Journal of Engineering and Applied Sciences
MEDWELL PUBLICATIONS SINCE 2000

The DKP Oscillator in a k-Minkowski Space-Time

[1]Yassine Chargui and [2]Bahri Cherif
[1]Department of Physics, College of Science and Arts at ArRass,
[2]Department of Mathematics, College of Science and Arts at ArRass, Qassim University, P.O. Box 53, ArRass 51921, Saudi Arabia

Key words: DKP equation, k-Minkowski space-time, DKP oscillator, energy spectrum

Corresponding Author:
Yassine Chargui
Department of Physics, College of Science and Arts at ArRass, Qassim University, P.O. Box 53, ArRass 51921, Saudi Arabia

Page No.: 3509-3513
Volume: 15, Issue 20, 2020
ISSN: 1816-949x
Journal of Engineering and Applied Sciences

Abstract: We study the dynamics of a Duffin-Kemmer-Petiau (DKP) oscillator, for a scalar boson in a (3+1)-dimensional k-Minkowski space-time. We use the Dirac derivatives approach to construct the k-DKP equation. We investigate the consequences of the k-deformation on the energy spectrum of the oscillator, and its eigenfunctions, for any value of the total angular momentum number using a perturbation method. In particular, we show that particle and antiparticle energies are asymmetric, a the charge conjugation symmetry for the k-DKP equation is broken by the deformation. Moreover, the equivalence between this system and the k-Klein-Gorden oscillator is discussed.

INTRODUCTION

The Duffin-Kemmer-Petiau (DKP) formalism[1] is based on a first-order relativistic covariant equation, `a la Dirac, describing in a unified manner spin-zero and spin-one particles and thereby providing an alternative to the conventional second-order wave equations of Klein-Gordon (KG) and Proca. However, the DKP theory presents the advantage of being much richer with respect to the introduction of interactions, since, it allows kinds of couplings which are not possible with the Klein-Gordon and Proca theories, albeit both theories remain equivalent in the case of minimally coupled vector interactions[2]. This benefit makes the DKP equation an excellent tool for physicist when studying phenologically several processes, especially in nuclear and particle physics. As a matter of fact, the relevance of the DKP formalism, particularly with non-minimal coupling, for modeling physical situations has been testified by many works. For instance, the DKP equation permits a better fitting of experimental data, relative to the scattering of mesons by nuclei at medium energies, than Klein-Gordon and Proca equation[3]. For the deuterons nucleus scattering too, this equation produces results which accords well with the findings of other approaches[4]. On the other hand, it has been found that the DKP theory can successfully describe the α-nucleus elastic scattering[5], the scattering of K^+ nucleus[6] and even quark confinement within QCD[7].

Besides, over the past years, the DKP equation has been considered within different other contexts such as the covariant Hamiltonian dynamics[8], the causal approach[2], the five- dimensional Galilean invariance[9], Bose-Einstein condensates[10], the noncommutative phase space[11], topological defects and the generalized uncertainty principle[12]. In addition, a lot of research has been conducted around exact and approximate solutions of the DKP equation in the presence of interactions with different structures (see for example and references therein). In particular, much attention has been given to the so-called DKP oscillator[13-15]. The latter is an analogous model of the Dirac oscillator[16] which has been introduced by Nedjadi and Barrett[13, 14]. They gave it this name because its non-relativistic limit produces a harmonic oscillator for spin-zero bosons and a harmonic oscillator plus a spin-orbit coupling term in the spin-one case.

Furthermore, recently, a great deal of importance is being attached to studying the Klein-Gordon and the Dirac equations in connection with quantum deformations related to theories of quantum gravity[17,18]. These theories share a common feature, namely, the notion of fundamental length scale. On the other hand, such a distance emerges naturally within noncommutative geometry[19, 20], therefore, in the literature, different types of Non-Commutative Space-Times (NCST) have been studied and their physical implications have been probed. Among them the so-called k-Minkowski space-time[21]. The latter is a NCST of Lie algebra type with coordinates satisfying commutation relations of the form:

$$\left[\tilde{x}^j, \tilde{x}^k\right] = 0, \left[\tilde{x}^0, \tilde{x}^k\right] = ia\tilde{x}^k, \; k = 1, 2, 3 \tag{1}$$

where, a = 1/k is the parameter of space-time deformation (a is real). In this research, we shall be interested in a particular k-Minkowski space-time which is characterized by the usual non-deformed Poincar´e algebra as a symmetry algebra[21]. The generators of this algebra are the operators $M_{\mu\nu}$ and D_μ where the latter are known as Dirac derivatives and they are satisfying the usual relations:

$$\begin{aligned}&\left[M_{\mu\nu}, D_\lambda\right] = g_{\nu\lambda}D_\mu - g_{\mu\lambda}D_\nu, \; \left[D_\mu, D_\nu\right] = 0\\&\left[M_{\mu\nu}, M_{\lambda\rho}\right] = g_{\mu\rho}M_{\nu\lambda} + g_{\nu\lambda}M_{\mu\rho} - g_{\nu\rho}M_{\mu\lambda} - g_{\mu\lambda}M_{\nu\rho}\end{aligned} \tag{2}$$

These derivatives D_μ transform as a vector representation under the action of $M_{\mu\nu}$ and admit infinite realizations in terms of commutative coordinates x^μ and their derivatives ∂_μ[21]. The simplest of them is given by:

$$D_0 = \frac{1}{a}\sin(a\partial_0) + i\frac{a}{2}\Delta e^A, \; D_k = \partial_k, \; k = 1, 2, 3 \tag{3}$$

with $A = -i\partial_0$ and $\Delta = \partial_k\partial_k$. In this study, we consider a k-DKP theory in this particular k-Minkowski space-time, constructed using Eq. 3. The latter equation is used to investigate first order effects of the k-deformation on the energy spectrum of the (3+1)-dimensional DKP oscillator, and the associated wave functions, for a scalar boson.

MATERIALS AND METHODS

In this study, we present the k-DKP equation and review its essential properties. First, let us recall that the ordinary DKP equation, describing a free particle of mass m is given by:

$$\left(i\beta^\mu\partial_\mu - mc^2\right)\psi(r, t) = 0 \tag{4}$$

where, ψ is the DKP spinor and β^μ are matrices fulfilling Kemmer's algebra defined by:

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = \beta^\mu g^{\nu\lambda} + \beta^\lambda g^{\mu\nu} \tag{5}$$

with $g^{\mu\nu}$ the Minkowski metric tensor in a (3+1)-dimensional flat space-time: $g^{\mu\nu}$ = diag (+, -, -, -). The algebra generated by the matrices β^μ have two nontrivial irreducible representations. The first is five dimensional and corresponds to spin-zero (scalar) fields whereas the second has ten dimensions and is associated to spin-one (vector) fields. For the scalar sector, an explicit form of matrices β^μ is given by:

$$\beta^0 = \begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}, \; \beta^i = \begin{pmatrix}0 & \rho^i\\ -\rho^i_T & 0\end{pmatrix}, \; i = 1, 2, 3 \tag{6}$$

where the block elements are defined as:

$$\theta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{7}$$

$$\rho^1 = \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \; \rho^2 = \begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \; \rho^3 = \begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix} \tag{8}$$

ρ^T being the transposed matrix of ρ and 0 is the zero (3×3) matrix. Then, using the realisation (3) of Dirac derivatives, results in the following free k-DKP equation:

$$\left[\frac{i}{a}\beta^0\sin(a\partial_t) - \frac{a}{2}\beta^0\Delta e^{ia\partial t} + i\beta.\nabla - m\right]\psi\left(r, t = 0\right) \tag{9}$$

Obviously, the non-deformed free DKP equation is recovered in the limit $a \to 0$. It is worth remembering that k-deformation breaks the charge conjugation symmetry of the DKP equation. To show this, let us first introduce a minimally-coupled electromagnetic field $A^\mu = (A_0, A)$, into Eq. 9. This leads to the equation:

$$\begin{aligned}&\left[\frac{i}{a}\beta^0\sin[a(\partial_t - iqA_0)]\right] - \frac{a}{2}\beta^0(\nabla - iqA)^2 e^{ia(\partial_t - iqA_0}) +\\&i\beta.(\nabla iqA) - m]\psi(r, t) = 0\end{aligned} \tag{10}$$

where, q is the electric charge of the particle. In the non-deformed DKP theory, the charge conjugation operator is given by: $C = \eta^5 K$ with $\eta^5 = \eta^0\eta^1\eta^2\eta^3$ where $\eta^0 = 2(\beta^0)^2 - 1$ and $\eta^k = 2(\beta^k)^2 + 1$ for k = 1, 2, 3. However, the matrix η^5 anticommutes with all the matrices β^μ which can be easily verified using Eq. 5. Acting then on Eq. 10 with operator C brings it into the form:

$$\left[\frac{i}{a}\beta^0 \sin[a(\partial_t\text{-}iqA_0)]\right]\text{-}\frac{a}{2}\beta^0(\nabla\text{-}iqA)^2 e^{ia(\partial_t\text{-}iqA_0)})+ \\ i\beta.(\nabla iqA)\text{-}m]\psi(r,t)=0 \quad (11)$$

Because of the a-dependent contributions (precisely the second term), re-versing the sing the electric charge in Eq. 11 does not produce Eq. 10. This shows that the charge conjugation is no longer a symmetry for the k-DKP equation. Hence, the particle and the antiparticle should now acquire different energies. This feature of the Deformed equation remains valid even in the particular case, when only first-order terms in the deformation parameter are retained. In that case, the free k-DKP equation, obtained by expanding Eq. 9 in powers of a is given by:

$$\left[i\beta^0\partial_t\text{-}\frac{a}{2}\beta^0\Delta+i\beta.\Delta\text{-}m\right]\psi(r,t)=0 \quad (12)$$

It is worth noting that, when only first order effects of the deformation are considered, using Eq. 12, we can still establish the continuity equation:

$$\partial_\mu J^\mu = 0 \quad (13)$$

where the four-current density is still given by $J^\mu = \bar{\psi}\beta^\mu\psi$ with $\bar{\psi} = \psi^\dagger\eta^0$.

RESULTS AND DISCUSSION

We investigate now the effect of the k-dependent contributions on the dynamics of the DKP-oscillator by considering only terms of first order in the deformation parameter a. The DKP-oscillator system, valid up to the first order in a, stems from Eq. 12 after performing the non-minimal substitution:

$$\nabla \to \nabla\eta^0 m\omega\tilde{r} \quad (14)$$

where ω is the oscillator frequency and ˜r is the position vector in the non- commutative k-Minkowski space-time. It is realized in terms of commutative coordinates as:

$$\tilde{r} = r(1\text{-}ia\partial_t) \quad (15)$$

where only terms of first order in a are retained and r is the position vector in the commutative space. Then, the resulting stationary equation describing a stationary state $\psi(r,t) = e^{-iEt}\varphi(r)$ with energy E is given by:

$$\left\{\beta^0 E\text{-}\frac{a}{2}\beta^0(\nabla+\eta^0\omega r)^2+i\beta[\nabla+\eta^0\omega(1\text{-}aE)r]\text{-}m\right\}\phi(r)=0 \quad (16)$$

Before proceeding further with the solution of Eq. 16, let us make some comments on the equivalence between the scalar DKP oscillator and the KG oscillator under k-deformation. We recall that the usual KG oscillator system stems from the substitution:

$$\nabla^2 \to (\nabla\text{-}m\omega r)(\nabla+m\omega r) \quad (17)$$

in the free KG equation. As a matter of fact, it is well known that both systems are equivalent in the non-deformed case[13, 14]. To this end, let us set:

$$\phi(r) = \begin{pmatrix}\varphi(r)\\ X(r)\\ \Theta(r)\end{pmatrix} \quad (18)$$

with Θ a three-component vector function. Then Eq. 16 yields the following system:

$$\begin{aligned}&\left[E\text{-}\frac{a}{2}(\nabla+m\omega r)^2\right]\phi(r)=m\varphi(r)\\ &\phi m(r)=\left[E\text{-}\frac{a}{2}(\nabla+m\omega r)^2\right]\varphi(r)+[\nabla\text{-}m\omega(1\text{-}aE)r].\Theta(r)\\ &m\Theta(r)=[\nabla+\omega(1\text{-}aE)r]\varphi(r)\end{aligned} \quad (19)$$

Thus, by eliminating X and Θ in favor of φ, we obtain an equivalent Klein- Gordon equation:

$$\left\{\left[E^2\text{-}aE(\nabla+\omega r)^2\right]\text{-}m^2\right\}\varphi(r)= \\ [\nabla\text{-}m\omega(1\text{-}aE)r][\nabla+m\omega(1\text{-}aE)r]\varphi(r) \quad (20)$$

Equation 20 is the same equation that would be obtained by putting the substitution:

$$\nabla^2 \to (\nabla\text{-}m\omega\tilde{r})\,(\nabla+m\omega\tilde{r}) \quad (21)$$

into the free k-deformed KG equation, which is given, in terms of Dirac derivatives by:

$$D_\mu D^\mu\psi\text{-}m^2\psi = 0 \quad (22)$$

Let us recall here that only terms of first order in a should be retained. We conclude then that the k-deformation preserves the equivalence between the scalar DKP oscillator and the KG oscillator, at least to the first order in a. Now, by returning to Eq. 16, we can easily verify that the total angular momentum J = L+S where, L is the orbital angular momentum and S is the spin operator, is still a constant of motion. Indeed, the operator $\beta^0[\nabla+\eta^0\omega r]$, related to the deformation, commutes with J. Thus, we will use spherical coordinates and search for solutions $\varphi(r)$ of the form:

$$\phi(r) = \frac{1}{r}\begin{pmatrix} F_{nJ}(r)Y_J^M(\Omega) \\ G_{nJ}(r)Y_J^M(\Omega) \\ \sum_L H_{nJL}(r)Y_{JL1}^M(\Omega) \end{pmatrix} \quad (23)$$

where $Y^M{}_J$ (Ω) is the spherical harmonics of order J while $Y^M{}_{JL1}$ (Ω) stands for the normalized vector spherical harmonics. Let us remember theta the total angular momentum for the above states is J = L. Substituting this expression of φ which is of parity $(-1)^J$, into Eq. 16, yields a set of five coupled radial differential equations involving the components $F_{nJ} = F$, $G_{nJ} = G$ and $H_{nJ\pm1} = H_{\pm1}$, given by:

$$\begin{aligned} &\left[E-\frac{a}{2}D_{\bar{J}}(\omega)D_{\bar{J}}^{+}(\omega)\right]F = mG \\ &\xi_J D_{\bar{J}+1}(\omega_a)F = -mH_1- \\ &\alpha_J D_{J+1}^{+}(-\omega a)H_1+\xi_J D_{\bar{J}}(-\omega_a)H_{-1} = mF- \\ &\left[E-\frac{a}{2}D_J^{-}(\omega)D_J^{+}(\omega)\right]G \end{aligned} \quad (24)$$

where: $\alpha_J = \sqrt{(J+1)/(2J+1)}$, $\xi J = \sqrt{J/(2J+1)}$, $\omega_a = (1-aE)\omega$ and $D^{\pm}{}_J(\omega)$ and $D^{\pm}{}_{J+1}(\omega)$ are given by:

$$D_q^{\pm}(\omega) = \frac{d}{dr}\pm\frac{q}{r}+m\omega r,\ q = J,\ J+1 \quad (25)$$

Eliminating G, H_1 and H_{-1} in favor of F and neglecting terms of higher orders in a, we end up with the following equation:

$$\begin{bmatrix} \frac{d^2}{dr^2}-2m\omega aEr\frac{d}{dr}-J\frac{J(J+1)}{r^2}-m^2\omega^2 r_2 + \\ m(1-eE)\omega+(E^2-m^2)(1+aE) \end{bmatrix}F(r) = 0 \quad (26)$$

Equation 26 can be brought into a confluent hypergeometric equation using the change of variable:

$$\rho = m\omega r^2 \quad (27)$$

along with the ansatz:

$$F(r) = e^{(aE-1)\rho/2}\rho^{\gamma/2}f(\rho) \quad (28)$$

where, the parameter γ should be fixed, so that, to peel off the term multiplied by $1/r^2$ in Eq. 26. Thus, by setting γ = J+1, we obtain the following equation for f:

$$\rho f''(\rho)+(d-\rho)f'(\rho)-bf(\rho) = 0 \quad (29)$$

where the parameters a and b are given by:

$$b = \frac{m^2-E^2}{4\,m\omega}(1+aE)+\frac{1+J}{2},\ d = J+\frac{3}{2} \quad (30)$$

Subsequently, the requirement that φ (r) be regular at the origin dictates a solution f proportional to Kummer's function of the first kind, F (b, d, ρ)[22]. Moreover, given the asymptotic form of F (b, d, ρ) as ρ→∞, the demand that the wave function be normalizable is full-filled by imposing the condition b = -n with n a positive integer. This constraint determines the energy spectrum as follows:

$$E_N^{\pm} = \pm\sqrt{m^2+2\,m\omega(N+1)}-2\,m\omega a(N+1),\ N = 0, 1, 2, ... \quad (31)$$

with the principal quantum number defined as N = 2n+J. Firstly, we note that, for a → 0, Eq. 31 yields the right energy spectrum of the non-deformed DKP scalar oscillator[13, 14]. This reveals the consistency of our calculations. We also note the asymmetry between particle and antiparticle energies introduced by the a-dependent contribution. Indeed, the correction brought by the deformation is the same for positive and negative energies. This fact reflects the breaking of the charge-conjugation symmetry for the k-DKP equation. In addition, we remark that this energy correction increases with the number N, hence, higher energy levels are more affected by the deformation. Moreover, it turns out that the presence of the a-term does not alter the known degeneracy of the oscillator energy levels. Thus we could surmise that the k-deformation preserve the underlying symmetry of the DKP oscillator coupling.

As for the oscillator eigenstates, since, F (n, d, ρ) is proportional to the generalized Laguerre polynomial of degree n, $L^{d-1}{}_n$ (ρ), the solution f can be rewritten as:

$$f_n(r) = C_n e^{m\omega(aE-1)r^2/2,1+J}L_n^{J+1/2}(m\omega r^2) \quad (32)$$

with C_n a normalization constant. The remaining radial components of the wave functions can be easily obtained from f using Eq. 24. Let us note that states with higher energies are more impacted by the deformation, since, the parameter a appears in the function f multiplied by the energy of the state. Moreover, for positive energy states, the a-dependent contribution softens the Gaussian decay of the function f whereas the latter is hastened for states with negative energies.

CONCLUSION

In summary, this research considers a k-DKP equation built in k-Minkowski space-time with a non-deformed k-Poincar´e algebra, using the approach of Dirac derivatives. This approach guarantees the covariance of the resulting equation with respect to the k-Poincar´e algebra as these derivatives transform as a vector representation under this algebra. We have studied the effect of the k-deformation on the dynamics of the (3+1)-dimensional DKP oscillator for a scalar boson. Thus, using a perturbation method, we derived the first-order corrections to the energy levels of the oscillator and its eigenstates, for any value of the angular momentum number. In particular, we have found that the deformation introduces an asymmetry between particle and antiparticle energies, thereby revealing the breaking of the charge-conjugation symmetry for the k-DKP equation. In addition, we have shown that the usual degeneracy of the oscillator energy eigenvalues is not affected by the k-dependent contribution.

ACKNOWLEDGMENTS

The researchers gratefully acknowledge Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number (3762-alrasscac-2018-1-14-S) during the academic year 1439 AH/2018 AD.

REFERENCES

01. Kemmer, N., 1938. Quantum theory of Einstein-Bose particles and nuclear interaction. Proc. R. Soc. London Ser. Math. Phys. Sci., 166: 127-153.
02. Lunardi, J.T., B.M. Pimentel, R.G. Teixeira and J.S. Valverde, 2000. Remarks on Duffin-Kemmer-Petiau theory and gauge invariance. Phys. Lett. A., 268: 165-173.
03. Barrett, R.C. and Y. Nedjadi, 1995. Meson-nuclear interactions in the Duffin-Kemmer-Petiau formalism. NuPhA, 585: 311-312.
04. Kozack, R.E., B.C. Clark, S. Hama, V.K. Mishra, R.L. Mercer and L. Ray, 1989. Spin-one Kemmer-Duffin-Petiau equations and intermediate-energy deuteron-nucleus scattering. Phys. Rev. C, Vol. 40, 10.1103/PhysRevC.40.2181
05. Ait-Tahar, S., J.S. Al-Khalili and Y. Nedjadi, 1995. A relativistic model for α-nucleus elastic scattering. Nucl. Phys. A, 589: 307-319.
06. Kerr, L.K., B.C. Clark, S. Hama, L. Ray and G.W. Hoffmann, 2000. Theoretical and experimental K++ nucleus total and reaction cross sections from the KDP-RIA model. Prog. Theor. Phys., 103: 321-335.
07. Gribov, V., 1999. QCD at large and short distances (annotated version). Eur. Phys. J. C-Particles Fields, 10: 71-90.
08. Kanatchikov, I.V., 2000. On the Duffin-Kemmer-Petiau formulation of the covariant Hamiltonian dynamics in field theory. Rep. Math. Phys., 46: 107-112.
09. Montigny, M.D., F.C. Khanna, A.E.D. Santana, E.S. Santos and J.D.M. Vianna, 2000. Galilean covariance and the Duffin-Kemmer-Petiau equation. J. Phys. A. Math. General, Vol. 33,
10. Abreu, L.M., A.L. Gadelha, B.M. Pimentel and E.S. Santos, 2015. Galilean DKP theory and Bose-Einstein condensation. Phys. A. Statistical Mech. Appl., 419: 612-621.
11. Hassanabadi, H., Z. Molaee and S. Zarrinkamar, 2012. DKP oscillator in the presence of magnetic field in (1+2)-dimensions for spin-zero and spin-one particles in noncommutative phase space. Eur. Phys. J. C., Vol. 72, 10.1140/epjc/s10052-012-2217-5
12. Chargu1i, Y., 2018. On the Duffin-Kemmer-Petiau equation with linear potential in the presence of a minimal length. Phys. Lett. A., 382: 949-953.
13. Nedjadi, Y. and R.C. Barrett, 1994a. The Duffin-Kemmer-Petiau oscillator. J. Phys. A. Math. General, Vol. 27, No. 12.
14. Nedjadi, Y. and R.C. Barrett, 1994b. Solution of the central field problem for a Duffin-Kemmer-Petiau vector boson. J. Math. Phys., 35: 4517-4533.
15. Hun, M.A. and N. Candemir, 2019. Relativistic quantum motion of the scalar bosons in the background space-time around a chiral cosmic string. Int. J. Mod. Phys. A., Vol. 34, No. 10. 10.1142/S0217751X19500568
16. Moshinsky, M. and A. Szczepaniak, 1989. The dirac oscillator. J. Phys. A. Mathe. General, Vol. 22, No. 17.
17. Lukierski, J., H. Ruegg, A. Nowicki and V.N. Tolstoy, 1991. q-deformation of Poincare algebra. Phys. Lett. B., 264: 331-338.
18. Chargui, Y., 2019. On the κ-deformed Dirac oscillator in (2+1)-dimensions. Mod. Phys. Lett. A., Vol. 34, No. 12. 10.1142/S0217732319500895
19. Doplicher, S., K. Fredenhagen and J.E. Roberts, 1994. Spacetime quantization induced by classical gravity. Phys. Lett. B., 331: 39-44.
20. Pramanik, S., M. Moussa, M. Faizal and A.F. Ali, 2015. Path integral quantization corresponding to the deformed Heisenberg algebra. Ann. Phys., 362: 24-35.
21. Meljanac, S. and M. Stojic, 2006. New realizations of Lie algebra kappa-deformed Euclidean space. Eur. Phys. J. C-Particles Fields, Vol. 47, No. 2. 10.1140/epjc/s2006-02584-8
22. Abramowitz, M. and I.A. Stegun, 1972. Handbook of Mathematical Functions. 10th Edn., Dover Publications Inc., USA., ISBN: 978-0-486-61272-0, pp: 1046.