)) \le \frac{\in M^2}{2a(a^3-M^2)}, \forall x \in X \tag{58}$$

Proof: Assuming $\nu(x, y) = \in$ in theorem 7.4, we arrive:

$$\nu(f(x)-C_2(x)) \le \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\in M^{2j}}{a^{3j}} \le \frac{\in M^2}{2a^4}\left(\frac{a^3-M^2}{a^3}\right)^{-1} \le \frac{\in M^2}{2a(a^3-M^2)} \tag{59}$$

$\forall x \in X$

Corollary 7.6: If f: $X \to X_\mu$ a mapping satisfies:

$$\mu(D_C f(x,y)) \le \in \left(\|x\|^m + \|y\|^m\right), \forall x, y \in X$$

For given real numbers $M^2<a^{m+3}$ and $\in>0$ then a unique cubic mapping C_2: $X \to X_\mu$ exists such that:

$$\mu(f(x)-C_2(x)) \le \frac{\in M^2}{2a(a^{m+3}-M^2)}\|x\|^m, \forall x \in X \tag{60}$$

where $x\neq 0$, if m<0.

Proof: Assuming $\nu(x, y) = \in (\|x\|^m + \|y\|^m)$ in theorem 7.1, we arrive:

$$\mu\left(f(x)-C_2(x)\right) \leq \frac{1}{2a}\sum_{j=1}^{\infty}\frac{M^{2j}}{a^{3j}}\left(\in\left\|\frac{x}{a^j}\right\|^m\right) \leq \frac{\in}{2a}\sum_{j=1}^{\infty}\left(\frac{M^2}{a^3.a^m}\right)^j\|x\|^m$$
$$\leq \frac{\in M^2}{2a\left(a^3.a^m\right)}\left(1-\frac{M^2}{a^3.a^m}\right)^{-1}\|x\|^m \leq \frac{\in M^2}{2a\left(a^{m+3}-M^2\right)}\|x\|^m \quad (61)$$

For all $x\in X$.

Stability of functional Eq. 1 in quadratic and quartic case can be analyzed by similar method were used in section-6 and 7.

CONCLUSION

We introduced a generalized mixed type of additive-quadratic-cubic-quartic functional equation with its general solution and various stabilities concerning Ulam problem in modular spaces by considering with and without Δ_a-condition.

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