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Development of Mathematical Model of Ground Unmanned Vehicle Movement

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Abstract: The study proposes a mathematical model of the Ground Unmanned Vehicle (GUV) which allows to determine the trajectory in two-dimensional coordinates and orientation angles of the unmanned vehicle. The moment reachability set for a third-order nonlinear controlled system, often called the "Dubins machine" is investigated. The object moves on a plane with a constant linear velocity and asymmetrically specified restrictions on turns to the right and to the left. The statement about the number and nature of control switches leading to the boundary of the reachability set is proved.

Key words: Unmanned vehicle, dubins machine, the trajectory, the angles of rotation of the wheels, optimize for speed, asymmetrically

INTRODUCTION

In applied works based on mathematical control theory, a model of a controlled object called the "Dubins machine" is very popular. This model is given by a nonlinear system of differential equations of the third order. Two phase variables characterize the geometric position of the controlled object on the plane, the third variable-the angle of direction of the velocity vector. The speed value is considered constant. The scalar control action constrained by the geometric constraint determines the instantaneous turning radius.

"Dubins machine" a controlled object (car or plane) with a simple model of movement in the horizontal plane. In 1957, the American mathematician L. Dubins published a theoretical research (Dubins, 1957) on a line of shortest length with a limited radius of curvature connecting two points on a plane with a given direction of exit from the first point and a given direction of entry into the second.

The results obtained by L. Dubins were very useful in the study of objects with a limited turning radius and constant speed of movement. That is why such objects became known as the Dubins machine. Later it turned out that in 1889 the Russian mathematician A. A. Markov studied close questions in the research (Markov, 1889) devoted to the problems of laying railways.

Dynamics of the simplest car was used by R. Isaacs in researches on differential games (Isaacs, 1965, 1967). The Dubins Model is used in the control of wheeled robots (Jean-Paul, 1998) for dispatching calculations in civil aviation as well as in applied works on the construction of trajectories of unmanned aerial vehicles in the horizontal plane (Meyer *et al*., 2015). In the book (Berdyshev, 2015) Y. I. Berdyshev used the model Dubins for optimal bypass of points in the plane.

The aim of this research is to develop a mathematical model of the movement of an Unmanned Vehicle (GUV) based on the Dubins machine as well as optimization of movement by the criterion of time.

MATERIALS AND METHODS

To set the trajectory of the GUV, it is necessary to set the coordinates of the control points and the coordinates of the velocity vector at each such point. The calculation algorithm based on the obtained data on the position and orientation of the GUV should generate a trajectory passing through the point of the current location of the GUV and the next checkpoint.

This trajectory has a number of natural limitations. First, it must be a continuously differentiable function (because the GUVcan not move in jumps) and secondly, at each point of its curvature should not exceed some predetermined value (since any vehicle has a limited turning radius).

To describe the dynamics of the movement of the machine Baton in this study, all movements are described through the composition of internal symmetries. Matrices and vectors are indicated in bold. The subscripts assigned to the start point of the movement with the upper end point of the movement, for example β_f or K_u^d except the index which clearly indicates the start or end point, for example, the expression (K_B-K_A) should be understood as the expression (K^d-K_u) or (K^u-K_u) where the value with the lower index K_B correspond to the upper indexes K^d or K^u.

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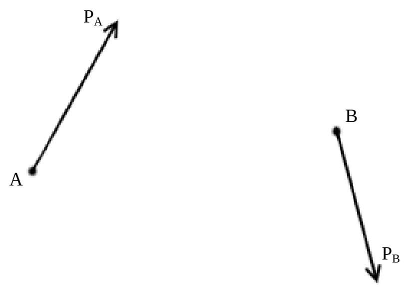

Fig. 1: Dubins machine

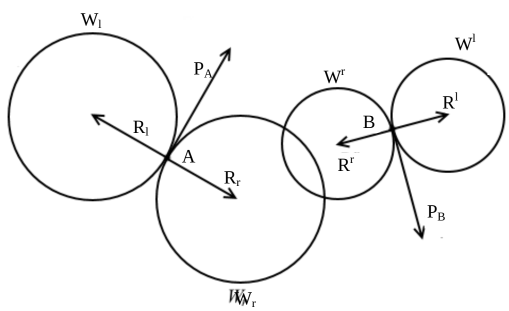

Fig. 2: Turning circles W_a and W_b of GUVat points A and B

In Fig. 1, points A and B are the start and end points of the movement, P_A and P_B are the orientation vectors of the GUV at these points. The next step in building models of the movement is to build circles of rotation W (with radiuses R_l and R_r), relating the vectors in the initial P_A and P_B end point. For each point and direction there are two circles of rotation GUV, denote the position of their centers R_l and R_r (Fig. 2) for point A (R^l and R^r for point B):

$$R_l = R \cdot M\left(\frac{\pi}{2}\right) \cdot \hat{P} \tag{1}$$

$$R_r = R \cdot M\left(-\frac{\pi}{2}\right) \cdot \hat{P} \tag{2}$$

where, R is the scalar value of the specified turning Radius ($R = R_A$ or $R = R_B$), $M(\alpha)$ is the linear operator of rotation on angle α:

$$M(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{3}$$

$\hat{P}$ the normalized vector P, the vector orientation of the vehicle (P_A or P_B):

$$\hat{P} = \frac{P}{|P|} \tag{4}$$

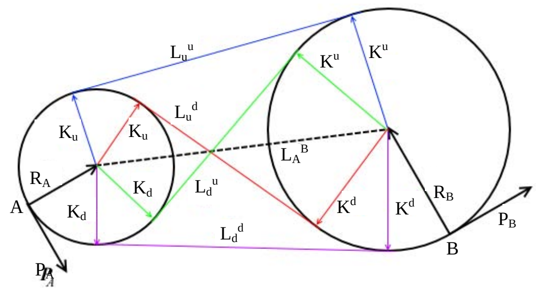

Fig. 3: Constructing tangents to circles of rotation

Table 1: Composition of the circles of rotation

Values	R^l	R^r
R_l	$R_l \cdot R^l$	$R_l \cdot R^r$
R_r	$R_r \cdot R^l$	$R_r \cdot R^r$

The compositions of the centers of the circles R for a pair of points A and B are given in Table 1 and the symbol "•" indicates the composition of objects standing to the left and right of it.

For the 2-dimensional Dubins Model, the following result is known: if two points are far enough away, the shortest path is one consisting of a segment of a radius circle, a straight line and another segment of a radius circle. It turns out that the desired trajectory is divided into three segments: the concatenation of two circles and a line segment. Each composition of rotation circles W has one of four paths $L_{A`}^{B`}$ (indicated by different colors in Fig. 3) formed by tangent lines to, simultaneously, two circles.

It is easy to notice that the vector K is symmetric with respect to the line L_A^B passing through the centers of rotation circles and is in two symmetric states within one rotation circle, let's call them K_u and K_d, respectively (Fig. 3):

$$K_A = R_A \cdot M(\sigma_K) \cdot \hat{L}_A^B \tag{5}$$

$$K_B = \begin{cases} K_A \cdot \dfrac{R_B}{R_A},\ K_A \bullet K_B = K_u^u V K_d^d \\ -K_A \cdot \dfrac{R_B}{R_A},\ K_A \bullet K_B = K_u^d V K_d^u \end{cases} \tag{6}$$

where, $\hat{L}_A^B$ is the normalized vector of the distance between the centers of the circles L_A^B given by the vectors R_A and R_B:

$$L_A^B = (B+R_B)-(A+R_A) \tag{7}$$

$$\hat{L}_A^B = -\hat{L}_B^A = \frac{L_A^B}{|L_A^B|} \tag{8}$$

Ehe angle σ_K between the vector K and the vector $\vec{L}_A^B$, depend on the indices K and inherits its indexing:

$$\begin{cases} \sigma_u^u = a\cos\left(\frac{R_A\text{-}R_B}{\left|L_A^B\right|}\right) \\ \sigma_d^d = \text{-}a\cos\left(\frac{R_A\text{-}R_B}{\left|L_A^B\right|}\right) \\ \sigma_u^d = a\cos\left(\frac{R_A+R_B}{\left|L_A^B\right|}\right) \\ \sigma_d^u = \text{-}a\cos\left(\frac{R_A\text{-}R_B}{\left|L_A^B\right|}\right) \end{cases} \tag{9}$$

It should be noted that since, $L_A{}^B$ is formed from the vectors R_B and R_A it inherits their indexing:

	R^l	R^r
R_l	$L_l{}^l$	$L_l{}^r$
R_r	$L_r{}^l$	$L_r{}^r$

Assuming that M (-π/2) = -M (π/2) and Eq. 7, the following relations (Eq. 10):

$$\begin{aligned} & L_l^l + 2\cdot R^r = L_l^r \\ & L_l^r \text{-} 2\cdot R_r = L_r^r \\ & L_r^r + 2\cdot R^l = L_r^l \\ & L_r^l \text{-} 2\cdot R_l = L_l^l \end{aligned} \tag{10}$$

Then the path of rectilinear movement between the two circles turn W_A and W_B (specified by vectors K_A and K_B from R_A and R_B, respectively) are as follows:

$$L_{A`}^{B`} = \left(B+R_B+K_B\right)\text{-}\left(A+R_A+K_A\right) \tag{11}$$

Given $K = K_B\text{-}K_A$ and Eq. 7, you can rewrite Eq. 11 as follows:

$$L_{A`}^{B`} = L_A^B + K. \tag{12}$$

The compositions of vectors K have the following form:

	K^u	K^d
K_u	$K_u{}^u$	$K_u{}^d$
K_d	$K_d{}^u$	$K_d{}^d$

That is $L_{A`}{}^{B`}$ inherits indexing from K (13):

$$\begin{aligned} & L_u^u = L_A^B + K_u^u \\ & L_u^d = L_A^B + K_u^d \\ & L_d^u = L_A^B + K_d^u \\ & L_d^d = L_A^B + K_d^d \end{aligned} \tag{13}$$

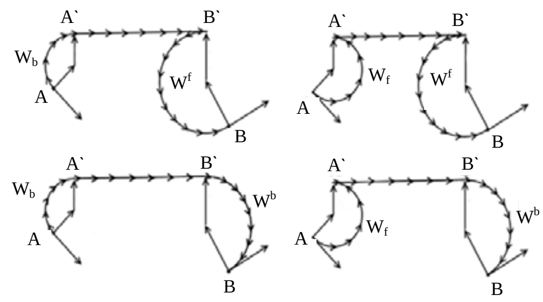

Fig. 4: Constructing tangents to circles of rotation WGUV

Given Eq. 5 and 6, the expression for K takes the following form:

$$\begin{aligned} & K_u^u = \left(R_B\text{-}R_A\right)\cdot M\left(\sigma_u^u\right)\cdot \vec{L}_A^B \\ & K_u^d = \left(R_B+R_A\right)\cdot M\left(\sigma_u^d\right)\cdot \vec{L}_B^A \\ & K_d^u = \left(R_B+R_A\right)\cdot M\left(\sigma_d^u\right)\cdot \vec{L}_B^A \\ & K_d^d = \left(R_B\text{-}R_A\right)\cdot M\left(\sigma_d^d\right)\cdot \vec{L}_A^B \end{aligned} \tag{14}$$

Based on Eq. 14 the following relations between the vectors K are valid:

$$\begin{aligned} & \frac{R_B+R_A}{R_B\text{-}R_A}\cdot M\left(\pi+\sigma_d^u\text{-}\sigma_u^u\right)\cdot K_u^u = K_d^u \\ & \frac{R_B\text{-}R_A}{R_B+R_A}\cdot M\left(\pi+\sigma_d^d\text{-}\sigma_u^d\right)\cdot K_u^d = K_d^d \\ & M\left(\sigma_d^d\text{-}\sigma_u^u\right)\cdot K_u^u = K_d^d \\ & M\left(\sigma_u^d\text{-}\sigma_d^u\right)\cdot K_d^u = K_u^d \end{aligned} \tag{15}$$

The next step in the construction of the trajectory of the GUV is the construction of all possible tangents to the circles of rotation W. Figure 4 shows examples of such tangents.

Since, the points start and end positions A and B lie on the circle of rotation W_A and W_B they allow movement in two directions: front and rear W_f speed W_b, thus, forming four possible trajectories for each of the known pathways expressed in the following compositions:

	W^f	W^b
W_f	$W_f{}^f$	$W_f{}^b$
W_b	$W_b{}^f$	$W_b{}^b$

Thus, for W fair the following relationship 16 and 17:

$$\begin{aligned} & W^f + W^b = 2\pi \\ & W_f + W_b = 2\pi \end{aligned} \tag{16}$$

				B							
				l				r			
				u		d		u		d	
				f	b	f	b	f	b	f	b
A	i	u	f								
			b								
		d	f								
			b								
	r	u	f								
			b								
		d	f								
			b								

Fig. 5: Motion combination matrix (A-Movement around point A, B-Movement around point B, l-Move left, r-Move right, u-Direction of the upward motion vector, d–direction of the motion vector to down, f–forward movement, b–reversing)

$$\begin{cases} W_f = a\cos\left(\dfrac{\langle R \cdot K\rangle}{|R|\cdot|K|}\right), \operatorname{sign}\left([P, R]_z\right) = \operatorname{sign}\left([K, R]_z\right) \\ W_b = a\cos\left(\dfrac{\langle R \cdot K\rangle}{|R|\cdot|K|}\right), \operatorname{sign}\left([P, R]_z\right) \neq \operatorname{sign}\left([K, R]_z\right) \end{cases} \quad (17)$$

Where:

$\langle R, K\rangle$: The scalar product of vectors R and K

$[P, R]_z$: The z component of the vector product of the vectors P and R

For the above compositions it is possible to write the length of the assumed trajectory when moving from point A to point B (Fig. 5):

$$S_A^B = S_A^{A`} + S_{A`}^{B`} + S_{B`}^{B`} \quad (18)$$

Then, given the features of the sites, the value of the traveled path S_A^B:

$$S_A^B = \beta_A^{A`} \cdot |R_A| + \left|L_{A`}^{B`}\right| + x_{B`}^{B} \cdot |R_B| \quad (19)$$

where, $\beta_A^{A`}$ is a sector of a circle A, overcome when moving along the planned trajectories:

$$\beta_A^{A`} = \begin{cases} \theta_A, \beta_A^{A`} = W_f \\ \omega_A, \beta_A^{A`} = W_b \end{cases} \quad (20)$$

$x_{B`}^{B}$– sector of a circle B overcome while moving in the intended trajectory x_(B`)^B-sector of a circle B overcome while moving in the intended trajectory:

$$x_{B`}^{B} = \begin{cases} \theta_B, x_{B`}^{B} = W^f \\ \omega_B, x_{B`}^{B} = W^b \end{cases} \quad (21)$$

Studies have shown that there are 64 possible trajectory and each trajectory is of length, travel time can have a number of stops different from zero, etc., there are different variants of the movement: of the 64 possible trajectories 32 can save the initial orientation and change it to 32; 32 trajectories suggest movement without stopping and 32 with the stop 32 to allow for the reversing between the stages of rotation and 32 allow movement forward course. The final structure of the motion combinations is shown in Fig. 5.

RESULTS AND DISCUSSION

To optimize the performance and search for equilibrium points, first of all it is necessary to determine the time of movement along the trajectory:

$$t = t_A^{A`} + t_{A`}^{B`} + t_{B`}^{B} \quad (22)$$

Where:

$t_A^{A`}$: The time of motion along the circle A

$L_{B`}^{B}$: The time of motion along the B

$L_{A`}^{B`}$: Time of movement along the segment tangent to the circles AиB

In the simplest case when the acceleration time to the maximum speed v_{max} and the braking time to a stop are negligible compared to the total travel time t, we obtain the following relations: forward speed:

$$v_f = v_{max} \quad (23)$$

Reversing speed:

$$v_b = \text{const} \leq v_f \quad (24)$$

Since, the points of the initial and final position lie on the rotation circles, they, together with the vector K, divide each circle S into two parts, forming two angles-θ and ω (Fig. 6) and the angle θ corresponds to the forward motion and the angle ω corresponds to the reverse motion. Since, θ by definition always coincides with the direction of forward motion we have:

$$t_f = \frac{|R_A| \cdot \theta_A}{v_f} \quad (25)$$

$$t_b = \frac{|R_A| \cdot \omega_A}{v_b} \quad (26)$$

where, t_f and t_b the time of movement (rotation) along the circle A when driving forward and reverse, respectively. In this case, we have an equilibrium value θ depending on the velocity ratio (the value θ in this case is equally characteristic of the rotation circles A and B):

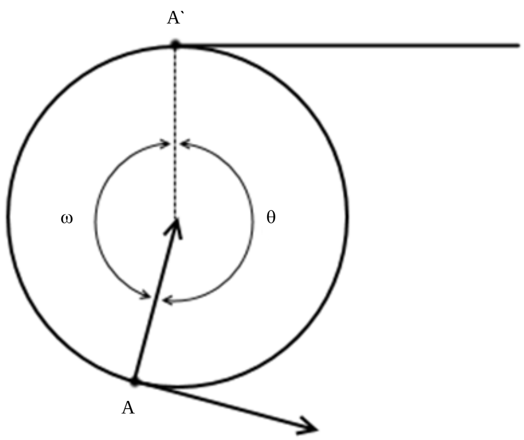

Fig. 6: Draw angles of the direction of movement

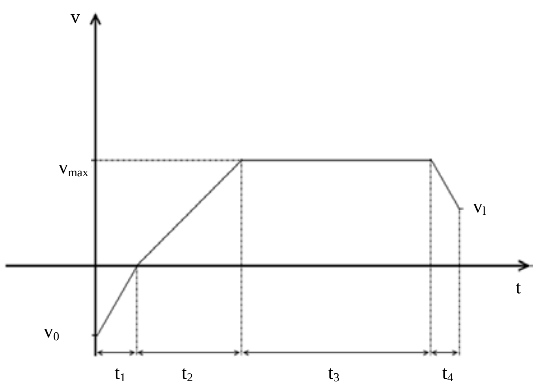

Fig. 7: Scheme of the time factor t of the forward motion on a circle

$$\theta = 2\pi \frac{v_f}{v_f + v_b} \quad (27)$$

$$\omega = 2\pi \frac{v_b}{v_f + v_b} \quad (28)$$

Considering characteristics of the trajectory, given a uniform and uniformly accelerated motion, the presence of phases of acceleration and deceleration at each of the sites and the restrictions start, end and maximum speed have motion (Fig. 7):

$$t = t_1 + t_2 + t_3 + t_4 \quad (29)$$

Where:

- t : The total time of forward or reverse motion ($t_A^{A`} = t_f$ or $t_A^{A`} = t_b$) on some section
- t_1 : Reverse braking time (if at the initial time the vehicle has a velocity vector accompanying the reverse movement)

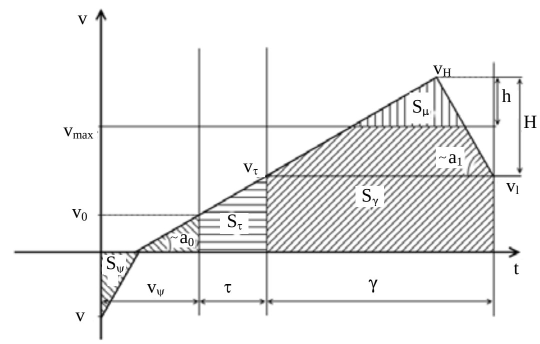

Fig. 8: Scheme determining the time of movement of a ground-based GUV along a trajectory

- t_2 : Forward acceleration time
- t_3 : Forward speed
- t_4 : Forward braking time

Given the features of the problem, you can change the intervals that determine the time of movement along the path as follows:

$$t = \psi + \tau + \gamma \quad (30)$$

Where:

- t : Total time spent on overcoming all sections;
- ψ : The section of preliminary braking and subsequent acceleration, i.e., the length of the braking distance is equal to half the distance S_ψ traveled in the region ψ and is equal to the acceleration distance to speed v_0 (in the explanatory diagram, this is shown in the form of a region composed of two triangles having the same area). If $v_\psi \geq 0$, then $v_0 = v_\psi$, $S_\psi = 0$, $\psi = 0$, since, preliminary braking is absent;
- τ : Section of pure acceleration or braking (depending on the ratio of speeds v_0 and v_1)
- γ : Section acceleration to maximum speed followed by braking to speed v_1

As a result of the presented in the research, a scheme is developed and proposed that determines the time of movement of the ground GUV along the trajectory (Fig. 8). In accordance with the proposed scheme of motion developed a mathematical model of the movement of ground GUV. The full path of the GUV movement can be represented as:

$$S = S_\tau + S_\gamma - S_\mu \quad (31)$$

Where:

- S : Total distance travelled
- S_τ : Total distance traveled on the site τ
- S_γ : Total distance traveled on the site γ
- S_μ : Normalizing distance, when exceeding the maximum speed v_{max}

The pre-braking section is described by the following dependencies:

$$v_0 = \begin{cases} v_\psi,\ v_\psi \geq 0 \\ -\sqrt{\dfrac{a_0}{a_\psi}} \cdot v_\psi,\ v_\psi < 0 \end{cases} \tag{32}$$

Where:

- v_0 : The speed that the vehicle achieves in ψ during acceleration or is the initial speed, the direction of this speed coincides with the direction of movement along the intended path
- v_ψ : The initial velocity of the vehicle, it is either co-directional with the movement along the intended trajectory or counter-directional with it
- a_0 : Acceleration during acceleration along the intended trajectory, a_0 = const
- a_ψ : Acceleration during braking on the site

The travel time ψ is described as:

$$\psi = \begin{cases} -v_\psi \cdot \left(\dfrac{a_\psi + \sqrt{a_\psi \cdot a_0}}{a_\psi \cdot \sqrt{a_\psi \cdot a_0}}\right),\ v_\psi < 0 \\ 0,\ v_\psi \geq 0 \end{cases} \tag{33}$$

From Eq. 33, it follows that the full path length S_ψ on the site prior braking ψ can be defined as:

$$S_\psi = \begin{cases} \dfrac{v_\psi^2}{a_\psi},\ v_\psi < 0 \\ 0,\ v_\psi \geq 0 \end{cases} \tag{34}$$

The time τ spent by the GUV for acceleration (deceleration) can be calculated as:

$$\tau = \begin{cases} \dfrac{v_1 - v_0}{a_0},\ v_1 \geq v_0 \\ \dfrac{v_0 - v_1}{a_1},\ v_1 < v_0 \end{cases} \tag{35}$$

Where:

- τ : Time to overcome the eponymous section
- v_1 : The final velocity of the vehicle to overcome all sections
- a_1 : Acceleration during braking to the speed v_1 with the current movement, it is aligned with the direction of movement along the assumed path

The full path of the traversed path S_τ on the section τ and the velocity $v_\tau (v_\tau \leq v_{max})$ are defined as:

$$S_\tau = \tau \cdot \frac{v_1 + v_0}{2} \tag{36}$$

$$v_\tau = \begin{cases} v_0,\ v_1 < v_0 \\ v_1,\ v_1 \geq v_0 \end{cases} \tag{37}$$

The last section in scheme 8 is the section of acceleration time γ to the maximum speed, followed by braking to the speed. For this phase characteristic difference H between the maximum speed developed on a plot of γ and the maximum speed v_τ develop at the site τ. Analysis of scheme 8 showed that this difference can be calculated as follows:

$$H = \gamma \cdot \frac{a_1 \cdot a_0}{a_1 + a_0} \tag{38}$$

The maximum velocity v_H which can develop GUV within the framework of this model on the site γ can be defined as:

$$v_H = v_\tau + H \tag{39}$$

The distance S_g that the GUV can cover when accelerating to the maximum permitted speed v_{max} and braking to the speed v_1 is determined from the following ratio:

$$S_g = \left(\frac{v_{max}^2 - v_\tau^2}{2}\right)\frac{a_1 + a_0}{a_1 \cdot a_0} \tag{40}$$

where, S_g the maximum, theoretically possible within the model, distance covered by the fastest acceleration to the maximum allowed velocity v_{max} and braking to speed v_1. Difference h (Fig. 8) between the maximum achievable speed v_H and the maximum allowed speed v_{max}, h≥0 can be defined as:

$$h = \begin{cases} v_H - v_{max},\ S \geq S_\tau + S_g \\ 0,\ S < S_\tau + S_g \end{cases} \tag{41}$$

Normalizing distance S_μ, limiting the traversed path within the movement with the maximum allowed speed v_{max} and full length S_γ distance traveled on a plot of γ, when acceleration to maximum attainable speed v_H and subsequent deceleration to final speed v_1 are determined from Eq. 42 and 43:

$$S_\mu = \gamma \cdot \frac{h}{2} \tag{42}$$

$$S_\gamma = \gamma \cdot \left(v_\tau + \frac{H}{2} \right) \tag{43}$$

From all the above relations, we can obtain the time of motion γ as a component of the total time t:

$$\tau \cdot \frac{v_1 + v_0}{2} + \gamma \cdot \left(v_\tau + \frac{H}{2} \right) - \gamma \cdot \frac{h}{2} - S = 0 \tag{44}$$

Substituting in Eq. 44 the necessary relations and synchronizing them under the conditions, we have the following set of relations:

$$\begin{cases} \gamma \cdot \left(\dfrac{v_1}{2} + \dfrac{v_{max}}{2} \right) + \left(\dfrac{v_1^2 - v_0^2}{2 \cdot a_0} - S \right) = 0, \; v_1 \geq v_0, \; S \geq S_g + S_\tau \\ \gamma \cdot \left(\dfrac{v_0}{2} + \dfrac{v_{max}}{2} \right) + \left(\dfrac{v_0^2 - v_1^2}{2 \cdot a_1} - S \right) = 0, \; v_1 < v_0, \; S \geq S_g + S_\tau \\ \gamma^2 \dfrac{a_0 \cdot a_1}{2 \cdot (a_0 + a_1)} + \gamma \cdot v_1 + \left(\dfrac{v_1^2 - v_0^2}{2 \cdot a_0} - S \right) = 0, \; v_1 \geq v_0, \; S < S_g + S_\tau \\ \gamma^2 \dfrac{a_0 \cdot a_1}{2 \cdot (a_0 + a_1)} + \gamma \cdot v_0 + \left(\dfrac{v_0^2 - v_1^2}{2 \cdot a_1} - S \right) = 0, \; v_1 < v_0, \; S < S_g + S_\tau \end{cases} \tag{45}$$

$$S_g + S_\tau = \frac{v_{max}^2 - v_1^2}{2 \cdot a_1} + \frac{v_{max}^2 - v_0^2}{2 \cdot a_0} \tag{46}$$

To shorten the entry, we introduce the following notation:

$$\gamma^{v_1}, \gamma_{v_1}, \gamma^{v_\psi}, \gamma_{v_\psi}, \gamma^S, \gamma_S$$

where:

γ^{v1} : Superscript if $v_1 \geq v_0$
γ_{v1} : Subscript if $v_1 < v_0$
γ^{v_ψ} : Superscript if $v_\psi \geq v_0$
γ_{v_ψ} : Subscript if $v_\psi < v_0$
γ^S : Superscript if $S \geq v_{max}{}^2 - v_1{}^2/2 \cdot a_1 + v_{max}{}^2 - v_0{}^2/2 \cdot a_0$
γ_S : Subscript if $S < v_{max}{}^2 - v_1{}^2/2 \cdot a_1 + v_{max}{}^2 - v_0{}^2/2 \cdot a_0$

Then, taking into account relation (32), we have the following set of solutions:

$$\begin{cases} \gamma^{v_\psi v_1 S} \cdot \left(\dfrac{v_1}{2} + \dfrac{v_{max}}{2} \right) + \left(\dfrac{v_1^2 - v_\psi^2}{2 \cdot a_0} - S \right) = 0 \\ \gamma_{v_\psi}^{v_1 S} \cdot \left(\dfrac{v_1}{2} + \dfrac{v_{max}}{2} \right) + \left(\dfrac{v_1^2}{2 \cdot a_0} - \dfrac{v_\psi^2}{2 \cdot a_\psi} - S \right) = 0 \\ \gamma_{v_1}^{v_\psi S} \cdot \left(\dfrac{v_\psi}{2} + \dfrac{v_{max}}{2} \right) + \left(\dfrac{v_\psi^2 - v_1^2}{2 \cdot a_1} - S \right) = 0 \\ \gamma_{v_\psi v_1}^{S} \cdot \left(\dfrac{v_{max}}{2} - \sqrt{\dfrac{a_0}{a_\psi}} \dfrac{v_\psi}{2} \right) + \left(\dfrac{a_0 \cdot v_\psi^2}{2 \cdot a_\psi \cdot a_1} - \dfrac{v_1^2}{2 \cdot a_1} - S \right) = 0 \end{cases}$$

$$\begin{cases} \left(\gamma_S^{v_\psi v_1} \right)^2 \cdot \dfrac{a_0 \cdot a_1}{2 \cdot (a_0 + a_1)} + \gamma_S^{v_\psi v_1} \cdot v_1 + \left(\dfrac{v_1^2 - v_\psi^2}{2 \cdot a_0} - S \right) = 0 \\ \left(\gamma_{v_\psi S}^{v_1} \right)^2 \cdot \dfrac{a_0 \cdot a_1}{2 \cdot (a_0 + a_1)} + \gamma_{v_\psi \eta}^{v_1} \cdot v_1 + \left(\dfrac{v_1^2}{2 \cdot a_0} - \dfrac{v_\psi^2}{2 \cdot a_\psi} - S \right) = 0 \\ \left(\gamma_{v_1 S}^{v_\psi} \right)^2 \cdot \dfrac{a_0 \cdot a_1}{2 \cdot (a_0 + a_1)} + \gamma_{v_1 S}^{v_\psi} \cdot v_\psi + \left(\dfrac{v_\psi^2 - v_1^2}{2 \cdot a_1} - S \right) = 0 \\ \left(\gamma_{v_\psi v_1 S} \right)^2 \cdot \dfrac{a_0 \cdot a_1}{2 \cdot (a_0 + a_1)} - \gamma_{v_\psi v_1 S} \cdot \sqrt{\dfrac{a_0}{a_\psi}} \cdot v_\psi + \left(\dfrac{\dfrac{a_0}{a_\psi} v_\psi^2 - v_1^2}{2 \cdot a_1} - S \right) = 0 \end{cases}$$

It is easy to notice that the desired time interval γ is the solution of a linear or quadratic equation, so, the total time t:

$$\begin{cases} t^{v_\psi v_1 S} = \dfrac{v_1 - v_\psi}{a_0} + \gamma^{v_\psi v_1 S} \\ t_{v_\psi}^{v_1 S} = -v_\psi \cdot \left(\dfrac{a_\psi + \sqrt{a_\psi \cdot a_0}}{a_\psi \cdot \sqrt{a_\psi \cdot a_0}} \right) + \dfrac{v_1 + \sqrt{\dfrac{a_0}{a_\psi}} \cdot v_\psi}{a_0} + \gamma_{v_\psi}^{v_1 S} \\ t_{v_1}^{v_\psi S} = \dfrac{v_\psi - v_1}{a_1} + \gamma_{v_1}^{v_\psi S} \\ t_{v_\psi v_1}^{S} = -v_\psi \cdot \left(\dfrac{a_\psi + \sqrt{a_\psi \cdot a_0}}{a_\psi \cdot \sqrt{a_\psi \cdot a_0}} \right) - \dfrac{\sqrt{\dfrac{a_0}{a_\psi}} \cdot v_\psi + v_1}{a_1} + \gamma_{v_\psi v_1}^{S} \\ t_S^{v_\psi v_1} = \dfrac{v_1 - v_\psi}{a_0} + \gamma_S^{v_\psi v_1} \\ t_{v_\psi S}^{v_1} = -v_\psi \cdot \left(\dfrac{a_\psi + \sqrt{a_\psi \cdot a_0}}{a_\psi \cdot \sqrt{a_\psi \cdot a_0}} \right) + \dfrac{v_1 + \sqrt{\dfrac{a_0}{a_\psi}} \cdot v_\psi}{a_0} + \gamma_{v_\psi S}^{v_1} \\ t_{v_1 S}^{v_\psi} = \dfrac{v_\psi - v_1}{a_1} + \gamma_{v_1 S}^{v_\psi} \\ t_{v_\psi v_1 S} = -v_\psi \cdot \left(\dfrac{a_\psi + \sqrt{a_\psi \cdot a_0}}{a_\psi \cdot \sqrt{a_\psi \cdot a_0}} \right) - \dfrac{\sqrt{\dfrac{a_0}{a_\psi}} \cdot v_\psi + v_1}{a_1} + \gamma_{v_\psi v_1 S} \end{cases}$$

Features of the optimal trajectory of the vehicle with a constant turning radius, suggest movement at a constant speed. Therefore, to remain within the framework of this model, it is necessary to take into account the angle of turning the wheels Θ during acceleration (or braking). For small acceleration values, the following ratio can be used (Selifonov *et al*., 2007):

$$\Theta = \frac{L}{R} - \frac{v^2}{g \cdot R} \cdot \left(\frac{G_1}{K_1} - \frac{G_2}{K_2} \right) \tag{47}$$

Where:

Θ	:	Wheel turning angle
L	:	Vehicle base
V	:	Vehicle center of mass Velocity
R	:	Vehicle turning Radius
$(G_1/K_1\text{-}G_2/K_2)$	:	The coefficient of the understeer
g	:	Gravitational acceleration

Given Eq. 47 the maximum speed V_{max} are developing in the area of the turn during acceleration or deceleration before depending on the maximum angle of rotation of the wheels Θ_{max} is as follows:

$$V_{max} = \sqrt{\frac{(L\text{-}\Theta_{max} \cdot R)g}{\frac{G_1}{K_1}\text{-}\frac{G_2}{K_2}}} \tag{48}$$

Naturally, V_{max} may not satisfy the condition of skidding and/or overturning, so as V_{max} you should choose the minimum speed that meets all the conditions.

CONCLUSION

A mathematical model of the GUV movement based on the Dubins machine is developed which allows determining the motion vector of a wheeled vehicle. It is established that there are 64 combinations of motion to construct the trajectory of the GUV. A scheme is developed that determines the time of movement along the trajectory and allows to determine the characteristics of the movement of the ground GUV. The proposed optimization criterion time and the determination of equilibrium points, given the 64 combinations of the trajectory.

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