Journal of Engineering and Applied Sciences
MEDWELL PUBLICATIONS SINCE 2000

A New Approach To Solve Burgers' Equation Using Runge-Kutta 6th Order Method Based On Cole-Hopf Transformation

[1]Abbas AL-Shimmary, [2]Sajeda Kareem Radhi and [3]Amina Kassim Hussain
[1]Department of Electrical Engineering, Faculty of Engineering, University of Al-Mustansiriyah, Baghdad, Iraq
[2]Department of Remote Sensing, College of Remote Sensing and Geophysics, AL-Karkh University for Science, Baghdad, Iraq
[3]Department of Material Engineering, Faculty of Engineering, University of Al-Mustansiriyah, Baghdad, Iraq

Key words: Nonlinear partial differential equations, Hopf Cole transformation, differential transform method, burger's equation, Burger

Corresponding Author:
Abbas AL-Shimmary
Department of Electrical Engineering, Faculty of Engineering, University of Al-Mustansiriyah, Baghdad, Iraq

Page No.: 2362-2369
Volume: 15, Issue 10, 2020
ISSN: 1816-949x
Journal of Engineering and Applied Sciences

Abstract: It is known that Non linear Partial Differential Equations (NPDEs) play an important role in engineering and applied physical sciences, Burger's equation has been the center of study for many researchers because of its vast application in physics and engineering problems and is modeled as one types of (NPDEs). There are several methods to solve Burger equation. The main purpose of this paper is to propose a new numerical method by applying the Hopf-Cole Transformation method (H-CT) which converts Burger's equation into heat equation coupling with Runge-Kutta 6th order method (RK6) and the help of Method of Line (MOL) that transpose the converted heat equation into a simple system of ordinary differential equations, the results were compared with the exact Differential Transform Method (DTM). Finally, this comparative study show the reliability and efficiency of the proposed method throw numerical examples which is closed to the exact solution.

INTRODUCTION

In the last years more scientists have interest in the investigation of the subject nonlinear waves which arise in various branches of natural sciences such as fluid dynamic, plasma physics, nonlinear optics, electromagnetic waves, propagation of light in fibers and many more. In this sense, the study of nonlinear partial differential equations NLPDEs and there analytic and numerical solutions has great relevance. Some of these equations are solved numerically using Hirota[1] method and scattering inverse method[2], the use of these methods is not easy task, therefore, new different computational methods used to obtain exact solutions such as the tanh, the generalized tanh, the extended tanh, the improved tanh-coth methods[3-7] all these methods are based on the reduction of the original equation into equation with fewer dependent or independent variables using the traveling wave solution. The strong parabolic second order nonlinear partial differential equation and the most well understood is the viscid Burger's equation which a rises in the theory of shock waves, turbulence problems and continuous stochastic processes. It has many different application as modeling of water in unsaturated oil, elasticity statics of flow problems, gas dynamics, etc. (see for example, Cordero *et al*.[8] and the references therein), the nonlinear Burger equation can be linearized into the heat equation by the interesting Hopf-Cole transformation this transformation named after Eberhard Hopf, in Hopf[9] and Cole[10] also more researches directly related to transformation has been substantial. In this study, we introduce a new different scheme for solving Burger's equation by applying Hopf-Cole transformation method that transform Burger's equation into linear heat equation coupling with Runge-Kutta of 6th order method were the transformed heat equation is reduced into a simple system of ODEs by using MOL, we test our proposed method in numerical illustrated examples and the results compared with exact solution of one of the well-known method (DTM), clearly that there are a variety methods such as Adomain Decomposition Method (ADM), see (Adomian), (Adomian)[11,12], Vibrational Iteration Method (VIM)[13], Tanh and first integral method[14], Differential Transform Method (DTM)[15] etc., these method provide the solutions in infinite series form and the obtained series may be converge to closed form solution if the exact solution exists. This study focus on the most popular (DTM) that is used for comparison with the proposed method. This comparison is important to investigate the quality and efficiency of the applied numerical scheme.

Burger's equation: Let us consider aone-dimensional quasilinear parabolic convection-diffusion Burger's equation with viscosity and without external force as an initial value problem:

$$\left.\begin{array}{l} u_t + u\,u_x = v\,u_{xx},\;\; v>0,\, t>0,\; x\in R \\ \text{with IC }\; u(x,0)=u_0(x) \quad \text{for}\;\; x\in R \end{array}\right\} \tag{1}$$

which is known viscid Burger's equation this equation is obtained as a result of combining linear diffusion with the nonlinear wave motion and regard the simplest model for diffusive wave in fluid dynamics, the simplest forms of the nonlinear advection term $\mu\,\mu_x$ causes either a shocking up effect or rarefaction, so, the presence viscous term helps to break the wave and the term v_{uxx} is a dissipation term similar to that occurring in the heat equation where $v = \mu/\rho$ is an arbitrary parameter (v is kinematic viscosity at sonic condition μ is the viscosity of fluid and is ρ the density) also the parameter ν related to the Reynolds number there is an important connection between Eq. 1 when the viscosity parameter tends to zero this fact will be studied in more detail under the title vanishing viscosity approach, in this case Eq. 1 is called inviscid Burger's equation. The properties of Eq. 1 have been studied by Hopf[9], the Burger's equation without viscosity is the simplest nonlinear example of a conservation law that appear in studies of gas dynamics, traffic flow and acoustic transmission. In Physics and Mathematics, the exact solutions of Eq. 1 is still important topic, thus, seeking for new methods[16-19], for this purpose, one of the numerical methods is studied by Runge-Kutta of 6th order, see[20]. In this study, we introduce a new technique for numerical solution of Burger's equation by coupling Hopf-Cole Transformation method (H-CT) which converts Burger's equation into heat equation coupling with Runge-Kutta 6th order method (RK6) and the help of Method of Line (MOL) that transpose the converted heat equation into a simple system of ordinary differential equations this will be presented in the next section.

Coupling Hopf-Cole transformation with Runge-Kutta 6th order method: In this study, we give a brief description of the Hopf-Cole transformation, this transformation is a trick discovered independently by Eberhard Hopf[9] and Cole[10], they showed that the transformation is generally recognized as a powerful approach which maps the solution of the nonlinear partial viscous Burger's equation to linear heat (diffusion) equation, therefore, we start with the transformed linear heat equation (Cauchy Problem):

$$\left.\begin{array}{l} w_t = v\,w_{xx},\; v>0 \\ w(x,0)=w_0(x)=e^{-\frac{1}{2v}\int_0^x u_0(y)dy} \end{array}\right\} \tag{2}$$

Now introduce the simplest nonlinear transformation:

$$w = e^{\alpha\phi} \tag{3}$$

where φ = φ(x, t), solving (Eq. 3) for φ implies:

$$\phi = \frac{1}{\alpha}\ln w, w(x,t)>0 \tag{4}$$

Equation 4 represent the common Hopf-Cole transformation, calculate the following terms by chain rules:

$$w_t = \alpha\,\phi_t\,e^{\alpha\phi}, w_x = \alpha\,\phi_x\,e^{\alpha\phi},\; w_{xx} = [\alpha\,\phi_{xx} + \alpha^2\phi_x^2]\,e^{\alpha\phi}$$

substituting this expressions in Eq. 2 yield

$$\alpha\,\phi_t\,e^{\alpha\phi} = v[\alpha\,\phi_{xx} + \alpha^2\phi_x^2]\,e^{\alpha\phi} \tag{5}$$

which simplifies to:

$$\phi_t = v\,\phi_{xx} + \alpha v \phi_x^2 \tag{6}$$

Equation 6 known as the potential Burger's equation the second step in this process is to differentiate (Eq. 6) wrt x the result is:

$$\phi_{tx} = v\phi_{xxx} + 2\alpha\, v\phi_x \phi_x^2 \tag{7}$$

Let, us introduce a potential function φ define by:

$$u = \varphi_x \text{ and } \alpha = -\frac{1}{2v} \tag{8}$$

$$\text{hence } w(x,t) = e^{\frac{-1}{2v}\int u(x,t)dx} \tag{9}$$

and the initial condition on Eq. 1 therefore, must be transformed by using (Eq. 9) into:

$$w(x,0) = e^{\frac{-1}{2v}\int_0^x u(y,0)dy} \tag{10}$$

then Eq. 7 is converted into Burger's Eq. 1. So, we got to the famous Hopf-Cole transformation:

$$u = \phi_x = \frac{\partial(-2v\ln w)}{\partial x} = -2v\frac{w_x}{w} \tag{11}$$

which reduce viscous Burger's Eq. 1 into the heat Eq. 2, Olver[21]. The next step related with the Method of Lines (MOL) that approximate the heat Eq. 2 to a simple system of ordinary differential equations, the idea of MOL is to discretized the spatial derivative only with an algebraic expression using central finite differences, typically the time is remain:

$$w_{xx} \approx \frac{w_{i+1} - 2w_i + w_{i-1}}{(h)^2} + O(h)^2, 1 \le i \le M \tag{12}$$

where, i is an index designating a position along a grid in x, h>0 is the spacing in x along the grid which has M points and $O(h)^2$, represents the truncation error, substituting Eq. 12 and 2 gives a system of m simultaneous ordinary differential equations of the first order:

$$\frac{dW_i}{dt} \approx \frac{v}{h^2}[w_{i+1} - 2w_i + w_{i-1}], 1 \le i \le M \tag{13}$$

Here, W_t is an approimate function of w_i:

$$\frac{dW_i(t)}{dt} = \frac{v}{h^2} A w_i(t),\ 1 \le i \le M \tag{14}$$

where $w_i(t) = [w_1(t), w_2(t),...,w_{M-1}(t)]^T$, A is the tridigonal matrix = Trid(1-21), solving the system of ODEs under given initial and boundary conditions:

$$\left.\begin{aligned} & w(x_i,0) = g(x_i)\ ,\quad 1 \le i \le M \\ & w(0,t) = w_1(x,t) = h_1(t) \quad \text{at grid } i = 1 \\ & w(L,t) = w_L(x,t) = h_L(t) \end{aligned}\right\} \tag{15}$$

by using Runge-Kutta of 6th order with seven stages method[20]:

$$\begin{aligned} W_i^1 = W_i^0 + h(b_1K_{i1} + b_2K_{i2} + b_3K_{i3} + \\ b_4K_{i4} + b_5K_{i5} + b_6K_{i6} + b_7K_{i7}) \end{aligned} \tag{16}$$

where:

$$\begin{cases} K_{i1} = \phi_i(t_0, W_i^0) \\ K_{i2} = \phi_i(t_0 + c_2h, W_i^0 + ha_{21}K_{i1}) \\ K_{i3} = \phi_i(t_0 + c_3h, W_i^0 + h(a_{31}K_{i1} + a_{32}K_{i2})\,) \\ K_{i4} = \phi_i(t_0 + c_4h, W_i^0 + h(a_{41}K_{i1} + a_{42}K_{i2} + a_{43}K_{i3})\,) \\ K_{i5} = \phi_i(t_0 + c_5h, W_i^0 + h(a_{51}K_{i1} + a_{52}K_{i2} + \\ a_{53}K_{i3} + a_{54}K_{i4})\,) \\ K_{i6} = \phi_i(t_0 + c_6h, W_i^0 + h(a_{61}K_{i1} + a_{62}K_{i2} + a_{63}K_{i3} + \\ a_{64}K_{i4} + a_{65}K_{i5})\,) \\ K_{i7} = \phi_i(t_0 + c_7h, W_i^0 + h(a_{71}K_{i1} + a_{72}K_{i2} + \\ a_{73}K_{i3} + a_{74}K_{i4} + a_{75}K_{i5} + a_{76}K_{i6})\,) \end{cases} \tag{17}$$

It is convenient to specify Eq. 17 by writing the parameters in an array called Butcher partitioned tableau. In the present case the array Fig. 1.

Coefficient of Butcher RK of order six with 7 stages:

$$\begin{array}{cccccccc}
0 & & & & & & & \\
\frac{1}{3} & \frac{1}{3} & & & & & & \\
\frac{2}{3} & 0 & \frac{2}{3} & & & & & \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{3} & -\frac{1}{12} & & & & \\
\frac{5}{6} & \frac{25}{48} & -\frac{55}{24} & \frac{35}{48} & \frac{15}{8} & & & \\
\frac{1}{6} & \frac{3}{20} & -\frac{11}{24} & -\frac{1}{8} & \frac{1}{2} & \frac{1}{10} & & \\
1 & -\frac{261}{260} & \frac{33}{13} & \frac{43}{156} & -\frac{118}{39} & \frac{32}{195} & \frac{80}{39} & \\
 & \frac{13}{200} & 0 & \frac{11}{40} & \frac{11}{40} & \frac{4}{25} & \frac{4}{25} & \frac{13}{200}
\end{array}$$

which reduce to the equation:

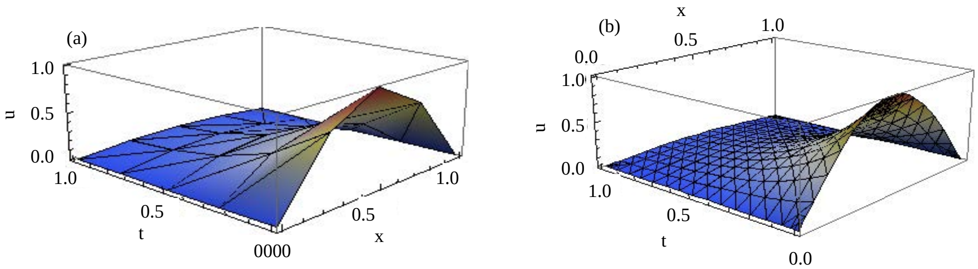

Fig. 1(a, b): (a) Plot of the $u_{app.}(x, t)$ and (b) Plot of $u_{exact}(x, t)$ for the Burger's equation

$$W_i^1 = W_i^0 + \frac{h}{200}(13K_{i1}+55K_{i3}+55K_{i4}+32K_{i5}+32K_{i6}+13K_{i7}) \quad (18)$$

with the help of Math Lab, get the approximate solution for the transformed heat equation, hence, the approximate solution of the Burger's problem is obtained using the transformation Eq. 11.

MATERIALS AND METHODS

Differential transform method: This method appear in different situations such as differential equation(s), eigenvalue problems, approximate solution of a system of ODEs, the method is firstly introduced by Zhou[22], associated with the method of lines, the method can be extended for solving system of linear and nonlinear PDEs, this method leads to an iterative procedure for obtaining an analytic series solutions of functional equations. The basic definition of two dimensional differential transform is defined as follows:

Definition 1: If u(x, y) is analytic and continuously differentiable function with respect to x and y then:

$$U(h,k) = \frac{1}{h!k!}\left[\frac{\partial^{h+k}u(x,y)}{\partial x^h \partial y^k}\right]_{x=0,y=0} \quad (19)$$

where the spectrum function U(h, k) is the transformed function.

Definition 2: The differential inverse transform of is U(h, k) defined as:

$$u(x,y) = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} x^h y^k U(h,k) \quad (20)$$

Combining Eq. 15 and 16, it can be obtained that:

$$u(x,y) = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{h!k!}\left[\frac{\partial^{h+k}u(x,y)}{\partial x^h \partial y^k}\right] x^h y^k \quad (21)$$

From the above definitions, it can be found that the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion some of the basic mathematical operations performed by DTM can be readily obtained and these are listed in Table 1.

Numerical examples: In this study, the solution of Burger's equation will be investigated by using the proposed Hopf-Cole transformation coupling with the RK6 order method, to clarify the accuracy of the present method. These example are chosen such that the exact solution can be given by applying DTM.

Example 1: Consider burger Eq. 1 subject to the IC and Dirichlet homogeneous BCs:

$$\left.\begin{aligned} &u(x,0) = \sin(\pi\ x), 0 \le x \le 1 \\ &u(0,t) = u(1,t) = 0,\ t>0 \end{aligned}\right\} \quad (22)$$

where the exact solution is:

$$u(x,t) = e^{-v\pi^2 t}\sin(\pi\ x) \quad (23)$$

By the Hopf-Cole transformation Eq. 11 and 1 transformed into the heat equation:

$$w_t = v\ w_{xx},\ \ v > 0, t > 0, 0 \le x \le 1 \quad (24)$$

with the transformed initial condition:

$$w(x,0) = e^{\frac{\cos(\pi x)-1}{2v\pi}},\ 0 \le x \le 1 \quad (25)$$

and transformed boundary conditions:

$$w_x(0,t) = w_x(1,t) = 0,\ t > 0 \quad (26)$$

Table 1: The fundamental operations of the two dimensional DT

Original function	Transformed function
$u(x,t)=f(x,t)\pm g(x,t)$	$U(h,k)=F(h,k)\pm G(h,k)$
$u(x,t)=c\ f(x,t)$	$Uh,k)=c\ F(h,k)$ where c is constant
$u(x,t)=\frac{\partial f(x,t)}{\partial x}$	$U(h,k)=(h+1)F(h+1,k)$
$u(x,t)=\frac{\partial f(x,t)}{\partial x}$	$U(h,k)=(k+1)F(h,k+1)$
$u(x,t)=\frac{\partial f(x,t)}{\partial x}$	$U(h,k)=\frac{(h+r)!(k+s)!}{h!k!}\ F(h+r,k+s)$
$u(x,t)=f(x,t)\otimes g(x,t)$	$U(h,k)=\frac{(h+r)!(k+s)!}{h!k!}\ F(h+r,k+s)$
$u(x,t)=\frac{\partial f(x,t)}{\partial x}\ \frac{\partial g(x,t)}{\partial t}$	$U(h,k)=\sum_{r=0}^{h}\sum_{s=0}^{k}(h\text{-}r+1)(k\text{-}s+1)F(h\text{-}r+1,s)\ G(h,k\text{-}s+1)$

Applying Eq. 12 for (Eq. 24), we obtained a system of M linear first order differential equation which can be written in matrix form as:

$$\frac{dW_i(t)}{dt}=\frac{v}{h^2}AW_i(t) \tag{27}$$

for the simple case $1\le i\le 3$ the system reduced to:

$$\begin{bmatrix}\frac{dw_1(t)}{dt}\\ \frac{dw_2(t)}{dt}\\ \frac{dw_3(t)}{dt}\end{bmatrix}=\frac{v}{h^2}\begin{bmatrix}w_0(t)-2w_1+w_2\\ w_1-2w_2+w_3\\ w_2-2w_3+w_4(t)\end{bmatrix} \tag{28}$$

$$=\phi_i(t,w_1,w_2,w_3)$$

$$\text{Where } A=\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -2\end{bmatrix} \text{ and } W_i=\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix} \tag{29}$$

using Runge-Kutta 6th order method with seven stages for the system of ODEs Eq. 28 and boundary conditions:

$$w_0(t)=w(0,t)=0,\ \ w_4(t)=w(1,t)=0 \tag{30}$$

and for simplicity, choose v = 0.3, h = 0.25, we get the approximate solution $w_i(x, t)$ given by Table 2. Now, calculate the values of $u_{app.}(x, t)$ at discrete points from the discrete version of Eq. 11:

$$u_{app.}(x,t)=u_i(x,t)=-2v\frac{(w_i)_x}{w_i} \tag{31}$$

Table 2: Exact solution by DTM and the approximate solution by coupling C-HM with RK6 order method of example

t-values	x	$u_{exact}(x,t)$	$u_{app.}(x,t)$
0.00	0.25	0.707107	0.707107
0.25		0.337296	0.350770
0.50		0.160893	0.174014
0.75		0.0767475	0.086425
1.00		0.0366092	0.043899
0.00	0.50	1.000000	1.000000
0.25		0.477009	0.496060
0.50		0.227537	0.246062
0.75		0.108537	0.121916
1.00		0.0517733	0.059025
0.00	0.75	0.707107	0.707107
0.25		0.337296	0.350770
0.50		0.160893	0.174014
0.75		0.0767475	0.086425
1.00		0.0366092	0.043899

where $(w_i)_x=\frac{\partial w_i}{\partial x}$ can be calculated from the first order centered difference formula:

$$(w_i)_x\approx\frac{\partial w_i}{\partial x}=\frac{w_{i+1}-w_{i-1}}{2h}1\le\ i\le 3 \tag{32}$$

the derivatives $(w_0)_x$ and $(w_4)_x$ at the end points are known, substituting Eq. 32 into 31 conclude equation approximate solution for the burger Eq. 1:

$$u_i\approx(\frac{-v}{h})\ \frac{w_{i+1}-w_{i-1}}{w_i}1\le\ i\le 3 \tag{33}$$

For comparative study, we have to apply the DTM for Burger Eq. 1 subject to the initial and boundary conditions Eq. 22, firstly, we discretizing Eq. 1 for the spatial first and second derivatives using the centered differences h>0 is the spacing in x along the grid which has M points and $O(h)^2$ represents the truncation error:

$$\left.\begin{aligned}\frac{\partial^2 u_i}{\partial x^2} &\approx \frac{1}{h^2}[u_{i+1} - 2u_i + u_{i-1}] + O(h)^2 \\ \frac{\partial u_i}{\partial x} &\approx \frac{u_{i+1} - u_{i-1}}{2h} + O(h)^2\end{aligned}\right\} \tag{34}$$

the system of ODEs of m-1 unknowns is obtained:

$$\frac{du_i}{dt} = \frac{v}{h^2}(Au_i + B(t)) - \frac{1}{2h}F(t, u) \tag{35}$$

where:

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}, B(t) = \begin{bmatrix} u(0,t) \\ 0 \\ 0 \\ \vdots \\ 0 \\ u(1,t) \end{bmatrix} \text{ and}$$

$$F(t, u_i) = \begin{bmatrix} u_1(u_0 - u_2) \\ u_2(u_1 - u_3) \\ \vdots \\ \vdots \\ u_{m-1}(u_{m-2} - u_m) \end{bmatrix}$$

Applying the DTM and operations of Table 1 to Eq. 35, we get the recurrence equation:

$$(k_2+1)U(k_1, k_2+1) = v\frac{(k_1+2)!}{k_1!}U(k_1+2, k_2) - u \otimes u_x\big|_{x=t=0} \tag{36}$$

or

$$U(k_1, k_2+1) = \frac{1}{(k_2+1)}\{v\frac{(k_1+2)!}{k_1!}U(k_1+2, k_2) - \sum_{r=0}^{k_1}\sum_{s=0}^{k_2}(k_1 - r + 1)U(r, k_2 - s)U(k_1 - r + 1, s)\} \tag{37}$$

then from the initial condition Eq. 22, we have:

$$\sum_{k=0}^{\infty} U(k,0)x^k = \sum_{k=0}^{\infty}\frac{u^{(k)}(0)}{k!}x^k \tag{38}$$

therefore, for $k_1, k_2 = 0,1,2,3,....,N$ we get

$$U(0,1) = 2!vU(2,0) - U(0,0)U(1,0) = 0$$

$$U(1,1) = 3!vU(3,0) - 2U(0,0)U(2,0) - (U(1,0))^2 = -\frac{3!}{1!}v\pi^3 - \pi^2$$

$$U(2,1) = \frac{4!}{2!}vU(4,0) - 3U(0,0)U(3,0) \text{ -}3U(1,0)U(2,0) = 0$$

$$U(3,1) = \frac{5!}{3!}vU(5,0) - 4U(0,0)U(4,0) \text{ -}4U(1,0)U(3,0) - 2(U(2,0))^2 = 20\ v\ \pi^5 + 4\ \pi^4$$

$$U(4,1) = \frac{6!}{4!}vU(6,0) - 5U(0,0)U(5,0) \text{ -}5U(1,0)U(4,0) - 5U(2,0)U(3,0 = 0$$

$$U(0,2) = \frac{1}{2}\{\frac{2!}{0!}vU(2,1) - U(0,1)U(1,0)\text{-}U(0,0)U(1,1)\} = 0$$

$$U(1,2) = \frac{1}{2}\{\frac{3!}{1!}vU(3,1) - 2U(0,1)U(2,0) \text{-}2U(0,0)U(2,1) - 2U(1,0)U(1,1)\} = 60\ v\ \pi^5 + 18\ v\ \pi^4 + \pi^3$$

$$U(2,2) = \frac{1}{2}\{\frac{4!}{2!}vU(4,1) - 3U(0,1)U(3,0)\text{-} 3U(0,0)U(3,1) - 3U(1,0)U(2,1) - 3U(2,0)U(1,1)\} = 0$$

In the same way, the rest of components were obtained by using Eq. 37, substituting the above quantities in Eq. 38, the approximate infinite series solution for each x and t) is:

$$u(x,t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} U(k_1,k_2)x^{k_1}t^{k_2} =$$

$$(\pi x - \frac{\pi^3 x^3}{6} + \frac{\pi^5 x^5}{120} +, \ldots,) + (-\pi^3 vx + \frac{1}{6}\pi^5 vx^3 - \frac{1}{120}(\pi^7 v)x^5 +, ...,)t + (\frac{1}{2}\pi^5 v^2 x - \frac{1}{12}(\pi^7 v^2)x^3 + \frac{1}{240}\pi^9 v^2 x^5 +, \ldots,)t^2 + (-\frac{1}{6}(\pi^7 v^3)x + \frac{1}{36}\pi^9 v^3 x^3 + (-\frac{1}{120}(\pi^{11} v^5)x + \frac{1}{720}\pi^{13} v^5 x^3 - \frac{(\pi^{15} v^5)x^5}{14400} +, \ldots,)t^5 +, \ldots, =$$

$$(1 - \pi^2 vt + \frac{1}{2}\pi^4 v^2 t^2 - \frac{1}{6}\left(\pi^6 v^3\right)t^3 + \frac{1}{24}\pi^8 v^4 t^4 - \frac{1}{120}\left(\pi^{10} v^5\right)t^5 +, \ldots,)(\pi x - \frac{\pi^3 x^3}{6} + \frac{\pi^5 x^5}{120} +, , \ldots) = e^{-v\pi^2 t}\sin(\pi\ x)$$

by substituting some values, we get the results of exact solution which we will compare with proposed numerical method given by Table 3 and shown in Fig. 1.

Example 2: Consider Burger Eq. 1 subject to the IC and Dirichlet homogeneous BCs:

$$\left.\begin{aligned}u(x,0) &= 2v\frac{\pi\ \sin(\pi\ x)}{3 + \cos(\pi\ x)}, 0 \le x \le 1 \\ u(0,t) &= u(1,t) = 0,\ t > 0\end{aligned}\right\} \tag{39}$$

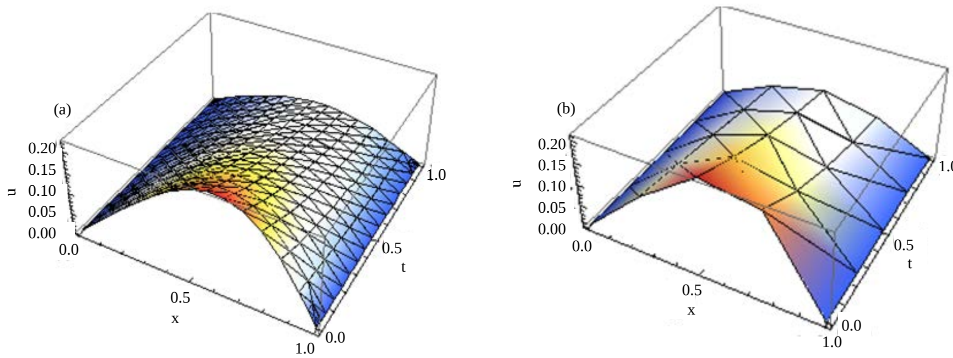

Fig. 2(a, b): (a) Plot of the $u_{exact}(x, t)$ and (b) Plot of for the $u_{app.}(x, t)$ for the Burger's equation example 2

Table 3: Exact solution by DTM and the approximate solution by combine C-HM with RK6 order method of example 2

t-values	x	$u_{exact}(x,t)$	$u_{app.}(x,t)$
0.00	0.25	0.11984800	0.119848
0.25		0.09771780	0.1051120
0.50		0.07903900	0.0882470
0.75		0.06350330	0.0722290
1.00		0.05073920	0.0582640
0.00	0.50	0.2094400	0.2094400
0.25		0.16364400	0.1690050
0.50		0.12786200	0.1345790
0.75		0.09990450	0.1067090
1.00		0.07805970	0.0844940
0.00	0.75	0.1937687	0.1937687
0.25		0.14183500	0.1384070
0.50		0.10560900	0.1032440
0.75		0.07959180	0.0789830
1.00		0.06051250	0.0613060

where the exact solution is:

$$u(x,t) = \frac{2v\pi\, e^{-v\pi^2 t} \sin(\pi\, x)}{3 + e^{-v\pi^2 t} \cos(\pi\, x)} \tag{40}$$

By Hopf-Cole transformation Eq. 11 and 1 transformed into the heat equation:

$$w_t = v\, w_{xx}\; v = 0.1, t > 0,\; 0 \le x \le 1 \tag{41}$$

with the transformed initial condition:

$$w(x,0) = \frac{3 + \cos(\pi\, x)}{4}, 0 \le x \le 1 \tag{42}$$

and transformed boundary conditions:

$$w_x(0,t) = w_x(1,t) = 0,\; t > 0 \tag{43}$$

We will do the same work as in example 1 ,we will get the results given in Table 3 and shown in Fig. 2.

CONCLUSION

The Hopf-Cole transformation method generally recognized as a powerful approach to transform the Burger's equation into the heat equation, combine with Runge-Kutta of order 6th method and the help of method of lines gives a suitable numerical method (the new proposed technique), the computed results with the use of this technique of the illustrate example show that this approach gives the required accuracy compared with exact solution by using differential transform method. As a future work, this proposed method can be applied to a system of Burger's equations if the condition of potential symmetry $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ is satisfied.

ACKNOWLEDGMENTS

The researcher would like to thank Al-Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad, Iraq for its support in the present research.

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