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Asymptotic Behavior of Eigenvalue and Eigenfunction of a Six Order Boundary Value Problem

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Abstract: We consider differential operators with separate boundary conditions in this study. And we find a new expression and their derivative of six linearly independent solution. And it will also prove the existence of uniqueness for separate boundary conditions. We obtain asymptotic formulas for the individual values and functions of these problems with boundary value where the potential q (x) is an arbitrary complex function valued in [a, b].

Key words: Differential operator, eigenvalue, eigenfunctions, spectral parameter, arbitrary complex function, potential

INTRODUCTION

It is well known that many researchers have investigated the spectral properties of the Sturm-Liouville operator generated by the separate boundary condition (Aigunov, 1996) and many researchers have found asymptotic formula for the Sturm-Liouville operator's eigenvalues and functions in the case of periodic andanti-periodic boundary conditions (Menken, 2010; Naimark, 1967; Moller and Zinsou, 2012; Jwamer and Aigounv, 2010; Aigounov and Tamila, 2009; Aigunov, 1996 and Tamarkin, 1928). Many researchers have been interested in the ongoing Sturm-Liouville issue in recent years as we see N.B. Kermov, H. Menken by Menken (2010), Karwan and Rando (2017) found upper bound with smooth coefficients for the proper functions of the fourth boundary value issue. In this study, we consider the differential operator:

$$y^{(6)}(x)+q(x)y(x)=\lambda^6 y(x)$$

$$U_i(y)=\sum_{j=0}^{5} a_{ij}y^{(j)}(b,\lambda),\ i = 0, 1, 2$$

where, a_{ij} are real numbers:

$$U_3(y)=\sum_{\iota=1}^{6}(\lambda)^{\iota-1}y^{(6-\iota)}(b,\lambda)$$

$$U_4(y)=\sum_{\iota=1}^{6}(-\lambda)^{\iota-1}y^{(6-\iota)}(b,\lambda)$$

$$U_5(y)=\sum_{\iota=1}^{6}\left(\frac{\lambda}{\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}\right)^{\iota-1}y^{(6-\iota)}(b,\lambda)$$

Where:
λ : A spectral parameter
q (x) : An arbitrary complex valued function

FUNDAMENTAL DIFFERENTIAL EQUATION SOLUTIONS SYSTEMS

The expressions of six linearly independent solutions and their derivatives can be found in this study.

Theorem 1: The fundamental system of solutions of linear differential equation:

$$y^{(6)}(x)+q(x)y(x)=\lambda^6 y(x) \tag{1}$$

Are y_0 (x, λ), y_1 (x, λ), y_2 (x, λ), y_3 (x, λ), y_4 (x, λ), y_5 (x, λ) that satisfy the initial conditions:

$$y_i^{(n)}(0,\lambda)=\begin{cases}1 & \text{if } i=n\\ 0 & \text{if } i\neq n\end{cases} \tag{2}$$

Where:

$$y_0=\frac{1}{3}\left[\cosh\lambda x+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x+\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}\int_0^x\left[\sinh\lambda(x-\xi)+\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]q(\xi)y_0(\xi)d\xi$$

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$$y_1=\left[\frac{1}{3\lambda}\sinh\lambda x+\frac{1}{6\lambda}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x+\frac{1}{6\lambda}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}$$
$$\int_a^b\left[\sinh\lambda(x-\xi)+\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]q(\xi)y_0(\xi)d\xi$$

$$y_2=\left[\frac{1}{3\lambda^2}\cosh\lambda x+\frac{1}{6\lambda^2}\left(-1+\frac{1}{32}i\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x+\frac{1}{6\lambda^2}\left(1-\frac{1}{32}i\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}$$
$$\int_0^x\left[\sinh\lambda(x-\xi)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]q(\xi)y_0(\xi)d\xi$$

$$y_3=\frac{1}{3\lambda^3}\left[\sinh\lambda x-\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x+\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}$$
$$\int_0^x\left[\sinh\lambda(x-\xi)+\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]q(\xi)y_0(\xi)d\xi$$

$$y_4=\left[\frac{1}{3\lambda^4}\cosh\lambda x+\frac{1}{6\lambda^4}\left(1-\frac{1}{32}i\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x+\frac{1}{6\lambda^4}\left(-1+\frac{1}{32}i\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}$$
$$\int_0^x\left[\sinh\lambda(x-\xi)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]q(\xi)y_0(\xi)d\xi$$

$$y_5=\left[\frac{1}{3\lambda^5}\sinh\lambda x-\frac{1}{6\lambda^5}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x-\frac{1}{6\lambda^5}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}$$
$$\int_0^x\left[\sinh\lambda(x-\xi)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]q(\xi)y_0(\xi)d\xi$$

Proof: Consider the linear differential operator:

$$l(y)=-y^{(6)}(x)+q(x)y(x) \tag{3}$$

We want to find a solution that is not zero:

$$l(y)-\lambda^6y(x)=0$$

Which satisfy the initial conditions Eq. 2. First, we reduce Eq. 3 to an integro-differential equations:

$$y^{(6)}(x)+\lambda^6y(x)=q(x)y(x)$$

$$m(y)=y^{(6)}(x)+\lambda^6y(x),m(y)=q(x)y(x) \tag{4}$$

The homogeneous linear differential equation $y^{(6)}(x)+\lambda^6y(x)=0$ has for $\lambda\neq0$ the solutions:

$$e^{\lambda w_0x},e^{\lambda w_1x},e^{\lambda w_2x},e^{\lambda w_3x},e^{\lambda w_4x},e^{\lambda w_5x}$$

Where:

$$w_0=1,\ w_1=\frac{1}{2}+i\frac{\sqrt{3}}{2},w_2=-\frac{1}{2}+i\frac{\sqrt{3}}{2},$$
$$w_3=-1,\ w_4=-\frac{1}{2}-i\frac{\sqrt{3}}{2},w_5=\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

Then by using the method of variation of parameters we can express the solutions of Eq. 4 for k = 0, 1, 2, 3, 4, 5 as:

$$y_k=c_0e^{\lambda x}+c_1e^{\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}+c_2e^{\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}+c_3e^{-\lambda x}+c_4e^{\lambda\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}+c_5e^{\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}+$$
$$\int_0^x\frac{\sinh(x-\xi)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)}{3\lambda^5}q(\xi)y_k(x,\lambda)d\xi \tag{5}$$

We apply Eq. 5 and in Eq. 2, then, we get. For k = 0 then:

$$y_0(0,\lambda)=1,\ y_0'(0,\lambda)=0,\ y_0''(0,\lambda)=0,$$
$$y_0'''(0,\lambda)=0,\ y_0^{(4)}(0,\lambda)=0,\ y_0^{(5)}(0,\lambda)=0$$

$$c_0+c_1+c_2+c_3+c_4+c_5=1$$

$$\lambda c_0+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\lambda c_1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\lambda c_2-$$
$$\lambda c_3+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\lambda c_4+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\lambda c_5=0$$

$$\lambda^2 c_0+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\lambda^2 c_1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\lambda^2 c_2+$$
$$\lambda^2 c_3+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^2\lambda^2 c_4+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^2\lambda^2 c_5=0$$

$$\lambda^3 c_0+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\lambda^3 c_1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\lambda^3 c_2-\lambda^3 c_3+$$
$$\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^3\lambda^3 c_4+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^3\lambda^3 c_5=0$$

$$\lambda^4 c_0+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\lambda^4 c_1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\lambda^4 c_2+$$
$$\lambda^4 c_3+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^4\lambda^4 c_4+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^4\lambda^4 c_5=0$$

$$\lambda^5 c_0+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\lambda^5 c_1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\lambda^5 c_2-\lambda^5 c_3+$$
$$\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^5\lambda^5 c_4+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^5\lambda^5 c_5=0$$

We can solve it for c_i and we get $c_i = 1/6$ for each k = 0:5 then y_0 has the form:

$$y_0=\frac{1}{3}\left[\cosh\lambda x+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x+\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]+\frac{1}{3\lambda^5}$$
$$\int_0^x\left[\begin{array}{l}\sinh\lambda(x-\xi)+\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-\xi)+\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\end{array}\right]q(\xi)y_0(\xi)d\xi$$

And we can use the same technique for $y_1(x,\lambda)$, $y_2(x,\lambda)$, $y_3(x,\lambda)$, $y_4(x,\lambda)$, $y_5(x,\lambda)$.

Corollary 1: For x in [a, b] and $\lambda\neq 0$, $y_0(x,\lambda)$ and its derivatives in theorem 1 can be written as:

$$y_0=\frac{1}{3}\left[\begin{array}{l}\cosh\lambda(x-a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\\\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+0\left(\frac{1}{\lambda^5}e^{|t|(x-a)}\right)$$

$$y_0'=\frac{1}{3}\lambda\left[\begin{array}{l}\sinh\lambda(x-a)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$
$$0\left(\frac{1}{\lambda^4}e^{|t|(x-a)}\right)$$

$$y_0''=\frac{1}{3}\lambda^2\left[\begin{array}{l}\cosh\lambda(x-a)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\\\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$
$$0\left(\frac{1}{\lambda^3}e^{|t|(x-a)}\right)$$

$$y_0'''=\frac{1}{3}\lambda^3\left[\begin{array}{l}\sinh\lambda(x-a)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\\\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$
$$0\left(\frac{1}{\lambda^2}e^{|t|(x-a)}\right)$$

$$y_0^{(4)}=\frac{1}{3}\lambda^4\left[\begin{array}{l}\cosh\lambda(x-a)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$
$$0\left(\frac{1}{\lambda}e^{|t|(x-a)}\right)$$

$$y_0^{(5)}=\frac{1}{3}\lambda^5\left[\begin{array}{l}\sinh\lambda(x-a)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$
$$0\left(e^{|t|(x-a)}\right)$$

Proof: Since, $y_0(x,\lambda)$ in theorem 1 has the form:

$$y_0 = \frac{1}{3}\left[\cosh\lambda(x-a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]+\frac{1}{3\lambda^5}$$
$$\int_a^x\left[\sinh\lambda(x-\xi)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]$$
$$q(\xi)y_0(\xi)d\xi$$

Since, $|\cosh z|\le e^{|re\,(z)|}$ and $|\sinh z|\le e^{|re\,(z)|}$:

$$|\sinh\lambda(x-\xi)|\le e^{|\sigma(x-\xi)|},\ \left|\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right|\le e^{\left|\left(\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}$$
$$\left|\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right|\le e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}$$

where, $\lambda = \sigma+i\tau$:

$$|\cosh\lambda x|\le e^{|\sigma x|},\left|\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right|\le e^{\left|\left(\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|},\ \left|\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right|\le e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}$$

$$y_0(x,\lambda)|\le\frac{1}{3}\left[|\cosh\lambda x|+\left|\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right|+\left|\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right|\right]+\frac{1}{3\lambda^5}$$
$$\int_a^x\left[|\sinh\lambda(x-\xi)|+\left|\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right|+\left|\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right|\right]q(\xi)y_0(\xi)d\xi$$

Then:

$$|y_0(x,\lambda)|\le\frac{1}{3}\left[e^{|\sigma x|}+e^{\left|\left(\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}+e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}\right]+\frac{1}{3\lambda^5}$$
$$\int_a^x\left[e^{|\sigma(x-\xi)|}+\left|\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right|+e^{\left|\left(\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}+\left|\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}\right|\right]$$
$$q(\xi)y_0(\xi,\lambda)d\xi$$

And since:

$$e^{|\sigma x|}\le e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|},e^{\left|\left(\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}\le e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}$$

$$|y_0(x,\lambda)|\le\frac{1}{3}3e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}+\frac{1}{3|\lambda|^5}$$
$$\int_a^x\left[e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}+e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}+e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}\right]$$
$$q(\xi)y_0(\xi,\lambda)d\xi$$

$$|y_0(x,\lambda)|\le e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}+\frac{1}{|\lambda|^5}\int_a^x e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}$$
$$q(\xi)y_0(\xi,\lambda)d\xi$$

Put:

$$|y_0(x,\lambda)| = e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)x\right|}f(x,\lambda)$$

$$f(x,\lambda) = y_0(x,\lambda)e^{\left|\left(\frac{1}{2}\sigma+\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}$$

$$f(x,\lambda) = y_0 = \frac{1}{3}\left[\cosh\lambda x + \cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x + \cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]e^{\left|\left(\frac{1}{2}\sigma+\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|} +$$

$$\frac{1}{3\lambda^5}\int_a^x\left[\sinh\lambda(x-\xi)+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-\xi)\right]e^{\left|\left(\frac{1}{2}\sigma+\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}$$

$$q(\xi)f(\xi,\lambda)d\xi$$

Let, M (λ) denote the maximum value of $|f(x, \lambda)|$ for x in [a, b] then we obtain:

$$M(\lambda)\le 1+\frac{1}{|\lambda|^5}M(\lambda)\int_a^x|q(\xi)|d\xi$$

$$M(\lambda)\left(1-\frac{1}{|\lambda|^5}M(\lambda)\int_a^x|q(\xi)|d\xi\right)\le 1$$

$$M(\lambda)\le\left\{1-\frac{1}{|\lambda|^5}M(\lambda)\int_a^x|q(\xi)|d\xi\right\}^{-1}$$

$|f(x, \lambda)|\le M(\lambda) = 0(1)$, $|\lambda|\to\infty$ and therefore:

$$y_0(x,\lambda) = 0\left\{e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}\right\}$$

So, the integral equation is:

$$0\left\{\frac{1}{\lambda^5}\int_a^x e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-\xi)\right|}e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(\xi-a)\right|}q(\xi)d\xi\right\} =$$

$$0\left\{|\lambda|^{-5}e^{\left|\left(-\frac{1}{2}\sigma-\frac{\sqrt{3}}{2}t\right)(x-a)\right|}\right\}$$

By the same way we get all derivatives.

Corollary 2: For x in [a, b] and $\lambda\neq 0$, y_1 (x, λ) and its derivatives in theorem 1 can be written as:

$$y_1 = \frac{1}{3\lambda}\left[\begin{array}{l}\sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$

$$0\left(\frac{1}{\lambda^6}e^{|t|(x-a)}\right)$$

$$y_1' = \frac{1}{3}\left[\begin{array}{l}\cosh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\\\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$

$$0\left(\frac{1}{\lambda^5}e^{|t|(x-a)}\right)$$

$$y_1'' = \frac{1}{3}\lambda\left[\begin{array}{l}\sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$

$$0\left(\frac{1}{\lambda^4}e^{|t|(x-a)}\right)$$

$$y_1''' = \frac{1}{3}\lambda^2\left[\begin{array}{l}\cosh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\\\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\\\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$

$$0\left(\frac{1}{\lambda^3}e^{|t|(x-a)}\right)$$

$$y_1^{(4)} = \frac{1}{3}\lambda^3\left[\begin{array}{l}\sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{array}\right]+$$

$$0\left(\frac{1}{\lambda^2}e^{|t|(x-a)}\right)$$

$$y_1^{(5)} = \frac{1}{3}\lambda^4$$

$$\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$

$$0\left(\frac{1}{\lambda}e^{|t|(x\text{-}a)}\right)$$

Corollary 3: For x in [a, b] and $\lambda\neq0$, y_2 (x, λ) and its derivatives in theorem 1 can be written as:

$$y_2 = \frac{1}{3\lambda^2}\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^7}e^{|t|(x\text{-}a)}\right)$$

$$y_2' = \frac{1}{3\lambda}\left[\sinh\lambda(x\text{-}a)+\frac{1}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^6}e^{|t|(x\text{-}a)}\right)$$

$$y_2'' = \frac{1}{3}\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)$$

$$y_2''' = \frac{1}{3}\lambda\left[\sinh\lambda(x\text{-}a)+\frac{1}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^4}e^{|t|(x\text{-}a)}\right)$$

$$y_2^{(4)} = \frac{1}{3}\lambda^2\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^3}e^{|t|(x\text{-}a)}\right)$$

$$y_2^{(5)} = \frac{1}{3}\lambda^3\left[\sinh\lambda(x\text{-}a)+\frac{1}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^2}e^{|t|(x\text{-}a)}\right)$$

Corollary 4: For x in [a, b] and $\lambda\neq0$, y_3 (x, λ) and its derivatives in theorem 1 can be written as:

$$y_3 = \frac{1}{3\lambda^3}\left[\sinh\lambda(x\text{-}a)-\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^8}e^{|t|(x\text{-}a)}\right)$$

$$y_3' = \frac{1}{3\lambda^2}\left[\cosh\lambda(x\text{-}a)-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^7}e^{|t|(x\text{-}a)}\right)$$

$$y_3'' = \frac{1}{3\lambda}\left[\sinh\lambda(x\text{-}a)-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$

$$0\left(\frac{1}{\lambda^6}e^{|t|(x\text{-}a)}\right)$$

$$y_3''' = \frac{1}{3}\left[\cosh\lambda(x\text{-}a)-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$

$$0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)$$

$$y_3^{(4)}=\frac{1}{3\lambda}\left[\sinh\lambda(x\text{-}a)\text{-}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^4}e^{|t|(x\text{-}a)}\right)$$

$$y_3^{(5)}=\frac{1}{3}\lambda^2\left[\cosh\lambda(x\text{-}a)\text{-}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^3}e^{|t|(x\text{-}a)}\right)$$

Corollary 5: For x in [a, b] and $\lambda\neq 0$, y_4 (x, λ) and its derivatives in theorem 1 can be written as:

$$y_4=\frac{1}{3\lambda^4}\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^9}e^{|t|(x\text{-}a)}\right)$$

$$y_4'=\frac{1}{3\lambda^3}\left[\sinh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^8}e^{|t|(x\text{-}a)}\right)$$

$$y_4''=\frac{1}{3\lambda^2}\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$
$$0\left(\frac{1}{\lambda^7}e^{|t|(x\text{-}a)}\right)$$

$$y_4'''=\frac{1}{3\lambda}\left[\sinh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$
$$0\left(\frac{1}{\lambda^6}e^{|t|(x\text{-}a)}\right)$$

$$y_4^{(4)}=\frac{1}{3}\left[\cosh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$
$$0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)$$

$$y_4^{(5)}=\frac{1}{3}\lambda\left[\sinh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$
$$0\left(\frac{1}{\lambda^4}e^{|t|(x\text{-}a)}\right)$$

Corollary 6: For x in [a, b] and $\lambda\neq 0$, y_5 (x, λ) and its derivatives in theorem 1 can be written as:

$$y_5=\frac{1}{3\lambda^5}\left[\sinh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+0\left(\frac{1}{\lambda^{10}}e^{|t|(x\text{-}a)}\right)$$

$$y_5'=\frac{1}{3\lambda^4}\left[\cosh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(-1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\right]+$$
$$0\left(\frac{1}{\lambda^9}e^{|t|(x\text{-}a)}\right)$$

$$y_5'' = \frac{1}{3\lambda^3}\begin{bmatrix} \sinh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\sinh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+ 0\left(\frac{1}{\lambda^8}e^{|t|(x\text{-}a)}\right)$$

$$y_5''' = \frac{1}{3\lambda^2}\begin{bmatrix} \cosh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3\cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+ 0\left(\frac{1}{\lambda^7}e^{|t|(x\text{-}a)}\right)$$

$$y_5^{(4)} = \frac{1}{3\lambda}\begin{bmatrix} \sinh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^4\sinh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+ 0\left(\frac{1}{\lambda^6}e^{|t|(x\text{-}a)}\right)$$

$$y_5^{(5)} = \frac{1}{3}\begin{bmatrix} \cosh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right)\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^5\cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+ 0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)$$

Theorem 2: The formula of eigenfunctions ψ_n (x) as $n\to\infty$ is:

$$\psi(x) = \mp\frac{1}{3}\left(\frac{36864}{3071(b\text{-}a)}\right)^2\begin{bmatrix} \sinh\lambda(x\text{-}a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)+\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right)\sinh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+0\left(\frac{1}{\lambda^2}\right)$$

where $a_{12} = 0$ and $a_{11}\neq 0$.

Proof: If ψ (x) = $c_0y_0+c_1y_1+c_2y_2+c_3y_3+c_4y_4+c_5y_5$ then from first boundary condition we get $a_{11}c_0+a_{12}c_1+a_{13}c_2+a_{14}c_3+a_{15}c_4+a_{16}c_5 = 0$, since, c_i cannot be zero for all i then. We can choose $c_0 = k_1a_{12}$ and $c_1 = -k_1a_{11}$ and $c_2 = k_2a_{14}$ and $c_3 = -k_2a_{13}$ and $c_4 = k_3a_{16}$ and $c_5 = -k_3a_{15}$ where $k_i\neq 0$ then from theorem 1 we obtain that:

$$\psi(x) = \frac{1}{3}k_1a_{12}\begin{bmatrix} \cosh\lambda(x\text{-}a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)+\cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)\text{-}$$
$$\frac{1}{3\lambda}k_1a_{11}\begin{bmatrix} \sinh\lambda(x\text{-}a)+\frac{1}{2}\left(1+\frac{1}{32}\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)+\frac{1}{2}\left(\text{-}1+\frac{1}{32}\right)\sinh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+$$
$$0\left(\frac{1}{\lambda^6}e^{|t|(x\text{-}a)}\right)+\frac{1}{3\lambda^2}k_1a_{14}\begin{bmatrix} \cosh\lambda(x\text{-}a)+\frac{1}{2}\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)+\frac{1}{2}\cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+$$
$$0\left(\frac{1}{\lambda^7}e^{|t|(x\text{-}a)}\right)\text{-}k_1a_{13}\frac{1}{3\lambda^3}\begin{bmatrix} \sinh\lambda(x\text{-}a)\text{-}\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)+\sinh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+$$
$$0\left(\frac{1}{\lambda^8}e^{|t|(x\text{-}a)}\right)\text{-}k_1a_{16}\frac{1}{3\lambda^4}\begin{bmatrix} \cosh\lambda(x\text{-}a)+\frac{1}{2}\left(1\text{-}\frac{1}{32}i\right)\cosh\lambda \\ \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)+\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right) \\ \cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+$$
$$0\left(\frac{1}{\lambda^9}e^{|t|(x\text{-}a)}\right)\text{-}k_1a_{15}\frac{1}{3\lambda^5}\begin{bmatrix} \sinh\lambda(x\text{-}a)\text{-}\frac{1}{2}\left(\text{-}1+\frac{1}{32}i\right) \\ \sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\text{-}\frac{1}{2}\left(1+\frac{1}{32}i\right) \\ \cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+$$
$$0\left(\frac{1}{\lambda^{10}}e^{|t|(x\text{-}a)}\right)$$

$$\psi(x) = \frac{1}{3}k_1a_{12}\begin{bmatrix} \cosh\lambda(x\text{-}a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x\text{-}a)+\cosh\lambda\left(\text{-}\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x\text{-}a)\end{bmatrix}+0\left(\frac{1}{\lambda^5}\right)$$

$$-0\left(\frac{1}{\lambda}\right)+0\left(\frac{1}{\lambda^6}\right)+0\left(\frac{1}{\lambda^2}\right)+0\left(\frac{1}{\lambda^7}\right)-0\left(\frac{1}{\lambda^3}\right)$$
$$+0\left(\frac{1}{\lambda^8}\right)+0\left(\frac{1}{\lambda^4}\right)+0\left(\frac{1}{\lambda^9}\right)-0\left(\frac{1}{\lambda^5}\right)+0\left(\frac{1}{\lambda^{10}}\right)$$

$$\psi(x)=\frac{1}{3}k_1a_{12}\begin{bmatrix}\cosh\lambda(x-a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\(x-a)+\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+0\left(\frac{1}{\lambda}\right)$$

If $a_{12}=0$ then $a_{11}\neq0$:

$$\psi(x)=-\frac{1}{3\lambda}k_1a_{11}\begin{bmatrix}\sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\\\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\\\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+0\left(\frac{1}{\lambda^2}\right)$$

If $a_{12}=0$, $a_{11}=0$ and $a_{14}\neq0$:

$$\psi(x)=-\frac{1}{3\lambda^2}k_2a_{14}\begin{bmatrix}\cosh\lambda(x-a)+\frac{1}{2}\cosh\lambda\\\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\\\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+0\left(\frac{1}{\lambda^7}e^{|t|(x-a)}\right)$$

If $a_{12}=0$, $a_{11}=0$, $a_{14}=0$ and $a_{13}\neq0$:

$$\psi(x)=-\frac{1}{3\lambda^3}k_2a_{13}\begin{bmatrix}\sinh\lambda(x-a)-\sinh\lambda\\\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\sinh\lambda\\\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+0\left(\frac{1}{\lambda^8}e^{|t|(x-a)}\right)$$

If $a_{12}=0$, $a_{11}=0$, $a_{14}=0$, $a_{13}=0$ and $a_{15}\neq0$:

$$\psi(x)=-\frac{1}{3\lambda^4}k_3a_{16}$$
$$\left[\cosh\lambda(x-a)+\frac{1}{2}\left(1-\frac{1}{32}i\right)\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]+$$
$$0\left(\frac{1}{\lambda^9}e^{|t|(x-a)}\right)$$

If $a_{12}=0$, $a_{11}=0$, $a_{14}=0$, $a_{13}=0$, $a_{16}=0$ and $a_{15}\neq0$:

$$\psi(x)=-\frac{1}{3\lambda^5}k_3a_{15}$$
$$\left[\sinh\lambda(x-a)-\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]+$$
$$0\left(\frac{1}{\lambda^{10}}e^{|t|(x-a)}\right)$$

If $a_{12}=0$ and $a_{11}\neq0$:

$$\psi(x)=-\frac{1}{3\lambda}k_1a_{11}$$
$$\left[\sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]-0\left(\frac{1}{\lambda^2}\right)$$

Then :

$$k_1^{-1}\psi(x)=-\frac{1}{3\lambda}a_{11}$$
$$\left[\sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]+0\left(\frac{1}{\lambda^2}\right)$$

$$k_1^{-2} = k_1^{-2}\int_a^b \psi^2(x)\,dx = \frac{1}{9}k_1^{-2}a_{11}^2\int_a^b \frac{1}{\lambda^2}\begin{bmatrix} \sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}^2 dx+0\left(\frac{1}{\lambda^2}\right)=$$

$$\frac{1}{9}k_1^{-2}a_{11}^2\Bigg(\frac{3071}{4096}\Bigg(\frac{\sin(\lambda(a-b)i)1025i}{3071\lambda^3}+\frac{\sin(\lambda(a-b)2\,i)1024i}{3071\lambda^3}+\frac{(a-b)}{\lambda^2}+\frac{1025\,3^{\frac{1}{2}}\sin\left(3^{\frac{1}{2}}\lambda(a-b)\right)}{9213\lambda^3}+$$

$$\frac{1}{\lambda^3}\sin\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\cos\left(\frac{\lambda(a-b)i}{2}\right)\left(\frac{2048\,3^{\frac{1}{2}}}{3071}+\frac{64}{3071}\right)+\frac{1}{\lambda^3}\cos\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)$$

$$\sin\left(\frac{\lambda(a-b)i}{2}\right)\left(\frac{3^{\frac{1}{2}}64i}{3071}-\frac{2048i}{3071}\right)-\frac{1}{\lambda^3}\sin\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\cos\left(\frac{\lambda(a-b)3i}{2}\right)$$

$$\left(\frac{2048\,3^{\frac{1}{2}}}{9213}-\frac{64}{3071}\right)+\frac{1}{\lambda^3}\cos\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\sin\left(\frac{\lambda(a-b)3i}{2}\right)\left(\frac{3^{\frac{1}{2}}64i}{9213}+\frac{2048i}{3071}\right)-$$

$$\frac{1}{\lambda^3}\sin\left(3^{\frac{1}{2}}\lambda(a-b)\right)\cos(\lambda(a-b)i)\left(\frac{1023\,3^{\frac{1}{2}}}{12284}-\frac{64}{3071}\right)+\frac{1}{\lambda^3}\cos\left(3^{\frac{1}{2}}\lambda(a-b)\right)$$

$$\sin(\lambda(a-b)i)\left(\frac{3^{\frac{1}{2}}16i}{3071}+\frac{1023i}{12284}\right)\Bigg)\Bigg)+0\left(\frac{1}{\lambda^2}\right)=$$

$$=\frac{1}{9}k_1^{-2}a_{11}^2\left(\frac{3071}{4096}\begin{pmatrix}0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^3}\right)+\frac{(a-b)}{\lambda^2}+0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^3}\right)- \\ 0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^3}\right)-0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^3}\right)\end{pmatrix}\right)+0\left(\frac{1}{\lambda^2}\right)=$$

$$\frac{1}{9}k_1^{-2}a_{11}^2\frac{3071}{4096}\frac{(a-b)}{\lambda^2}+0\left(\frac{1}{\lambda^3}\right)+0\left(\frac{1}{\lambda^2}\right)$$

$$k_1^{-2}=\frac{1}{9}k_1^{-2}a_{11}^2\frac{3071}{4096}\frac{(a-b)}{\lambda^2}+0\left(\frac{1}{\lambda^2}\right)$$

Then:

$$a_{11}=\pm\lambda\left(\frac{36864}{3071(b-a)}\right)^2$$

$$\psi(x)=\mp\frac{1}{3\lambda}k_1\lambda\left(\frac{36864}{3071(b-a)}\right)^2\begin{bmatrix} \sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+ 0\left(\frac{1}{\lambda^2}\right)$$

$$\psi(x)=\mp\frac{1}{3}\left(\frac{36864}{3071(b-a)}\right)^2\begin{bmatrix} \sinh\lambda(x-a)+\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x-a)+\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+0\left(\frac{1}{\lambda^2}\right)$$

$$\psi(x)=-\frac{1}{3\lambda^5}k_3a_{15}\begin{bmatrix} \sinh\lambda(x-a)-\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ (x-a)-\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{bmatrix}+0\left(\frac{1}{\lambda^{10}}\right)$$

Then:

$$k_3^{-1}\psi(x) = -\frac{1}{3\lambda^5}a_{15}\left[\cosh\lambda(x-a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]+0\left(\frac{1}{\lambda^{10}}\right)$$

$$k_1^{-2} = k_1^{-2}\int_a^b \psi^2(x)dx = \frac{1}{9}a_{15}^2\int_a^b \frac{1}{\lambda^{10}}\left[\cosh\lambda(x-a)+\cosh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)+\cosh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\right]^2 dx+0\left(\frac{1}{\lambda^{10}}\right) = -\frac{1}{9}a_{15}^2\frac{1}{\lambda^{11}}$$

$$\left(\begin{aligned}
&-\frac{\sin(\lambda(a-b)i)1025i}{4096}-\frac{\sin(\lambda(a-b)2i)i}{4}-\frac{\cos\left(3^{\frac{1}{2}}\lambda(a-b)\right)\sin(\lambda(a-b)i)1023\,i}{16384}+\frac{\sin\left(3^{\frac{1}{2}}\lambda(a-b)\right)\cos(\lambda(a-b)i)}{256}+\\
&\frac{\cos\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\sin\left(\frac{\lambda(a-b)i}{2}\right)i}{2}+\frac{\sin\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\cos\left(\frac{\lambda(a-b)i}{2}\right)}{64}-\frac{\cos\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\sin\left(\frac{\lambda(a-b)3i}{2}\right)i}{2}+\\
&\frac{\frac{\sin\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\cos\left(\frac{\lambda(a-b)3\,i}{2}\right)i}{2}}{64}\lambda\left(\frac{3071\,a}{4096}-\frac{3071\,b}{4096}\right)-\frac{1025\,3^{\frac{1}{2}}\sin\left(3^{\frac{1}{2}}\lambda(a-b)\right)}{12288}+\frac{3^{\frac{1}{2}}\cos\left(3^{\frac{1}{2}}\lambda(a-b)\right)\sin(\lambda(a-b)i)i}{256}+\\
&\frac{1023\,3^{\frac{1}{2}}\sin\left(3^{\frac{1}{2}}\lambda(a-b)\right)\cos(\lambda(a-b)i)}{16384}+\frac{3^{\frac{1}{2}}\cos\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\sin\left(\frac{\lambda(a-b)i}{2}\right)i}{64}-\frac{3^{\frac{1}{2}}\sin\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\cos\left(\frac{\lambda(a-b)i}{2}\right)}{2}+\\
&\frac{3^{\frac{1}{2}}\cos\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\sin\left(\frac{\lambda(a-b)3i}{2}\right)i}{192}+\frac{3^{\frac{1}{2}}\sin\left(\frac{3^{\frac{1}{2}}\lambda(a-b)}{2}\right)\cos\left(\frac{\lambda(a-b)3i}{2}\right)}{6}
\end{aligned}\right)+0\left(\frac{1}{\lambda^{10}}\right)$$

Then:

$$k_1^{-2} = \frac{1}{9}\frac{3071}{4096}a_{15}^2\frac{1}{\lambda^{10}}(a-b)+0\left(\frac{1}{\lambda^{10}}\right)$$

If $k_3 = 1$ then $a_{15} = \pm\lambda^5\left(\frac{36864}{3071(b-a)}\right)^2$

$$\psi(x) = \mp\frac{1}{3}\left(\frac{36864}{3071(b-a)}\right)^2$$

$$\left[\begin{aligned}&\sinh\lambda(x-a)-\frac{1}{2}\left(-1+\frac{1}{32}i\right)\sinh\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\&(x-a)-\frac{1}{2}\left(1+\frac{1}{32}i\right)\sinh\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x-a)\end{aligned}\right]+$$

$$0\left(\frac{1}{\lambda^{10}}\right)$$

Theorem 3: Asymptotic formula for eigen values λ_n as $n\to\infty$ is:

$$\lambda = \frac{i\theta+2n\pi i}{b-a}$$

Proof: Since:

$$\Delta(\lambda) = \begin{vmatrix}
U_0(y_0) & U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) & U_0(y_5)\\
U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(y_5)\\
U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(y_5)\\
U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5)\\
U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5)\\
U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5)
\end{vmatrix}$$

Then, we can see that:

$$\Delta(\lambda) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5) \\ U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5) \\ U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5) \end{vmatrix}$$

If we suppose that $X = \begin{vmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{33} & a_{35} & a_{36} \end{vmatrix} \neq 0$ and since, the boundary conditions are linearly independent then:

$$\Delta(\lambda) = X\begin{vmatrix} U_3(y_0) & U_3(y_1) & U_3(y_3) \\ U_4(y_0) & U_4(y_1) & U_4(y_3) \\ U_5(y_0) & U_5(y_1) & U_5(y_3) \end{vmatrix}$$

Then we calculate $U_j\ (y_i)$ for j = 3, 4, 5 and i = 0, ..., 5 by using corollary 2-7:

$$U_3(y_0) = \frac{1}{3}\lambda^5\left[3\cosh(\lambda(a\text{-}b))\text{-}3\sinh(\lambda(a\text{-}b))\right]+0\left(e^{|t|(x\text{-}a)}\right)$$

$$U_3(y_1) = \frac{1}{3}\lambda^4\left[3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+0\left(\frac{1}{\lambda}e^{|t|(x\text{-}a)}\right)$$

$$U_3(y_2) = \frac{1}{3}\lambda^3$$

$$\left[\begin{array}{l} 3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))+\sin\left(\frac{\frac{1}{32}\lambda(a\text{-}x)}{2}\right) \\ \cosh\left(\frac{\lambda(a\text{-}x)}{2}\right)i+\frac{1}{4}\cosh\left(\frac{\lambda\left(\frac{1}{32\,i}\text{-}1\right)(a\text{-}x)}{2}\right)\left(\frac{1}{32\,i}\text{-}1\right)+ \\ \frac{1}{4}\cosh\left(\frac{\lambda\left(\frac{1}{32\,i}+1\right)(a\text{-}x)}{2}\right)\left(\frac{1}{32\,i}\text{-}1\right) \end{array}\right]+$$

$$0\left(\frac{1}{\lambda^2}e^{|t|(x\text{-}a)}\right)$$

$$U_3(y_3) = \frac{1}{3}\lambda^2\left[3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+0\left(\frac{1}{\lambda^3}e^{|t|(x\text{-}a)}\right)$$

$$U_3(y_4) = \frac{1}{3}\lambda\left[3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+0\left(\frac{1}{\lambda^4}e^{|t|(x\text{-}a)}\right)$$

$$U_3(y_5) = \frac{1}{3}\left[3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)$$

$$U_4(y_0) = \text{-}\frac{1}{3}\lambda^5\left[3\cosh(\lambda(a\text{-}b))+3\sinh(\lambda(a\text{-}b))\right]+0\left(e^{|t|(x\text{-}a)}\right)$$

$$U_4(y_1) = \text{-}\frac{1}{3}\lambda^4\left[\text{-}3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+$$
$$0\left(\frac{1}{\lambda}e^{|t|(x\text{-}a)}\right)$$

$$U_4(y_2) = \text{-}\frac{1}{3}\lambda^3$$

$$\left[\begin{array}{l} 3\cosh(\lambda(a\text{-}x))+3\sinh(\lambda(a\text{-}x))\text{-}\sin\left(\frac{\frac{1}{32}\lambda(a\text{-}x)}{2}\right) \\ \cosh\left(\frac{\lambda(a\text{-}x)}{2}\right)i+\frac{1}{4}\cosh\left(\frac{\lambda\left(\frac{1}{32i}\text{-}1\right)(a\text{-}x)}{2}\right)\left(\frac{1}{32i}\text{-}1\right)+ \\ \frac{1}{4}\cosh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a\text{-}x)}{2}\right)\left(\frac{1}{32i}\text{-}1\right) \end{array}\right]+$$

$$0\left(\frac{1}{\lambda^2}e^{|t|(x\text{-}a)}\right)$$

$$U_4(y_3) = \text{-}\frac{1}{3}\lambda^2\left[\text{-}3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+$$
$$0\left(\frac{1}{\lambda^3}e^{|t|(x\text{-}a)}\right)$$

$$U_4(y4) = \text{-}\frac{1}{3}\left[\lambda 3\cosh(\lambda(a\text{-}x))+3\sinh(\lambda(a\text{-}x))\right]+0\left(\frac{1}{\lambda^4}e^{|t|(x\text{-}a)}\right)$$

$$U_4(y_5) = \text{-}\frac{1}{3}\lambda\left[-3\cosh(\lambda(a\text{-}x))\text{-}3\sinh(\lambda(a\text{-}x))\right]+0\left(\frac{1}{\lambda^5}e^{|t|(x\text{-}a)}\right)$$

$$U_5(y_0) = \frac{1}{3}\lambda^5$$

$$\left[\begin{array}{l} 3\cosh\frac{\left(\lambda\left(\sqrt{3}i\text{-}1\right)(a\text{-}b)\right)}{2}+3\cosh\frac{\left(\lambda\left(\sqrt{3}i+1\right)(a\text{-}b)\right)}{2}\text{-} \\ 3\sinh\frac{\left(\lambda\left(\sqrt{3}i\text{-}1\right)(a\text{-}b)\right)}{2}\text{-}3\sinh\frac{\left(\lambda\left(\sqrt{3}i+1\right)(a\text{-}b)\right)}{2} \end{array}\right]+$$

$$0\left(e^{|t|(x\text{-}a)}\right)$$

$$U_5(y_1)=\frac{1}{3}\lambda^4\left[\begin{array}{l}\cosh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\left(-\frac{3}{2}+\frac{3i}{64}\right)+\frac{1}{4}\cosh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(\frac{3}{2}+\frac{3i}{64}\right)+\sinh\\ \left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\left(\frac{3}{2}-\frac{3i}{64}\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(-\frac{3}{2}-\frac{3i}{64}\right)+\end{array}\right]+$$
$$0\left(\frac{1}{\lambda}e^{|t|(x-a)}\right)$$

$$U_5(y_2)=\frac{1}{3}\lambda^3\frac{1}{4}$$
$$\left[\begin{array}{l}-\cosh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\left(\frac{1}{32i}-5\right)-\cosh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(\frac{1}{32i}-5\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\\ \left(\frac{1}{32i}-5\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(\frac{1}{32i}-5\right)\end{array}\right]+0\left(\frac{1}{\lambda^2}e^{|t|(x-a)}\right)$$

$$U_5(y_3)=\frac{1}{3}\lambda^2$$
$$\left[3\cosh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)-3\cosh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)-3\sinh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\right]+$$
$$0\left(\frac{1}{\lambda^2}e^{|t|(x-a)}\right)$$

$$U_5(y_4)=\frac{1}{3}\lambda$$
$$\left[\begin{array}{l}\cosh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\left(-\frac{3}{2}+\frac{3i}{64}\right)+\cosh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(\frac{3}{2}+\frac{3i}{64}\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\\ \left(\frac{3}{2}-\frac{3i}{64}\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(-\frac{3}{2}+\frac{3i}{64}\right)\end{array}\right]+0\left(\frac{1}{\lambda^4}e^{|t|(x-a)}\right)$$

$$U_5(y_5)=\frac{1}{3}$$
$$\left[\begin{array}{l}\cosh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\left(-\frac{3}{2}-\frac{3i}{64}\right)+\cosh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(\frac{3}{2}-\frac{3i}{64}\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}-1\right)(a-x)}{2}\right)\\ \left(\frac{3}{2}+\frac{3i}{64}\right)+\sinh\left(\frac{\lambda\left(\frac{1}{32i}+1\right)(a-x)}{2}\right)\left(-\frac{3}{2}+\frac{3i}{64}\right)\end{array}\right]+0\left(\frac{1}{\lambda^5}e^{|t|(x-a)}\right)$$

$$\Delta(\lambda) = X$$
$$\left(\frac{\lambda^{11}}{27}\left(\begin{matrix}\cosh\frac{\left(\lambda\left(\sqrt{3}i-1\right)(a-x)\right)}{2}(96-i)+\cosh\frac{\left(\lambda\left(\sqrt{3}i+1\right)(a-x)\right)}{2}(-96-i)+\sinh\frac{\left(\lambda\left(\sqrt{3}i-1\right)(a-x)\right)}{2}(-96+i)+\\ \sinh\frac{\left(\lambda\left(\sqrt{3}i+1\right)(a-x)\right)}{2}(96+i)\begin{pmatrix}\cosh\lambda(a-x)\\+\sinh\lambda(a-x)\end{pmatrix}\left(\cosh\lambda(a-x)-\sinh\lambda(a-x)\right)\end{matrix}\right)\right)$$
$$\frac{1}{32}+0\left(\lambda^{6}e^{|t|(x-a)}\right)+0\left(\lambda e^{|t|(x-a)}\right)+0\left(\frac{1}{\lambda^{2}}e^{|t|(x-a)}\right)$$

$$\Delta(\lambda) = \frac{27}{32}X$$
$$\left(\lambda^{11}\left(\cosh\frac{\left(\lambda\left(\sqrt{3}i-1\right)(a-x)\right)}{2}(96-i)+\cosh\frac{\left(\lambda\left(\sqrt{3}i+1\right)(a-x)\right)}{2}(-96-i)+\sinh\frac{\left(\lambda\left(\sqrt{3}i-1\right)(a-x)\right)}{2}(-96+i)+\sinh\frac{\left(\lambda\left(\sqrt{3}i+1\right)(a-x)\right)}{2}(96+i)\right)\right)+$$
$$0\left(\lambda^{6}e^{|t|(x-a)}\right)+0\left(\lambda e^{|t|(x-a)}\right)+0\left(\frac{1}{\lambda^{2}}e^{|t|(x-a)}\right)$$

$$\Delta(\lambda) = \frac{27}{32}Xe^{\left(-\frac{\sqrt{3}\lambda(a-b)i}{2}\right)}e^{\left(-\frac{\lambda(a-b)}{2}\right)}\lambda^{11}$$
$$\left[\left(\left(e^{(\lambda(a-b))}(96-i)-96-i\right)\right)+0\left(\frac{1}{\lambda^{5}}e^{|t|(x-a)}\right)+0\left(\frac{1}{\lambda^{10}}e^{|t|(x-a)}\right)+0\left(\frac{1}{\lambda^{12}}e^{|t|(x-a)}\right)\right]$$

$$\Delta(\lambda) = \frac{27}{32}Xe^{\left(-\frac{\sqrt{3}\lambda(a-b)i}{2}\right)}e^{\left(-\frac{\lambda(a-b)}{2}\right)}\lambda^{11}\left[\left(\left(e^{(\lambda(a-b))}(96-i)-96-i\right)\right)+0\left(\frac{1}{\lambda^{5}}e^{3|t|(b-a)}\right)\right]$$

If $\Delta(\lambda) = 0$ then:

$$\left(e^{(\lambda(a-b))}(96-i)-(96+i)+0\left(\frac{1}{\lambda^{5}}e^{3|t|(b-a)}\right)\right) = 0$$

Then $e^{(\lambda(a-b))}(96-i)-(96+i) = 0\left(\frac{1}{\lambda^{5}}e^{3|t|(b-a)}\right)$ for large value of $|\lambda|$ we get:

$$e^{(\lambda(a-b))} = \frac{96+i}{96-i}$$

If $\theta = \tan^{-1}(96+i/96-i)$ then:

$$\lambda = \frac{1}{b-a}\left\{\ln\left|\frac{96+i}{96-i}\right|+i\theta+2n\pi i\right\},\ \text{for } n = 0, \pm 1, \pm 2, ...$$

$$\lambda = \frac{1}{b-a}\left\{\ln 1+i\theta+2n\pi i\right\},\ \text{for } n = 0, \pm 1, \pm 2, ...$$

$$\lambda = \frac{i\theta+2n\pi i}{b-a},\ \text{for } n = 0, \pm 1, \pm 2, ...$$

CONCLUSION

In mathematics, spectral theory is an inclusive term for theories that extend a single square matrix's eigenvector and eigenvalue theory to a much broader theory of operator structure in a variety of mathematical spaces.

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