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On Direct Integration of First, Second and Third Order ODES

E.O. Adeyefa, O.A. Olajide, L.S. Akinola, O.E. Abolarin, A.A. Ibrahim and Y. Haruna

Department of Mathematics, Federal University Oye-Ekiti, P.M.B. 373, Oye-Ekiti, Ekiti State, Nigeria

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Corresponding Author:
E.O. Adeyefa
Department of Mathematics, Federal University Oye-Ekiti, P.M.B. 373, Oye-Ekiti, Ekiti State, Nigeria

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Abstract: A step hybrid block method for the solution of Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) is presented in this study. The method is formulated from continuous schemes obtained via. collocation and interpolation techniques and applied in a block-by-block manner as numerical integrator for first, second and third order ODEs. The convergence properties of the method are discussed via. zero-stability and consistency. Numerical examples are included and comparisons are made with existing methods in the literature.

INTRODUCTION

In this study, we focus on direct integration of initial value problems of the form:

$$\begin{pmatrix} \dfrac{d^m y}{dx^m}\text{-}f(x, y(x)) \\ \dfrac{d^{m+1} y}{dx^{m+1}}\text{-}f(x, y(x), y'(x)) \\ \dfrac{d^{m+2} y}{dx^{m+2}}\text{-}f(x, y(x), y'(x), y''(x)) \end{pmatrix} = \begin{pmatrix} z \\ 2z \\ 3z \end{pmatrix} \qquad (1)$$

With the conditions $y(a) = y_0$, $y'(a) = y'_0$ $y''(a) = y``_0$ $a \le x \le b$ where m is the order of the Ordinary Differential Equations (ODEs). We obtain the numerical solution of (Eq. 1) by constructing a step block method:

$$\begin{cases} \sum_{i=0}^{k} \alpha_i(t) y_{n+i} = h\left(\sum_{i=0}^{k} \beta_i(t) f_{n+1} + \beta_j(t) f_{n+j}\right) + \\ h^2\left(\sum_{i=0}^{k} \lambda_i(t) g_{n+i} + \lambda_j(t) g_{n+j}\right) + h^3\left(\sum_{i=0}^{k} \delta_i(t) w_{n+i} + \delta_j(t) w_{n+j}\right) \end{cases} \qquad (2)$$

where either of α_0 (t) and β_0 (t) do not varnish α_k (t) = I, β_k (t) $\neq 0$ and k = 1.

The solution of (1) for $m \ge 2$ has been extensively discussed in the literature using different approaches. Lambert (1973, 1991), Abdullahi (1999) and Brugnano and Trigiante (1998), among others reduced higher order Initial Value Problems (IVPs) to a system of first order equations. Resulting from this are the setback highlighted in their works.

Several numerical methods have been proposed to improve on the efficiency and convergence of the existing methods (Butcher, 2003; Yahaya *et al*., 2016; Adeniyi and Alabi, 2011; Jator, 2007, 2010a, 2010b; Olusola, 2018; Kuboye *et al*., 2018; Ismail, 2009; Ramos *et al*., 2016; Simos, 2002; Sagir, 2014; Vigo-Aguiar and Ramos, 2006; Olabode, 2009, 2013; Adeyefa, 2017).

The formulation of block method to integrate IVPs of order one or higher order has been widely reported in the literature. However, to use a formulated block method for integration of several order IVPs, say first, second and third order ODEs has not been commonly reported. Thus, the focus of this study is to formulate a self-starting method for the numerical integration of first, second and third order IVPs. In what immediately follows in the next section, we consider the formulation of the proposed block method.

MATERIALS AND METHODS

Here, we formulate a step hybrid method capable of solving first, second and third order ODEs employing and choose Chebyshev polynomials as our basis function. In Eq. 1 and 2, we set m = 1, z = 0 and I = 0, 1, j = 1/4. Thus, we introduce the Chebyshev polynomials:

$$y(x)=\sum_{j=0}^{k+8} a_j T_j(x) \qquad (3)$$

Equation 3 is interpolated at $x = x_n$, its first and second derivatives are collocated at $x = x_{n+v}$, v = 0, 1/4, 1 while its third derivative is collocated at $x = x_{n+c}$, c = 0, 1/4. As a result, we have:

$$\left.\begin{aligned}&\sum_{j=0}^{k+8} a_j T_j(x)=y_n\\&\sum_{j=1}^{k+8} ja_j T_{n+m}^{j-1}=f_{n+m}\\&\sum_{j=2}^{k+8} j(j\text{-}1)a_j T_{n+m}^{j-2}=g_{n+m}\\&\sum_{j=3}^{k+8} j(j\text{-}1)(j-2)a_j T_{n+c}^{j-3}=w_{n+c}\end{aligned}\right\} \qquad (4)$$

Solving Eq. 4 using Gaussian elimination approach in order to get the unknown variables α which are substituted into Eq. 3. This yields a continuous implicit scheme of the form:

$$\begin{aligned}y(x)=h\left(\sum_{i=0}^{1}\beta_i(t)f_{n+i}+\beta_{\frac{3}{4}}(t)f_{n+\frac{3}{4}}\right)+h^2\left(\sum_{i=0}^{1}\lambda_i(t)g_{n+i}+\lambda_{\frac{3}{4}}(t)g_{n+\frac{3}{4}}\right)+\\ h^3\left(\delta_0(t)w_0+\delta_{\frac{3}{4}}(t)w_{n+\frac{3}{4}}\right)\end{aligned} \qquad (5)$$

Where, $t = 2x\text{-}2x_n\text{-}h/h$

Equation 5, when evaluated at $x = x_{n+cj}$, c_j = 1, 1/4, i.e., t = 1,-½, respectively, yields:

$$\begin{pmatrix}y_{n+\frac{1}{4}}\\ y_{n+1}\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}y_n+hD\begin{pmatrix}f_n\\ f_{n+\frac{1}{4}}\\ f_{n+1}\end{pmatrix}+h^2E\begin{pmatrix}g_n\\ g_{n+\frac{1}{4}}\\ g_{n+1}\end{pmatrix}+h^3F\begin{pmatrix}w_n\\ w_{n+\frac{1}{4}}\\ w_{n+1}\end{pmatrix} \qquad (6)$$

Where the values of D, E and F are:

$$D=\begin{pmatrix}\frac{34107}{286720} & \frac{1321}{10080} & -\frac{19}{2584800}\\ -\frac{456}{35} & \frac{4352}{315} & \frac{67}{315}\end{pmatrix}$$

$$E=\begin{pmatrix}\frac{159}{28672} & -\frac{853}{120960} & \frac{1}{1105920}\\ -\frac{39}{28} & -\frac{1664}{945} & -\frac{7}{540}\end{pmatrix}$$

$$F=\begin{pmatrix}\frac{179}{17203200} & \frac{11}{64512} & 0\\ -\frac{19}{420} & \frac{8}{63} & 0\end{pmatrix}$$

Equation 6 is our proposed first, second and third order IVPs solver.

Basic properties of the method: We shall consider in this section, the analysis of basic properties of this method such as order, error constant, zero stability and consistency is investigated.

Order and error constant: Equation 6 derived is a discrete scheme belonging to the class of LMMs of the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h\sum_{j=0}^{k}w_j f_{n+j}+h^2\sum_{j=0}^{k}\beta_j g_{n+j}+h^3\sum_{j=0}^{k}\gamma_j G_{n+j} \qquad (7)$$

By Fatunla (1991) and Lambert (1991), we define the local truncation error associated with Eq. 7 by the difference operator:

$$L[y(x):h]=\sum_{j=0}^{k}\begin{bmatrix}\alpha_j y(x_n+jh)\text{-}hw_j f(x_n+jh)\text{-}\\ h^2\beta_j g(x_n+jh)\text{-}h^3\gamma_j G(x_n+jh)\end{bmatrix} \qquad (8)$$

Where, y(x) is an arbitrary function, continuously differentiable on [a, b]. Equation 8 in Taylor series about point , we obtain the expression the expression:

$$\begin{aligned}L[y(x);h]&=C_0y(x)+C_1hy'(x)+C_2h^2y''(x)\\&+,...,+C_{p+2}h^{p+2}y^{p+2}(x)\end{aligned}$$

In the spirit by Lambert (1991) (Eq. 8) is of order p if $C_0 = C_1 = C_2 = ,..., C_{p+1} = C_{p+2}$ and $C_{p+3} \neq 0$. The $C_{p+3} \neq 0$ is called the error constant and $C_{p+3}\, h^{p+3}\, y^{p+3}\, (x_n)$ is the principal local truncation error at the point x_n. Thus, the block (6) is of order p = 6 and error constants:

$$C_{p+3}=\left[\frac{-221}{926635508121600},\frac{83}{93251404800}\right]^T$$

Zero stability of the method: To analyze the zero-stability of the method, we (Eq. 10) in vector notation form of column vectors $e = (e_1, ..., e_r)^T$, $d = (d_1, ..., d_r)^T$, $y_m = (y_{n+1}, ..., y_{n+r})^T$, $F(y_m) = (f_{n+1}, ..., f_{n+r})$, $G(ym) = (g_{n+1}, ..., g_{n+r})$, $W(y_m) = (w_{n+1}, ..., w_{n+r})$ and matrix $A = (a_{ij})$, $B = (b_{ij})$. Thus, Eq. 6 forms the block equation:

$$A_0 y_m = hBF(y_m)+A^1 y_n+hbf_n+h^2DG(y_m)+ h^2dg_n+h^3VW(y_m)+h^3uT_n \quad (9)$$

where, h is a fixed mesh size within a block.

In line with (Eq. 9): $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

The first characteristic polynomial of the block hybrid method is given by:

$$\rho(R) = \det\left(RA^0\text{-}A^1\right) \quad (10)$$

Substituting $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

In Eq. 10 and solving for R, the values of Rare obtained as 0 and 1. According to Simeon (1988, 1991), the block formulae represented by (Eq. 6) are zero-stable, since from Eq. 10, $\rho(R) = 0$ satisfy $|R| \leq 1$, j = 1 and for those roots with $|R_j| = 1$ the multiplicity does not exceed two.

Consistency and convergence of the method: The linear multistep method (7) is said to be consistent if it has order $\rho \geq 1$. The method is consistent being of order 6.

According to the theorem of by Dahlquist (1979), the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since, the method satisfies the two conditions, hence, it is convergent.

Numerical experiment: We consider in this section, four test problems which includes first, second and third order ordinary differential equations to test the effectiveness of this new scheme.

Problem 1: We consider the third order IVP :

$$y''' = 3\sin x,\; y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$$

With exact solution $y(x) = 3\cos x + x^2/2 - 2$which has been solved by Olabode (2009) with step number k =6. The numerical solution is displayed in Table 1.

Table 1: Comparison of error of the proposed method and error in Olabode (2009)

t-values	Error in new method, k = 1	Error in Olabode (2009), k = 6
0.1	2.0E-10	4.172279744E-09
0.2	4.0E-10	9.578546178E-08
0.3	2.0E-10	3.991586710E-07
0.4	3.0E-10	1.036864440E-06
0.5	9.0E-10	2.128509889E-06
0.6	1.1E-09	3.789539851E-06
0.7	1.5E-09	6.130086676E-06
0.8	1.3E-09	9.253867047E-06
0.9	1.5E-09	1.325714643E-05
1.0	2.0E-09	1.822777782E-05

Table 2: Comparison of errors of the proposed method and the existing methods Mohammed and Adeniyi (2014), Mohammed

t-values	Error in new method	Error in Mohammed and Adeniyi (2014)	Error in Mohammed
0.1	1.1×10^{-10}	2.004×10^{-7}	2.198×10^{-5}
0.2	9.1×10^{-11}	5.386×10^{-7}	6.0704×10^{-6}
0.3	6.1×10^{-7}	8.84×10^{-7}	1.0051×10^{-5}
0.4	3.4×10^{-10}	1.2297×10^{-6}	1.40253×10^{-5}
0.5	1.45×10^{-6}	1.5750×10^{-6}	1.79934×10^{-5}
0.6	1.46×10^{-6}	1.9204×10^{-6}	2.16162×10^{-5}
0.7	1.47×10^{-6}	2.506×10^{-6}	2.993×10^{-5}
0.8	1.49×10^{-6}	3.106×10^{-6}	3.4561×10^{-5}
0.9	1.5×10^{-6}	3.705×10^{-6}	4.1114×10^{-5}
1.0	1.52×10^{-6}	4.304×10^{-6}	4.7656×10^{-5}

Table 3: Comparison of errors of the proposed method and the existing methods Ajileye *et al*. (2018), Sunday *et al*. (2013)

t-values	Error in new method	Error in Ajileye *et al.* (2018)	Error in Sunday *et al.* (2013)
0.1	0	1.218026×10^{-13}	5.574430×10^{-12}
0.2	1×10^{-10}	1.399991×10^{-13}	$3.9461.77\times10^{-12}$
0.3	1×10^{-10}	1.184941×10^{-12}	8.183232×10^{-12}
0.4	2×10^{-10}	1.538991×10^{-12}	3.436118×10^{-15}
0.5	3×10^{-10}	1.110001×10^{-12}	1.929743×10^{-10}
0.6	3×10^{-10}	5.270229×10^{-12}	1.879040×10^{-10}
0.7	2×10^{-10}	2.10898×10^{-12}	1.776835×10^{-10}
0.8	3×10^{-10}	1.297895×10^{-11}	1.724676×10^{-10}
0.9	3×10^{-10}	3.08229×10^{-11}	1.847545×10^{-10}
1.0	2×10^{-10}	4.121925×10^{-11}	3.005770×10^{-10}

Problem 2: We consider the IVP y" = y', y (0) = 0, y' (0) = -1, h = 0.1 with exact solution $y(x) = 1\text{-}05e^{-0.5x}$ which has been solved in Mohammed and Adeniyi (2014) with step number k = 5. The numerical solution is displayed in Table 2.

Problem 3: We consider first order IVP y' = 0.5 (1-y), y (0) = 0.5, h = 0.1 with the exact solution $y(x) = 1\text{-}0.5e^{-0.5x}$. This IVP was solved by Ajileye *et al*. (2018), Sunday *et al*. (2013). The numerical solution is displayed in Table 3.

Problem 4: We consider non-linear IVPs $y''\text{-}x\,x\,(y')^2 = 0$, y (0) = 1, y' (0) = ½, h = 0.003125 whose exact solution is:

$$y(x) = 1+\frac{1}{2}\text{In}\left(\frac{2+x}{2\text{-}x}\right)$$

The numerical solution is displayed in Table 4.

Table 4 : Comparing the errors of the new block and existing methods for problem 4

x	Error in new method, k=1	Error by Kuboye *et al*. (2018), k = 3	Error by Kuboye (2015), k = 6	Error in Adeniyi and Alabi (2011) k = 6
0.1	0	5.850875E-13	9.577668E-10	0.1329867326E-09
0.2	0	2.848832E-12	2.368709E-09	0.5872691257E-08
0.3	0	6.328715E-12	3.732243E-09	0.1327845616E-07
0.4	0	6.756392E-09	5.475119E-09	0.2317829012E-07
0.5	0	1.380119E-08	1.142189E-08	0.3218793564E-07
0.6	1.0E-09	2.174817E-08	4.567944E-08	0.6871246012E-07
0.7	1.0E-09	1.073052E-07	2.055838E-06	0.1012728156E-06
0.8	1.0E-09	2.001340E-07	4.248299E-06	0.1231093271E-06
0.9	1.0E-09	3.088383E-07	6.660458E-06	0.2019286712E-06
1.0	2.0E-09	9.805074E-07	9.445166E-06	0.2990871645E-06

RESULTS AND DISCUSSION

The results obtained from the four test problems considered are summarized in Table 1-4. The proposed method is of step number k = 1 and it compares favourably with existing methods despite their k>1 methods. In problem 2, our step length is h = 0.1 against h = 0.01 used in Mohammed and Adeniyi (2014). The proposed method still gives better accuracy even with larger h. In Table 3, the methods developed by Ajileye *et al*. (2018), Sunday *et al*. (2013) performed better than the new method in terms of accuracy but their methods do not have the ability to solve higher order ordinary differential equations.

CONCLUSION

A step block method has been formulated and applied to solve first, second and third order ordinary differential equations directly without construction of additional schemes or employing existing predictors for implementation. Numerical experiments performed using this method show that the method is consistent, efficient and accurate. We therefore, recommend the method for direct integration of first, second and third order ordinary differential equations.

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