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Non-One Sided Bounded Variation Sequences Order r

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Abstract: Trigonometric series in partial differential equations contain coefficients of the Fourier series which is decreasing monotone and convergence to zero. The properties of the Fourier series coefficient are sufficient conditions for the series to convergence uniformly. The coefficients in the Fourier series have been developed into several classes, such as General Monotone Sequences (GMS) and non-one sided bounded variation sequences (NBVS). Not long after that there was a new class, namely General Monotone Sequences order r (GMS(r)). Of the several classes mentioned above and still meet the convergence requirements, so, they are still guaranteed to be in the Fourier series. This study will discuss the development of the non-one sided bounded variation sequences class into order r such as general monotone sequences order r.

INTRODUCTION

The concept of single-sequence monotone classes in this section will present theories about the quasi monotone class and the rest bounded variation class. Here, are some definitions and theorems that discuss the concept of a single-sequence monotonous class:

Theorem 1: Suppose that $a_n \geq a_{n+1}$ and $a_n \to 0$, then necessary and sufficient conditions for the uniform convergence of the series is:

$$\sum_{n=1}^{\infty} a_n \sin nx$$

Is $na_n \to 0$[1]. The coefficient sequence of theorem 1 belongs to a class called the monotone sequences class, written by MS.

Definition 2: Let $a = \{a_n\}$ be a complex sequences. Sequences a called General Monotone Sequences (GMS), if there are positive constant C such that:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C|a_n|$$

for all $n \in N$[2].

Definition 3: Let $a = \{a_n\}$ be a complex sequences. Sequences a called Non-one Sided Bounded Variation Sequences (NBVS), if there are positive constant C such that:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C(|a_n| + |a_{2n}|)$$

For all $n \in N$[3].

Definition 4: Let $\beta = \{\beta_n\}$ be a positive sequence. The sequence of complex numbers $a = \{a_n\}$ is said to be β-general monotone sequences or $a \in GM(\beta)$, if the relation:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \le C\beta_n$$

for all n∈N and positive constant C[4].

Definition 5: Let $\beta = \{\beta_n\}$ be a nonnegative sequence and r a natural number. The sequences of complex $a = \{a_n\}$ is said to be (β, r)-General Monotone or a∈GMS(β, r), if the relation:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+r}| \le C\beta_n$$

for all n∈N[5].

Theorem 6: Let a nonnegative sequence (a_n)∈GM(β, r) where r≥1. If the series (1) converges uniformly then $na_n \to 0$ as $n \to \infty$.

Theorem 7: Let a sequence (a_n)∈GM(β, 2). If $n|a_n| \to 0$ as $n \to \infty$ then the series 1 converges uniformly.

Theorem 8: Let a∈$SBVS_p(\beta, \Delta^n)$, $1 \le p < \infty$ with β real non-negative sequence, if $\{a_n\}$ decreasing monotone:

$$\alpha_m = m^{-1/p} \sup_{i \ge b_m} \sum_{k=i}^{2i} \beta_k$$

And:

$$m\left(\sum_{k=m}^{\infty} |\Delta^t a_k|^p\right)^{1/p} < \left(\frac{C}{m} \sup_{i \ge b_m} \sum_{v=i}^{2i} \beta_v\right)$$

for 1≤t≤n-1, m≥n, then:

$$|g(x) - S_{m-1}(g, x)| \le 6C(m+2M)m^{-\frac{1}{p}} \sup_{i \ge b_m} \sum_{k=i}^{2i} \beta_k$$

where, C positive constant only depending on $SBVS_p$ (β, Δ^n) with m≥n, x = π/M and x∈(0, π).

MATERIALS AND METHODS

This research uses literature study by utilizing several books and several scientific journals. From these books and journals, we get several definitions and theorems to develop broader definitions and theorems.

RESULTS AND DISCUSSION

In this study, we research a new class that is a generalization of the class non one sided bounded variation sequences.

Definition 9: Let $a = \{a_n\}$ is complex number sequences and r is a natural number. Sequences of a belongs to class non one sided bounded variation sequences order r, we write NBVS(r), if there is positive constant C such that:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+r}| \le C(|a_n| + |a_{2n}|)$$

if r = 1, then NBVS(1) = NBVS.

Theorem 10: Let r∈N, then NBVS(1)⊆NBVS(r).

Proof: Suppose given a∈NBVS(1), then there is positive constant C, then for each r∈N such that:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+r}| = \sum_{k=n}^{2n-1} \left|\sum_{p=0}^{r-1} (a_{k+p} - a_{k+p+1})\right|$$
$$\le \sum_{k=n}^{2n-1} \sum_{p=0}^{r-1} |a_{k+p} - a_{k+p+1}| = \sum_{p=0}^{r-1} \sum_{k=n+p}^{2n+p-1} |a_k - a_{k+1}|$$
$$\le C(|a_{n+p}| + |a_{2n+p}|)$$

Therefore, a∈NBVS(r) proven NBVS(1)⊆NBVS(r).

Theorem 11: If r_1, r_2∈N, $r_1 < r_2$ and $r_1|r_2$, then NBVS(r).

Proof: If $r_1|r_2$ there i∈N is such that $r_2 = i.r_1$. Furthermore given a∈NBVS(r_2), then there are positive constant C such that:

$$\sum_{k=n}^{2n-1} |a_k - a_{k+r_2}| = \sum_{k=n}^{2n-1} \left|\sum_{i=0}^{p-1} \left(a_{k+ir_1} - a_{k+(i+1)r_1}\right)\right|$$
$$\le \sum_{k=n}^{2n-1} \sum_{i=0}^{p-1} |a_{k+ir_1} - a_{k+(i+1)r_1}| = \sum_{i=0}^{p-1} \sum_{k=n+ir_1}^{2n+ir_1-1} |a_k - a_{k+r_1}|$$
$$\le C(|a_{n+pr_1}| + |a_{2n+pr_1}|)$$

Therefore, a∈NBVS(r_2) proven NBVS(r_1)⊆NBVS(r_2).

Theorem12: If a∈NBVS(r) and $\{n|a_n|\}$ decreasing monoteone, then:

$$\sum_{m=n}^{\infty} |a_m - a_{m+r}| \le C(|a_n| + |a_{2n}|)$$

Proof: For each n∈N because a∈NBVS(r) then obtained:

$$n\sum_{m=n}^{\infty} |a_m - a_{m+r}| = \sum_{s=0}^{\infty} \sum_{m=2^s n}^{2^{s+1}n-1} |a_m - a_{m+r}|$$
$$\le \sum_{s=0}^{\infty} 2^s n \frac{|a_{2^s n}|}{2^s} \le \sum_{s=0}^{\infty} n \frac{|a_n|}{2^s} = 2C'n|a_n|$$

So:

$$\sum_{m=n}^{\infty} |a_m - a_{m+r}| \le C(|a_n| + |a_{2n}|), C = 2C'$$

Theorem 13: If $a \in NBVS(r)$ dan $\{n|a_n|\}$ convergence to 0, then:

$$\lim_{n\to\infty} n \sum_{m=n}^{\infty} |a_m - a_{m+r}| = 0$$

Proof: Let $d_n = \sup_{m \geq n}(m|a_m|)$, then d_n decreasing monoteone and $\lim_{n\to\infty} d_n = \lim_{n\to\infty} n|a_n| = 0.$ Suppose $a \in NBVS(r)$ given according to theorem 12, we have:

$$n \sum_{m=n}^{\infty} |a_m - a_{m+r}| \leq 2C'd_n$$

so:

$$\lim_{n\to\infty} n \sum_{m=n}^{\infty} |a_m - a_{m+r}| = 0$$

Definition 14: Let $a = \{a_n\}$ is complex number sequences and r is a natural number. Sequences of a belongs to class Non one Sided Bounded Variation Difference Sequences order r ($NBVS(\Delta^{r+1})$, if there are positive constant a C such that:

$$\sum_{k=n}^{2n-1} |\Delta^r a_k| \leq C(|a_n| + |a_{2n}|), n \geq r \geq 1$$

with $\Delta^r a_k = \Delta^{r-1} a_k - \Delta^{r-1} a_{k+1}$.

Theorem 15: If $r \in N$ and $|\Delta^r a_k| \geq |\Delta^r a_{k+1}|$, then $NBVS(\Delta^r) \subseteq NBVS((\Delta^{r+1})$.

Proof: Suppose given $a \in NBVS(\Delta^r)$, then there are positive constant C such that:

$$\sum_{k=n}^{2n-1} |\Delta^r a_k| \leq C(|a_n| + |a_{2n}|), n \geq r$$

noted that:

$$|\Delta^{r+1} a_k| = |\Delta^r a_k - \Delta^r a_{k+1}| \leq |\Delta^r a_k| + |\Delta^r a_{k+1}|$$

Then:

$$\sum_{k=n}^{2n-1} |\Delta^{r+1} a_k| \leq \sum_{k=n}^{2n-1} |\Delta^r a_k| + |\Delta^r a_{k+1}|$$
$$\leq \sum_{k=n}^{2n-1} |\Delta^r a_k| + |\Delta^r a_k| \leq C(|a_n| + |a_{2n}|), n \geq r$$

So, $a \in NBVS(\Delta^{r+1})$ and $NBVS(r) \subseteq a \in NBVS(\Delta^{r+1})$.

Theorem 16: If $r \in N$ $a \in NBVS(\Delta^r,)$ and $\{n|a_n|\}$ decreasing monotone, such that:

$$\sum_{m=n}^{\infty} |\Delta^r a_m| \leq C(|a_n| + |a_{2n}|)$$

Proof: Suppose given $a \in NBVS(\Delta^r)$, we write:

$$n \sum_{m=n}^{\infty} |\Delta^r a_m| = n \sum_{s=0}^{\infty} \sum_{m=2^s n}^{2^{s+1}n-1} |\Delta^r a_v| \leq C' n \sum_{s=0}^{\infty} \frac{|a_{2^s n}| 2^s n}{2^s n}$$
$$\leq C' \sum_{s=0}^{\infty} \frac{n|a_n|}{2^s} \leq 2C' n(|a_n| + |a_{2n}|)$$
$$= Cn(|a_n| + |a_{2n}|), C = 2C'$$

So,

$$\sum_{m=n}^{\infty} |\Delta^r a_m| \leq C(|a_n| + |a_{2n}|)$$

Lemma 17: Let given $r \in N$, if $a \in NBVS(\Delta^r)$ and $\lim_{n\to\infty} n|a_n| = 0$, then $\lim_{n\to\infty} \sum_{m=n}^{\infty} |\Delta^r a_m| = 0$.

Proof: Let $d_n = \sup_{m \geq n}(m|a_m|)$ then d_n decreasing monotone and $\lim_{n\to\infty} d_n = \lim_{n\to\infty} n|a_n| = 0.$ according to theorem 16:

$$n \sum_{m=n}^{\infty} |\Delta^r a_m| \leq Cd_n$$

and

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty}(|a_n| + |a_{2n}|) = 0$$

and

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} |\Delta^r a_m| = 0$$

Theorem 18: Let $r \in N$, if $a \in NBVS(\Delta^r)$ and $\lim_{n\to\infty} n|a_n| = 0$:

- with r = 1, then sinus deries convergence uniform on $[0, \pi]$
- $\lim_{n\to\infty} n \sum_{m=n}^{\infty} |\Delta a_m + \Delta^2 a_m +, \ldots, + \Delta^{r-1} a_m| = 0,$ with $r \geq 2$, then sinus deries convergence uniform on $[0, \pi]$

Proof:

- Let $a \in NBVS(\Delta^r)$ for $r \in N$. Then, we have:

$$g(x) - S_m(g, x),$$

with

$$S_m(g, x) = \sum_{j=1}^{m} a_j \sin jx$$

Then, we calculate:

$$g(x) - S_{m-1}(g, x) = \sum_{k=m}^{\infty} a_k \sin kx$$

To calculate $\sum_{k=m}^{\infty} a_k \sin kx$ with any $x \in (0, \pi)$ can be selected $N \in N$, so that, $x \in \left[\frac{\pi}{N+1}, \frac{\pi}{N}\right]$. Therefore:

$$\sum_{v=1}^{m} \sin vx = \frac{\cos\frac{1}{2}x - \cos\left(m+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} \le \frac{1}{\sin\frac{1}{2}x} \le \frac{\pi}{x}$$

if $N \le m$, then $D^*_m(x) = \sum_{v=1}^{m} \sin vx \le \frac{\pi}{x}$. Furthermore sum:

$$\sum_{k=m}^{\infty} a_k \sin kx = A+B$$

separated by:

$$A = \sum_{k=m}^{m+N-1} a_k \sin kx$$

$$B = \sum_{k=m+N}^{\infty} a_k \sin kx$$

Part A toward 0 because:

$$\begin{aligned}\left|\sum_{k=m}^{m+N-1} a_k \sin kx\right| &\le x\left|\sum_{k=m}^{m+N-1} ka_k\right| \\ &= x\left|\sum_{k=m}^{m+N-1} k\sum_{s=k}^{\infty} \Delta a_s\right| \\ &= x\sum_{k=m}^{m+N-1} k\sum_{s=k}^{\infty} |\Delta a_s|\end{aligned} \tag{2}$$

According to 17:

$$|A| \le x\, No(1) \le \pi\, o(1)$$

Part B, according to Abel transform obtained:

$$B = \sum_{k=m+N}^{\infty} \Delta a_k D^*_{m+N}(x) - a_{m+N} D^*_{m+N-1}(x) = S+T \tag{3}$$

With $D^*_m(x) = \sum_{j=1}^{m} \sin jx$ and:

$$S = \sum_{k=m+N}^{\infty} \Delta a_k D^*_k(x)$$

$$T = -a_{m+N} D^*_{m+N-1}(x)$$

$$|S| \le \frac{\pi}{x}|a_{m+N}| \text{ and } |T| \le \frac{\pi}{x}|a_{m+N}|$$

So, that:

$$\begin{aligned}|B| &\le \frac{2\pi}{x}|a_{m+N}| \le 2(N+1)|a_{m+N}| \\ &\le 2(m+N)\left|\sum_{s=m+N}^{\infty} \Delta a_s\right| \le 2(m+N)\sum_{s=m+N}^{\infty} |\Delta a_s|\end{aligned} \tag{4}$$

According to lemma 17, we obtained $|B| \le 2o(1)$. So:

$$A+B \le (2+\pi)o(1),\ \forall x \in (0, \pi)$$

So for any $\varepsilon>0$, there are $n_1 \in N$ such that for $m \ge n_1$:

$$|g(x) - S_{m-1}(g,x)| \le A+B \le K\varepsilon$$

with $K = (2+\pi)$. For $x = 0$, $S_m(g, 0) = 0$. So, $S_m(g, x)$ convergence uniform on $[0, \pi]$.

- For $n \ge 2$ and according to Eq. 2 obtained:

$$\begin{aligned}|A| &= x\sum_{k=m}^{m+N-1} k\sum_{s=k}^{\infty} |\Delta a_s| \\ &= x\sum_{k=m}^{m+N-1} k\sum_{s=k}^{\infty} \left|\Delta a_{s+1} + \Delta^2 a_{s+1} +, \dots, + \Delta^{n-1} a_{s+1} + \Delta^r a_s\right| \\ &\le x\sum_{k=m}^{m+N-1} k\sum_{s=k}^{\infty} \left|\Delta a_{s+1} + \Delta^2 a_{s+1} +, \dots, + \Delta^{n-1} a_{s+1}\right| + \left|\Delta^r a_s\right| \\ &\le x\, No(1) + (Cn|a_n|) \le \pi o(1) + Co(1)\end{aligned}$$

from (3) obtained:

$$|B| = \sum_{k=m+N}^{\infty} \Delta a_k D^*_{m+N}(x) - a_{m+N} D^*_{m+N-1}(x) = S+T$$

and from (4) obtained:

$$B \le 2(m+N)\sum_{s=m+N}^{\infty} |\Delta a_s|$$

or

$$\begin{aligned}|B| &\le 2(n+N)\sum_{s=m+N}^{\infty} \left|\Delta a_{s+1} + \Delta^2 a_{s+1} +, \dots, + \Delta^{n-1} a_{s+1} + \Delta^r a_s\right| \\ &\le 2(m+N)\sum_{s=m+N}^{\infty} \left|\Delta a_{s+1} + \Delta^2 a_{s+1} +, \dots, + \Delta^{n-1} a_{s+1}\right| \\ &\quad + 2(m+N)\sum_{s=m+N}^{\infty} |\Delta a_s|\end{aligned}$$

According to condition (i) and lemma 17we obtained:

$$|B| \le 2o(1) + 2o(1)$$

So:

$$A+B \le (4+C+\pi)o(1), \forall x \in (0, \pi)$$

For any $\varepsilon>0$ there are $n_1 \in N$, so for $m \ge n_1$

$$|g(x) - S_{m-1}(g,x)| \le A+B \le K\varepsilon$$

With $K = 4+C+\pi$. On $x = 0$ and $S_m(g, 0) = 0$, so, $S_m(g, x)$ convergence uniform on $[0, \pi]$.

CONCLUSION

In this study, we have introduced the class. We have obtained some conclusion:

- Let r_1, $r_2 \in N$, $r_1 < r_2$ and $r_1 | r_2$, then $NBVS(r_1) \subseteq NBVS(r_2)$
- If $r \in N$ and $|\Delta^r a_k| > |\Delta^r a_{k+1}|$, then $NBVS(\Delta^r) \subseteq NBVS(\Delta^{r+1})$
- Non one sided Bounded Variation Difference Sequences order r ($NBVS(\Delta^r)$) convergence uniform on $[0, \pi]$

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