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Studying the Chaotic of Modified Jerk Map Based on Lyapunov Exponents, Topological Entropy and Sensitivity

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Abstract: In the last four decades, Chaos has been studied intensively as an interesting practical phenomenon. Hence, it is considered to be one of the most important branches in mathematics science that deals with the dynamic behavior of systems which are sensitive to the initial conditions. It has therefore been used in many scientific applications in the sciences of chemistry, physics, computers, communications, cryptography and engineering as well as in bits generators and psychology. However, there are many issues that need to be considered and highlighted such as future prediction, computational complexities and unstable behavior of dynamic system. The dynamic system must contain three characteristics in order to be considered a chaotic system which is first, to be sensitive to the initial conditions; second to have dense periodic orbits and finally to be topologically mixing. In the previous work, we studied the fixed point of a modified Jerk Map with the form $MJ_{a,\ b} = (y\text{-}ax\text{+}by^2)$ in order to find the contracting and expanding area of this map as well as to define the area in which the fixed points of attracting, repelling or saddle are located. In this study, we continue to address the same problem by modified Jerk Map. We prove that it has a positive Lypaunov exponent if $|a| = 1$ and has sensitivity dependence to initial condition if $|a|>1$ and we give an estimate of topological entropy. Finally, to simulate our equations and obtain related results, we have used MATLAB program by implementing a Lypaunov exponent and drawing the sensitivity of $MJ_{a,\ b}$.

INTRODUCTION

The dynamical system is a theory that has mostly been studied as an abstract concept subject in branches of Mathematics, Physics and Computer Science[1]. Mostly, it is considered Chaotic according to either the metric properties or topology of the system[2]. Guirao and Lampart[3] addressed how three-period orbits with a dynamical system emphasize that the dynamical system is chaotic. Hence, chaos can generally be defined as an existing state between a specific and randomized state[4].

One of the most widely accepted and popular definitions of anarchy was defined by R.L. Devaney. In this definition, the systems must be based on exhibit topological transitivity sensitive dependency to initial conditions and dense periodic orbits[5]. Later, several works such as Hunt and Ott[4] and Dai[6] were implemented to prove that if a system is transitive with dense periodic orbits then it should obviously show sensitivity dependency to an initial condition.

Lyapunov defines the Chaos as follows: the continuously differentiable map can be chaotic if and only if the has a positive Lyapunov exponent and if it is topologically transitive[4, 7]. Chaos has been studied intensively as an interesting practical phenomenon in the last four decades. Hence, it is considered as one of the most important branches in mathematics science and also can be used in many significant applications in the sciences of computers and cryptography[8], bits generators[9, 10], Ecology[11], Economy[9, 10, 12], Biology[13, 14] and Communications[15, 16].

The Lyapunov exponents give the average exponential rate of convergence or divergence for near orbits in the phase-space. Thus, Chaotic will be defined for any dynamics system containing at least one positive Lyapunov's exponent. Any small initial differences initially in a system may affect its ability, consequently leading to less predictability. The dynamics system becomes unpredictable with the magnitude of the exponent indicating the time scale[17]. One of the important measures that are used to measure the complexity of the dynamics system is topological entropy. It represents the exponential growth rate of the number of distinguishable orbits iterates. It must, therefore be a non-negative real number[18].

In general, sensitivity is employed to nonlinear equations models. The idea of sensitivity is derived from the effect of the butterfly. The reason behind it is that lost patterns and the great effects of inputs are as marginal or negligible as the flap of the butterfly wings. Any change in initial condition, even if it were small, may lead to an undesired result. It is therefore, impossible to predict future behavior. However, this does not mean that the system is not deterministic[2]. Mendoza *et al*.[19] which is referred to Sprott[20] presented a new form in the explicit third order called Jerk Map as in the following Eq. 1:

$$x^{\cdots} = J\left(x,\ x^{\cdot},\ x^{\cdot\cdot}\right) \tag{1}$$

In term of physics, it mean that the Jerk Map is presented as a third derivative of the position with respect to time. In other words, according to Eq. 1, x"' is the third derivative of x that represents the rate of change of the acceleration in a mechanical system. Jerk dynamics can be described by a set of three first-order synchronous differential equations where the dependent variables are the position x, velocity $x^{\cdot}$ and acceleration $x^{\cdot\cdot}$. It is generally as follows[19]:

$$\frac{dx}{dt} = x^{\cdot} = u \tag{2}$$

$$\frac{d^2x}{dt^2} = x^{\cdot\cdot} = c \tag{3}$$

$$\frac{d^3x}{dt^3} = x^{\cdots} = -Ax^{\cdot\cdot}-Bx^{\cdot}+Q\left(x\right) \tag{4}$$

Sprott called Eq. 4, the last of these three equations as the Jerk equation[20] where the parameters A and B are numerical constantsand Q(x) is a nonlinear function. As is customary, the Jerk Map is a 3-D dynamical system. In our previous research by Al-Hashemi *et al*.[21], we transformed Jerk Map 3-D dynamical system into a 2-D dynamical system to reduce the computational time and space based on Eq. 5 as follows:

$$MJ_{a,\,b} = \begin{pmatrix} y \\ -ax+by^2 \end{pmatrix} \tag{5}$$

where, variables and are the states and prime indicates as differentiation. The work addresses and studies the fixed points of $DMJ_{a,\,b}$ as well as its general properties. It also found the contracting and expanding area of this map and is thus made to determine the fixed points of attracting, repelling or saddle. Continuing with the previous work, the proposing study aims to prove the properties of chaotic in dynamical system including sensitivity, Lyapunov exponents and topological entropy.

MATERIALS AND METHODS

Lyapunov exponents: In the first instance, Lyapunov exponents is developedas follows: let F be a continuous differential map at x, $\forall x \in X$ in direction u. Also Lyapunov exponent of a map F is defined by:

$$L\pm\left(x,u\right) = \lim_{n\to\infty}\frac{1}{n}\ln\left\|DF_x^n u\right\| \tag{6}$$

whenever the limit exists. In higher dimensions, map has Lyapunov exponents, based on:

$$L_1^{\pm}\left(x,\,u_1\right),\ L_2^{\pm}\left(x,\,u_1\right),\ L_3^{\pm}\left(x,\,u_3\right),\ ...,\ L_n^{\pm}\left(x,\,u_n\right) \tag{7}$$

for a minimum Lyapunov exponent that is:

$$L^{\pm}\left(x,\,u\right) = \max\left\{L_1^{\pm}\left(x,\,u_1\right),\ L_2^{\pm}\left(x,\,u_2\right),\ ...,\ L_n^{\pm}\left(x,\,u_n\right)\right\} \tag{8}$$

where $u = (u_1, u_2, ..., u_n)$[22]. So, we proposed a new important theorem of Lyapunov exponents as follows:

Proposition (1): For $\forall \begin{vmatrix} x \\ y \end{vmatrix} \in R^2$, if $|a| = 1$, $y \neq \frac{\sqrt{a}}{b}$ and by $>\sqrt{a}$ then the $MJ_{a,b}$ has positive Lyapunov exponents.

Proof: Let $\begin{vmatrix} x \\ y \end{vmatrix} \in R^2$ the Lyapunov exponents of $MJ_{a,b}$ given by the Eq. 9 as follows:

$$L_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_1\right) = \lim_{n\to\infty}\frac{1}{n}\ln\left\| DMJ^n{}_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_1 \right\| \tag{9}$$

From proposition (3-6) by $|\lambda_1| = \frac{1}{|\lambda_2|}$ where $y \neq \frac{\sqrt{a}}{b}$. Continuous, if $|\lambda_1|<1$, then:

$$L_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_1\right) = \lim_{n\to\infty}\frac{1}{n}\ln\left\| \left(DMJ_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_1 \right)^n \right\| > \ln\left\| by \mp \sqrt{(by)^2 - a} \right\| \tag{10}$$

by hypothesis $L_1>0$. So: if $|\lambda_1|>1$, then:

$$L_2\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_2\right) = \lim_{n\to\infty}\frac{1}{n}\ln\left\| \left(DMJ_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_2 \right)^n \right\| < \ln\left\| by \mp \sqrt{(by)^2 - a} \right\| \tag{11}$$

This Lyapunov exponent $L^{\pm}(x, v) = \max\{L_1^{\pm}(x, v_1)\ L_2^{\pm}(x, v_2)$ hence, Lyapunov exponent of $MJ_{a,b}$ is positive. In order to implement the Lyapunov exponents dependence on the initial condition of a map, the points (x_i, y_i) are changed or fixed where, i = 1, 2 control parameters (a, b). MATLAB program is used to simulate and obtain results as illustrated in Table 1 and 2. In Table 1, all the values of parameter a are <1 and the parameter b is set as 0.05, since, it may not may affect the results. So, when $|a|<1$, all Lyapunov exponents results are negative.

In Table 2, all the values of parameter a are set to be 1 or more than 1 and the parameter is set as 0.005, since, it may not affect the results. So, wherever $|a|>1$, all Lyapunov exponent results are positive but when $|a| = 1$, there are two Lyapunov exponents results; one is positive and the other is negative.

Topological entropy: In this study, the topological entropy is defined as follows: let $F: X\to X$ be a continuous map of a compact metric space X for $\in>0$ and $n\in Z^+$, we

Table 1: |a|<1

A	B	(x, y)	L_1	L_2
0.99	0.05	(0.1,0.2)	-0.0050152848	-0.0050350510
0.79	0.05	(0.1,0.2)	-0.1178607200	-0.1178616130
0.59	0.05	(0.1,0.2)	-0.2638161836	-0.2638165585
0.39	0.05	(0.1,0.2)	-0.6189371307	-0.6189372253
-0.99	0.05	(0.1,0.2)	0.0049733468-	-0.0050769890
-0.79	0.05	(0.1,0.2)	-0.1178610782	-0.1178612553
-0.59	0.05	(0.1,0.2)	-0.2638163575	-0.2638163846
-0.39	0.05	(0.1,0.2)	-0.4708042671	-0.4708042727

Table 2: |a|>1 or |a| = 1

A	B	(x,y)	L_1	L_2
1.001	0.005	(0.1,0.2)	0.0039907258	0.0039774438
1.004	0.005	(0.1,0.2)	0.0019976051	0.0019944162
1.008	0.005	(0.1,0.2)	0.0039907258	0.0039774438
1	0.005	(0.1,0.2)	0.0000004999	-0.0000004999
-1.001	0.005	(0.1,0.2)	0.0009925325	0.0000069679
-1.004	0.050	(0.1,0.2)	0.0030486413	0.0009433799
-1.008	0.050	(0.1,0.2)	0.0071595456	0.0008086240

say $E\subset x$ is an $(n, \in)$ separated set, if for every $x, y\in E$, then exists, i.e., $0\leq i\leq n$ such that $f^i(x), f^i(y))>\in$ then the topological entropy of f, denoted by $h_{top}(f)$ is defined to be:

$$h_{top}(f) = \lim_{n\to\infty}\left\{\lim_{n\to\infty}\sup\frac{1}{n}\log_n(n, \in)\right\} \tag{12}$$

where, $N(n, \in)$ is the maximum cardinal of all $(n, \in)$-separated sets[23]. Thus, we proposed two new important theorems of topological entropy as follows:

Theorem (1): Let the $MJ_{a,b}: R^2\to R^2$ be a continuous map $h_{top}|DMJ_{a,b}|\geq\log|\lambda|$ where, λ is the largest eigen value of $DMJ(v)$ where $v\in R^2$.

Proposition (2): If a>0, then: $h_{top}|MJ_{a,b}|> \frac{1}{2}\log(a^2+a+1)$.

Proof: Since:

$$a+1+\sqrt{a^2+a+1} > \sqrt{a^2+a+1} \tag{13}$$

so:

$$by+\sqrt{b^2y^2-2bya+a^2} > \sqrt{a^2+a+1} \tag{14}$$

therefore,

$$\log|\lambda_1| > \log\sqrt{a^2+a+1}$$

then:

$$h_{top}|MJ_{a,b}| > \frac{1}{2}\log\left(a^2+a+1\right) \tag{15}$$

According to theorem (3-2) by:

Theorem (2): Let $MJ_{a,b}: R^2\to R^2$ be $h_{top}|MJ_{a,b}|\leq \text{lag}\ \max_{x\in R^2}\max_{LC\ T\in R^2}|\det DMJ_{a,b}L|$

Proposition (3): The upper estimate of topological entropy of $MJ_{a,b}$ IS $h_{top}|MJ_{a,b}|\leq\log|a|$. By theorem (3-2) we get:

$$h_{top}\left|MJ_{a,b}\right| < \log \max_{x\in R^2} \max_{LC\ T\in R^2} \left|\det\ DMJ_{a,b}(x) L\right| \le \log|a| \quad (16)$$

RESULTS AND DISCUSSION

Sensitive dependence on initial condition: This section defines the sensitive dependence on initial condition as follows:

Let X be a compact metric space and T a continuous map. A dynamical system (X, T) has sensitivity dependence on initial conditions if $\exists\delta>0$ such that, for $x\in X$ and each $\in>0$, there is $y\in X$ with (x, y)< and $n\in N$ such that (Tnx, Tny)>δ[2].

So, we proposed one new important theorem of the sensitive dependence on initial condition as follows:

Proposition (4): If |a|>1 then, $MJ_{a,\ b}$ has a sensitive dependence on initial condition.

Proof: Let:

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$$

since:

$$MJ_{a,b}(x) = \begin{vmatrix} x_2 \\ -ax_1+bx_2^2 \end{vmatrix} \quad (17)$$

if $|x|\le 1$ by definition as well as by hypothesis:

$$MJ_{a,b}(x) < \begin{vmatrix} x_2 \\ -ax_1 \end{vmatrix} \quad (18)$$

and

$$MJ^2_{a,b}(x) < \begin{vmatrix} -ax_1 \\ (-a)^2 x_1 \end{vmatrix} \quad (19)$$

that is:

$$MJ^n_{a,b}(x) < \begin{vmatrix} (-a)^{n-1} x_1 \\ (-a)^n x_1 \end{vmatrix} \quad (20)$$

thus:

$$\text{if } |a| > 1,\ n \to \infty, \text{ then: } MJ^n_{a,b}(x) \to \infty$$

$$\text{Let } y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \in R^2 \text{ such that } d(x, y) < \delta$$

$$d\left(MJ_{a,b}(x).MJ^2_{a,b}(y)\right) = \sqrt{(x_2)^2 + (-ax_1)^2} \quad (21)$$

$$d\left(MJ^2_{a,b}(x).\ MJ^2_{a,b}(y)\right) = \sqrt{(-ax_1)^2 + \left((-a)^2 x_1\right)^2}$$
$$d\left(MJ^n_{a,b}(x).\ MJ^n_{a,b}(y)\right) = \sqrt{\left((-a)^{n-1} x_1\right)^2 + \left((-a)^n x_1\right)^2} \quad (22)$$
$$d\left(MJ^n_{a,b}(x).\ MJ^n_{a,b}(y)\right) = \sqrt{\left(a^{2n-2}+a^n\right)x_1)^2}$$

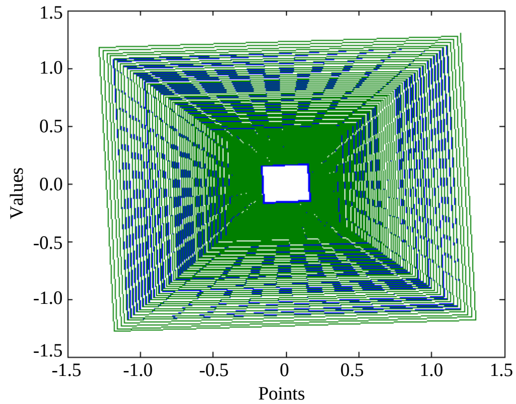

Fig. 1: a = 0.99, b = 0.005, points (1.1, 1.2) and (1.2, 1.3)

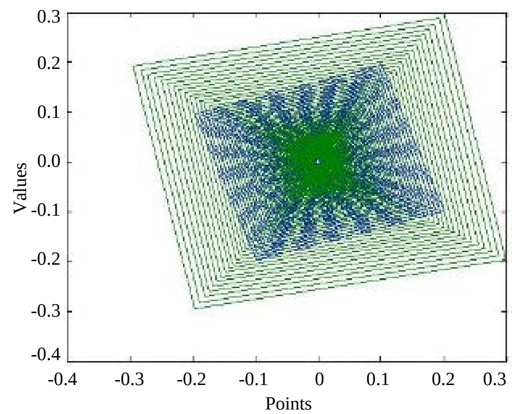

Fig. 2: a = 0.98, b = 005, points (1.1, 1.2) and (1.2, 1.3)

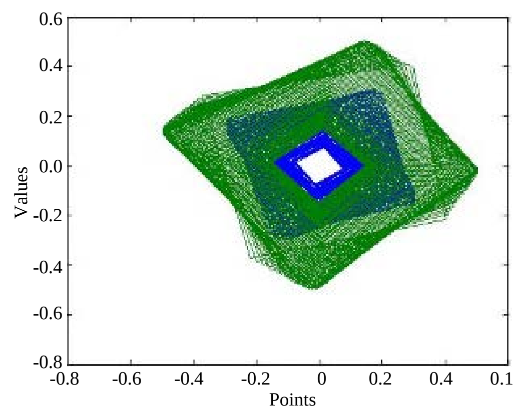

Fig. 3: a = 0.99, b = 0.5, points (0.2, 0.3) and (0.3, 0.4)

if $|a|>1$ and $n\to\infty$, $d\left(MJ^n_{a,b}(x).\ MJ^n_{a,b}(y)\right)\to\infty$ hence, $MJ_{a,b}$ has a sensitive dependence on initial condition. In order to simulate and obtain results, for |x|>1, the iterates of $MJ_{a,\ b}$ are diverge. Thus, its dependence on initial condition is sensitive (Fig. 1-6). As well, studying the sensitive dependence on map initial conditions is done by changing

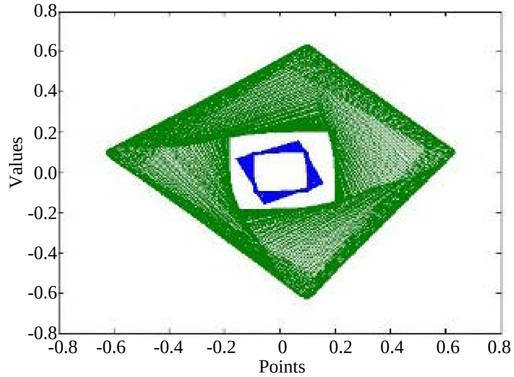

Fig. 4: a = 1.00001, b = 0.5, points (0.1, 0,2) and (0.2, 0.4)

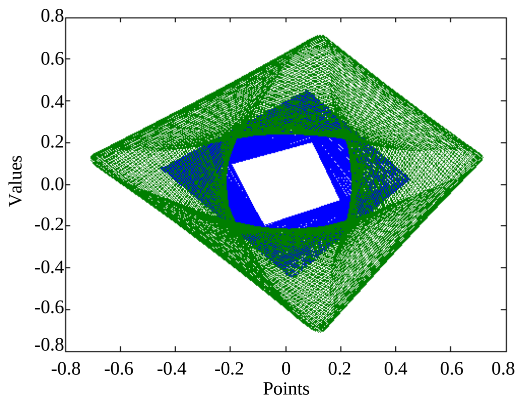

Fig. 5: a = 1.0000006, b = 0.5, points (0.1, 0.1) and (0.2, 0.2)

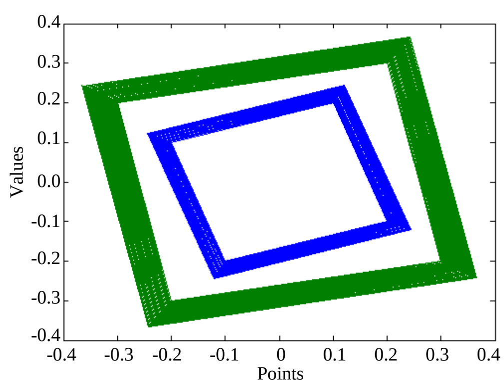

Fig. 6: a = 1.00099, b = 0.05, points (0.1, 0.2) and (0.2, 0.3)

the points (x_I, y_I) where, i = 1, 2 control parameters (a, b). For this purpose, the MATLAB program is used as illustrated in Fig. 1-6.

CONCLUSION

Chaos has many issues that have to be considered when it is employed for a dynamic system that includes future predictability, computational complexities and unstable behavior. Although, there were many previous studies that addressed to prove the chaos in dynamic systems, yet, these studies generally, dealt with (3-D) systems and some of them dealt with the (2-D) systems. Moreover, they did not address all the Chaos characteristics and prove them by means of dynamic systems which we addressed and managed to prove in this research. In our previous paper, we developed a nonlinear function approach by translating 3-D to 2-D of dynamical behavior of such modified Jerk Map which was adopted in this research in order to prove the chaotic properties of the dynamic systems. In case that the sensitive dependence on initial condition of $MJ_{a,b}$ is satisfied when |a|>1 and n→∞.

$$d\left(MJ^{n}_{a,b}(x).\ MJ^{n}_{a,b}(y)\right) \to \infty$$

We also proved that $MJ_{a,b}$ either had two positive Lyapunov exponents if |a|>1 or had one positive and another negative value if |a| = 1. Finally, if a>0 the topological entropy of $MJ_{a,b} = h_{top}\ |MJ_{a,b}| \geq \log|\lambda| > 1/2\log(a^2+a+1)$ as well as upper estimate of topological entropy of $MJ_{a,b}$ is $h_{top}|MJ_{a,b}| \leq \log|a|$. We then also simulated our study by implementing it in practice using the MATLAB program. The obtained results were identical to what was proven mathematically.

RECOMMENDATION

As future work, we can develop our study by employing one of the optimization methods in order to obtain more accurate results.

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