Journal of Engineering and Applied Sciences 15 (3): 815-818, 2020
ISSN: 1816-949X

An Efficient Approach of the Fractional Lagrange Interpolation

Mohammad H. Al-Towaiq and Yousef S. Abu Hour
Department of Mathematics and Statistics,
Jordan University of Science and Technology, Irbid, Jordan

Abstract: In this study, we propose an efficient approach of the fractional Lagrange interpolation based on new definition of the fractional derivative. We analyze the error and the properties of the new approach based on the new definition. Numerical applications are given to illustrate the applicability and the efficiency of the proposed approach.

Key words: Fractional interpolation, approximation, lagrange interpolation, fractional differential equation, error, numerical applications

INTRODUCTION

Recently, the fractional calculus has found in many real applications in science and engineering such as the control theory, fluid mechanics, bioengineering and biophysics (Dalir and Bashour, 2010; Grace, 2015; Jia *et al*., 2016; Maleki and Kajani, 2015; Soczkiewicz, 2002). Most of the fractional problems do not have analytical (exact) solutions, so, approximation and numerical techniques have been used (Grace, 2015; Garrappa, 2015; Jia *et al*., 2016; Khder, 2015). For example; Jia *et al*. (2016) introduced numerical scheme to solve the nonlinear differential equation of fractional order. The numerical examples they gave demonstrated the accuracy and the efficiency of their technique. Khder (2015) presented an approximate method for solving a certain class of fractional variational problems. He used the properties of Rayleigh-Rits method and the chain rule for fractional calculus to reduce the fractional variational problems to solve a system of algebraic equations. Maleki and Kajani (2015) presented a multi-domain Legendre-Gauss psedudospectral to find approximate solutions for the fractional Volterra's Model for the population growth of species in closed system.

Different definitions of the fractional derivatives and fractional integrals are presented (Dalir and Bashour, 2010; Khalil *et al*., 2014) . Khalil *et al*. (2014) introduced a new definition of the fractional derivative (conformable fractional derivative):

$$f^{(\alpha)}(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \tag{1}$$

This definition satisfies Rolle's theorem of fractional calculus which we will use to analysis the error of the new approach.

The problem of interpolate distinct points (x_0, y_0) to (x_n, y_n) with a polynomial of degree at most n, say $P_n(x)$ that passes through of them is the same as approximating a function f for which:

$$f(x_i) = y_i,\ i = 0, ..., n$$

This called polynomial interpolation (Berrut and Trefethen, 2004; Dvornikov, 2008; Elsaid, 2010; Hamasalh and Muhammad, 2015; Zahra and Elkholy, 2012). Hamasalh and Muhammad (2015) presented a study of three interpolator fractional splines. They extended the fractional splines function with equally spaced knots to approximate the solution of the fractional equation. They discussed and analyzed the convergence of the method and they estimated the error bound. The most popular technique is Lagrange's interpolation, Berrut and Trefethen (2004) used to compute $P_n(x)$ which defined by: Let

$$P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x) \tag{2}$$

Where:

$$L_i(x) = \prod_{j=0,\, j\neq i}^{n}\frac{x-x_i}{x_i-x_j} \tag{3}$$

To analysis the error of the α-root interpolation we will use the above fractional derivative definition 1.

MATERIALS AND METHODS

Theorem: (Rolle's Theorem for Conformable Fractional Differentiable Functions) (Khalil *et al*., 2014). Let a>0 and f: [a, b]→R be a given function that satisfies:

Corresponding Author: Mohammad H. Al-Towaiq, Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan

- f is continuous on [a, b]
- f is α-differentiable for some $\alpha \in (0,1)$
- f(a) = f(b)

Then, there exists $c \in (a, b)$ such that $f^{(\alpha)}(c) = 0$.

The proposed method: In this section we will introduce α Lagrange's polynomial and using it to define α root interpolation and its properties. In addition, error analysis will be discussed in this section.

Definition: We say $L_{i,\alpha}(x)$ is Lagrange's α-polynomial if:

$$L_{i,\alpha}(x) = \prod_{j=0,\, j\neq i}^{n} \frac{(x\text{-}x_i)^{\alpha}}{(x_i\text{-}x_j)^{\alpha}}$$

Choosing α is very important issue here to avoid complex or undefined values, so that, the best choice for α is α = (1/k) where k is odd positive integer. So, the root function to interpolate distinct points (x, y) to (x_n, y_n) is given by:

$$P_{n,\alpha}(x) = \sum_{i=0}^{n} y_i L_{i,\alpha}(x)$$

Notice that in case α = 1 then $P_{n,1}(x) = P_n(x)$ is the classical Lagrange interpolation. Moreover, $L_{i,\alpha}(x)$ satisfies the cardinal function.

$$L_{i,\alpha}(x_j) = \delta_{i,j} = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$$

Error analysis: Before we analyze the error of this interpolation technique, we introduce the following notation.

Notation: The n-times sequential fractional derivative of f is given by:

$$f_n^{(\alpha)}(t) = f^{(\alpha)}\left(f^{(\alpha)}\left(\ldots\left(f^{(\alpha)}(t)\right)\right)\right)$$

Theorem: Suppose $P_{n,\alpha}(x)$ interpolates f(x) at the n+1 distinct nodes x_0, x_1, …, x_n on [a, b] and let f(x) continuously α-differentiable on [a, b], then:

$$f(x) - P_{n,\alpha}(x) = \frac{1}{(n+1)!\alpha^{n+1}} f_n^{(\alpha)}(\tau) \prod_{j=0}^{n} (x - x_j)^{\alpha}$$

for some $\tau \in (a, b)$.

Proof: Let:

$$h^{(\alpha)}(t) = \prod_{j=0}^{n} (t - x_j)^{\alpha}$$

$$c = \frac{f(x) - P_{n,\alpha}(x)}{h^{(\alpha)}(x)}$$

and:

$$\varnothing^{(\alpha)}(t) = f(t)\text{-}P_{n,\alpha}(t)\text{-}ch(t)$$

Observe that c is a constant and well defined because $h^{(\alpha)}(t) \neq 0$. $\varnothing^{(\alpha)}(t)$ has n+2 roots which satisfies Roll's theorem, then $\varnothing^{(\alpha)}(t)$ has at least n+1 roots. That is $\varnothing^{(\alpha)}(t) = f^{(\alpha)}(t)\text{-}P_{n,\alpha}{}^{(\alpha)}(t)\text{-}ch^{(\alpha)}(t)$ satisfies Roll's theorem and $\varnothing_2{}^{(\alpha)}(t)$ has n roots and so on. Finally, $\varnothing_{n+1}{}^{(\alpha)}(t)$ must have at least one root, say $\tau \in (a,b)$. Thus:

$$\varnothing_{n+1}^{(\alpha)}(\tau) = f_{n+1}^{(\alpha)}(\tau)\text{-}\left(P_{n,\alpha}\right)_{n+1}^{(\alpha)}(\tau)\text{-}ch_{n+1}^{(\alpha)}(\tau) = 0$$

$$\left(P_{n,\alpha}\right)_{n+1}^{(\alpha)}(\tau) = 0$$

and:

$$h_{n+1}^{(\alpha)}(\tau) = (n+1)!$$

Then:

$$f_{n+1}^{(\alpha)}(\tau) = c(n+1)!\,\alpha^{n+1} = \frac{f(x) - P_{n,\alpha}(x)}{h^{(\alpha)}(x)}(n+1)!\,\alpha^{n+1}$$

which implies that:

$$f(x)\text{-}P_{n,\alpha}(x) = \frac{1}{(n+1)!\alpha^{n+1}} f_n^{(\alpha)}(\tau) \prod_{j=0}^{n} (x\text{-}x_j)^{\alpha}$$

Therefore, the upper bound error of α-root interpolation when $|f^{(\alpha)}(x)<M$ and α = (1/k) for odd integer k is:

$$\frac{k^{n+1}}{(n+1)!} M \max \prod_{J=0}^{n} |\, x\text{-}x_j \,|^{\alpha}$$

So, this upper bound goes to zero as $n \to \infty$ because $\frac{k^{n+1}}{(n+1)!} \to 0$ for large n.

RESULTS AND DISCUSION

Numerical demonstration: In this section, we will use α-root interpolation to solve some numerical applications such as the Fredholm linear integral equation, ordinary differential equation and fractional differential equation and use α-root interpolation to interpolate the first quadrant of the unit circle and compare it using different values for α.

Example 1: Consider the following Fredholm linear integral equation using α = 1/3 and α = 1,

$$\frac{3}{2}\left(s-\frac{1}{2}\right)=\int_0^1 f(t)(s-t)\,dt \tag{4}$$

Solution: $L_0 = (1-s)^\alpha$, $L_1 = s^\alpha$ then $P_{1,\alpha} = f(0)L_0+f(1)L_1$. Using Trapezoidal rule with 2 nodes to approximate the integral term of 4. Let $s_i = t_i = 0.1*i$, i = 1, 2, we obtain the following corresponding linear equations:

$$\frac{3s_i}{2}=\frac{3}{4}+\frac{1}{2}\left[f(0)s_i+f(1)(s_i-1)\right] \tag{5}$$

Solving (5), we get $f(0) = f(1) = \frac{3}{2}$, then $P_{1,1}=\frac{3}{2}$ is the classical Lagrange polynomial and

$$P_{1,\frac{1}{3}}=\frac{3}{2}\left(\sqrt[3]{(1-s)}+\sqrt[3]{s}\right)$$

Figure 1 shows the exact and the approximate solutions of example 1 with α = 1 and α = 1/3. From Fig. 1, we conclude that when we use (1/3)-root interpolation is more accurate than the classical Lagrange interpolation. One can easily see that $\|P_{1,1}-f(s)\|_\infty = 2.5$ but:

$$\left\|P_{1,\frac{1}{3}}-f(s)\right\|_\infty = 0.5, \forall s\in[0,1]$$

Example 2: Solve the following system:

$$f_1^1 f_1^4 = \frac{2x-1.5}{5},\quad f_1(0)=1$$
$$f_2^1 f_2^4 = \frac{1-2x}{5}\quad f_2(0)=0$$
$$f_3^1 f_3^4 = \frac{2x-0.5}{5}\quad f_3(0)=0$$

To find $G(s) = f_1(s)+f_2(s)+f_3(s)$; where G(0) = 1,G(0.5) = 1,G(1) = 1. We interpolate G for different values of α. We found that:

$$\left\|P_{2,\frac{1}{3}}-G(s)\right\|_\infty \cong 0.03$$

$$\left\|P_{2,\frac{1}{7}}-G(s)\right\|_\infty \cong 0.026, \left\|P_{2,\frac{3}{5}}-G(s)\right\|_\infty \cong 0.057, \left\|P_{2,\frac{7}{9}}-G(s)\right\|_\infty \cong 0.091$$

and $\|P_{2,1}-G(s)\|_\infty \cong 0.15$. From this and Fig. 2 we conclude that for smaller values of α gives better accuracy than the classical Lagrange interpolation.

Example 3: Interpolate $f(x)=\sqrt{1-x^2}, x\in[0,1]$. When α = 1, we obtain the classical Lagrange polynomial $P_{1,1}(x) = 1-x$. For α =1/3 we obtain:

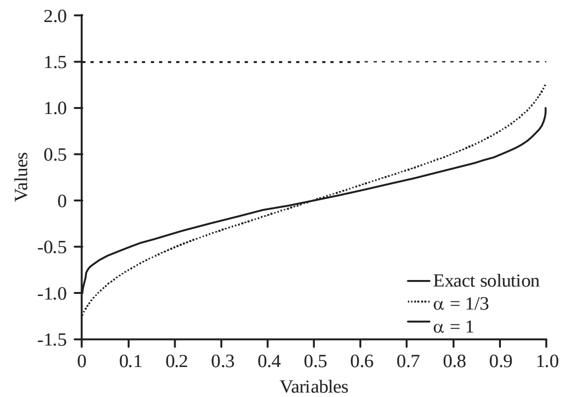

Fig. 1: Exact and approximate solutions of example 1 with α = 1 and α = 1/3

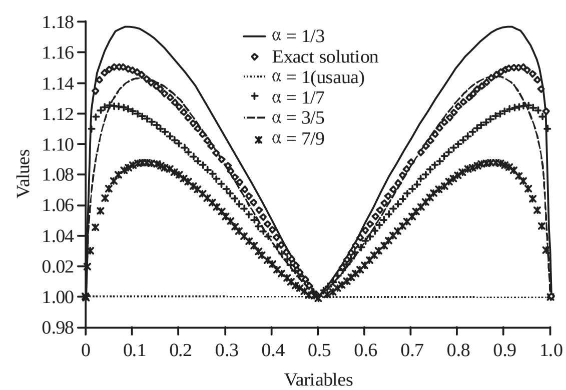

Fig. 2: Exact and approximate solutions of example 2 for different values of α

$$P_{1,\frac{1}{3}}(x)=\sqrt[3]{1-x}$$

Figure 3 shows that when α = 1/3 it gives more accurate solution than the classical one.

Example 4: Consider the following nonhomogeneous fractional differential equation with homogeneous initial condition:

$$D^{\frac{1}{2}}y(x)=\frac{1}{6}, D^{-\frac{1}{2}}, y(0)=0$$

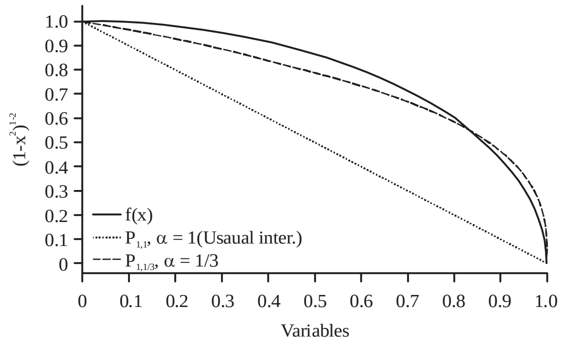

Fig. 3: Exact and approximate solutions of example 3 for $\alpha = 1$ and $\alpha = 1/3$

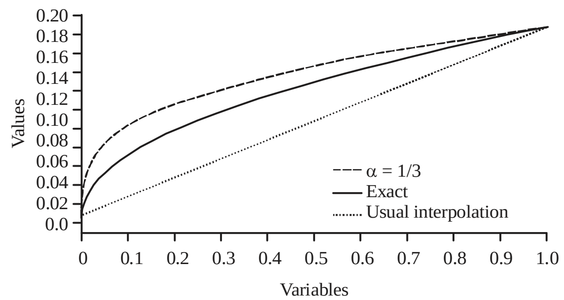

Fig. 4: Exact and approximate solutions of example 4, for α=1 and α= 1/3

where, D^{σ} is Riemann-Liouville α-derivative. We interpolate the approximate solution for y (0) = 0 and:

$$y(1) = \frac{1}{6\Gamma(\frac{3}{2})}$$

When $\alpha = 1$, we obtain:

$$P_{1,1}(x) = \frac{1}{6\Gamma(\frac{3}{2})}x$$

and when $\alpha = 1/3$, we obtain:

$$P_{1,\frac{1}{3}}(x) = \frac{1}{6\Gamma(\frac{3}{2})}\sqrt[3]{x}$$

We plot the curves of $P_{1,1}(x)$, $P_{1,\frac{1}{3}}(x)$ and y(x) curves in Fig. 4. Notice that when $\alpha = 1/3$ the interpolation is more accurate than the classical Lagrange one.

CONCLUSION

We propose a new approach of fractional Lagrange interpolation called α-root interpolation which agree with the classical interpolation when $\alpha = 1$. We analyze the convergence of the new approach and drive an error bound. We used this interpolation to solve some numerical applications such as the Fredholm integral equation and a fractional differential equation. The numerical results demonstrated the efficiency, simplicity and the applicability of the new approach.

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