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Efficient Decomposition Method for Integro-Differential Equations

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Abstract: Different methods have been used in the solution of integro-differential equations. Many of these methods such as Standard Adomian Decomposition Method (SADM) take several iterations which might be difficult to solve and also consume time before getting an approximation. This present study developed a new Modified Adomian Decomposition Method (MADM) for Integro-Differential Equations. The modification was carried out by decomposing the source term function into series. The newly Modified Adomian Decomposition Method (MADM) accelerates the convergence of the solution (MADM) faster the Standard Adomian Decomposition Method (SADM). This study recommends the use of MADM for solving Integro-Differential Equations.

INTRODUCTION

Several analytical solutions have been used to handle the problem of polynomial problems. These methods include Hirota's bilinear method, the Darboux transformation, the symmetry method, inverse scatting transformation, the Adomian decomposition method and other asymptotic methods. These methods have been used to solve nonlinear problems[1-5]. Amongst the method, the Adomian Decomposition method has been proved to be effective and reliable for handling different equations, linear or non-linear[5, 6].

The Adomian Decomposition Method (ADM) has been described as a method for solving both the linear and nonlinear differential equations and Boundary Value Problems (BVPs) seen in different fields of science engineering[7]. This method has been found that it is mainly depends upon the calculation of Adomian polynomials for nonlinear operators. The use of Adomian Decomposition method faces some problems which may arise from the nature of equations in consideration. Wazwaz introduced the modified Adomian decomposition method to solve some identified problems related to the nature of problems considered.

This present work introduces a new modification to Adomian Decomposition Method for integro-differential equations by using Adomian Polynomials. This new modification for integro-differential equations introduces a change in the formulation of Adomian polynomials; it provides a qualitative improvement over the standard Adomian method. The new modified Adomian Decomposition Method (MADM) can effectively improve the accuracy, speed of convergence and calculations.

MATERIALS AND METHODS

Consider the following integral equation:

$$y(x) = g(x) + \lambda\int_a^b k(x,t)\big[L\big(y(t)\big) + N\big(y(t)\big)\big]dt,\ \lambda \neq 0 \qquad (1)$$

where the kernel k(x, t) and the function g(x) are given real functions, λ is a numerical parameter L(y(t)) and N(y(t)) are linear and non-linear operator of y(x), respectively and the unknown y(x) is the solution to be determined. The solution of (Eq. 1) is usually expressed in the form:

$$y(x)=\sum_{j=0}^{+\infty} y_j(x) \quad (2)$$

and the method identifies the non-linear term N(y(t)) by decomposing the series:

$$N(y(x))=\sum_{j=0}^{+\infty} A_j(x) \quad (3)$$

where $A_j(x)$, j = 1, 2, 3,..., is called the Adomian Polynomials which are evaluated by:

$$A_r=\frac{1}{r!}\frac{d^r}{dx^r}N\left[\sum_{j=0}^{r}\lambda^r y_j\right];\ r=0,1,2,\ldots, \quad (4)$$

Substituting (2) and (3) into (1), we have:

$$\sum_{j=0}^{+\infty} y_j(x)=g(x)+\lambda\int_a^b k(x,t)\left[L\left(\sum_{j=0}^{+\infty} y_j(t)\right)+\sum_{j=0}^{+\infty} A_j(t)\right]dt \quad (5)$$

By the SADM:

$$y_0=g(x),\ y_{j+1}(x)=\lambda\int_0^x k(x,t)\left[L(y_j)+A_j\right]dt,\ j\geq 0 \quad (6)$$

In what follows, equations in (6) are the standard ADM.

The Modified ADM by Wazwaz[8]: This modification is based on the assumption that the function g(x) can be divided into two parts namely, $g_1(x)$ and $g_2(x)$ under the assumption that:

$$g(x)=g_1(x)+g_2(x) \quad (7)$$

thereby making a slight variation on the components $y_0(x)$ and $y_2(x)$. It says that only the part $g_1(x)$ will be assigned to the zeroth component $y_0(x)$ whereas the remaining part $g_2(x)$ will be combined with other terms given in (6). Consequently, the modified recursive relation is:

$$y_0(x)=g_1(x) \quad (8)$$

$$y_1(x)=g_2(x)+\lambda\int_0^x k(x,t)\left[L(y_0)+A_0\right]dt \quad (9)$$

$$y_{j+1}(x)=\lambda\int_0^x k(x,t)\left[L(y_j)+A_j\right]dt,\ j\geq 1 \quad (10)$$

From this approach, it was observed that the slight variation in reducing the number of will result in a reduction of computational work and will accelerate the convergence, the slight variation in the definition of the component y_0 and y_1 may provide solution using two iterations only and sometimes may not need the computations of Adomian polynomials which are required for non-linear term. However, the approach fails whenever the functions in the integral equation can not be evaluated analytically and also fails whenever the source term g(x) can not be split into divisions.

The Modified ADM by Wazwaz and El-Sayed[9]: Here, the function g(x) is expressed in Taylor series as:

$$g(x)=\sum_{j=0}^{\infty} g_i(x) \quad (11)$$

Thereby producing a new recursive relation:

$$y_0(x)=g_0(x) \quad (12)$$

$$y_1(x)=g_1(x)+\lambda\int_0^x k(x,t)\left[L(y_0)+A_0\right]dt \quad (13)$$

$$y_{j+1}(x)=g_{j+1}(x)+\lambda\int_0^x k(x,t)\left[L(y_j)+A_j\right]dt,\ j\geq 1 \quad (14)$$

The term $y_0(x)$, $y_1(x)$, $y_2(x)$ of the solution y(x) follow immediately and the solution y(x) can be easily obtained using (3). It is evident that this algorithm reduces the number of terms involved in each component and hence the size of calculations is minimized compared to SADM. This reduction of terms in each component facilitates the construction of Adomian polynomials for non-linear operators.

The Modified ADM by Xie[10]: The effectiveness of Wazwaz[8] depends on the proper choice of $g_1(x)$ and $g_2(x)$ which may need quite a little computational work. Also, the computation of $y_1(x)$ may be complicated to continue or analytically impossible, hence, Xie[10] suggests that $y_1(x)$ be expressed in Taylor series of the form:

$$y_1(x)=\sum_{j=0}^{+\infty} y_{1j}(x) \quad (15)$$

and:

$$y_2(x)=\lambda\int_0^x k(x,t)\left[L(y_1)+A_1\right]dt \quad (16)$$

where the Adomian polynomial A_1 can be evaluated by (4) with $y_0(x)$ and $y_1(x)$ are defined below:

$$y_0(x)=g(x) \quad (17)$$

$$y_1(x)=\lambda\int_0^x k(x,t)\left[L(y_0)+A_0\right]dt \quad (18)$$

Also:

$$y_2(x)=\sum_{j=0}^{+\infty} y_{2j}(x) \quad (19)$$

and:

$$y_{j+1}(x)=\lambda\int_0^x k(x,t)\left[L(y_j)+A_j\right]dt \quad (20)$$

and their Taylor series by:

$$y_j(x)=\sum_{j=0}^{+\infty} y_{ji}(x)\ j\geq 3 \quad (21)$$

In practice, some problems need the determination of a few terms in the series such as:

$$y(x)=\sum_{j=0}^{n} y_j(x)\ \ j\geq 3 \quad (22)$$

by truncating the series at certain terms . This produces a unformly convergence in the inifinite series, therefore, a few terms will attain the maximum accuracy.

The new approach to ADM (NADM): In recent times, several modifications of ADM have been proposed and duly applied to integral equations. However, these methods exist with their various drawbacks. To improve on the accuracies and subsequently the convergence of SADM and other modifications as mentioned, we shall based our assumption on the decomposition of the source term g(x) in series of the form:

$$g(x)=\sum_{j=0}^{+\infty} g_j(x)$$

and the new recursive relation obtained as:

$$y_0(x)=g_0(x) \quad (23)$$

$$y_1(x)=g_1(x)+g_2(x)+\lambda\int_0^x k(x,t)\left[L(y_0)+A_0\right]dt \quad (24)$$

$$y_2(x)=g_3(x)+g_4(x)+\lambda\int_0^x k(x,t)\left[L(y_1)+A_1\right]dt \quad (25)$$

$$\begin{aligned} y_{j+1}(x)&=g_{2(j+1)}(x)+g_{2(j+1)-1}(x)+\\ &\lambda\int_0^x k(x,t)\left[L(y_j)+A_j\right]dt,\ j\geq 1 \end{aligned} \quad (26)$$

and subsequently the function y(x) is obtained as:

$$y(x)=\sum_{j=0}^{+\infty} y_j(x)$$

Assuming that the non-linear function is F(y(x)), the polynomial below are few Adomial polynomials for the expansion of the non-linear term:

$$\begin{aligned} &A_0=F(y_0),\\ &A_1=y_1F'(y_0),\\ &A_2=y_2F'(y_0)+\frac{1}{2!}y_1^2F''(y_0),\\ &A_3=y_3F'(y_0)+y_1y_2F''(y_0)+\frac{1}{3!}y_1^3F'''(y_0),\\ &A_4=y_4F'(y_0)+\left(\frac{1}{2!}y_2^2+y_1y_3\right)F''(y_0)+\\ &\quad \frac{1}{2!}y_1^2y_2F'''(y_0)+\frac{1}{4}y_1^4F^{(iv)}(y_0) \end{aligned}$$

Two important observations can be made here. First, A_0 depends only on y_0, A_1 depends only on y_0 and y_1, A_2 depends only on y_0, y_1 and y_2 and so on. Secondly, substituting these A_j's in (4) gives:

$$\begin{aligned} &F(y)=A_0+A_1+A_2+A_3+,\ldots,=\\ &F(y_0)+(y_1+y_2+y_3+,\ldots,)\,F'(y_0)+\\ &\frac{1}{2!}\left(y_1^2+2y_1y_2+2y_1y_3+y_2^2\right)F''(y_0)+\\ &\frac{1}{3!}\left(y_1^3+3y_1^2y_3+6y_1y_2y_3+,\ldots,\right)F'''(y_0)+,\ldots,=\\ &F(y_0)+(y-y_0)F'(y_0)+\frac{1}{2!}(y-y_0)^2F''(y_0)+,\ldots, \end{aligned}$$

In the following, we will calculate Adomian polynomials for several linear terms that may arise in nonlinear integral equations.

Case 1: The first four Adomian polynomials for $F(y)=y^2$ are given by:

$$\begin{aligned} &A_0=y_0^2\\ &A_1=2y_0y_1\\ &A_2=2y_0y_2+y_1^2\\ &A_3=2y_0y_3+2y_1y_2 \end{aligned}$$

Case 2: The first four Adomian polynomials for $F(y)=y^3$ are given by:

$$\begin{aligned} &A_0=y_0^3,\\ &A_1=3y_0^2y_1,\\ &A_2=3y_0^2y_2+3y_0y_1^2,\\ &A_3=3y_0^2y_3+6y_0y_1y_2+y_1^3 \end{aligned}$$

Case 3: The first four Adomian polynomials for $F(y)=y^4$ are given by:

$$A_0 = y_0^4,$$
$$A_1 = 4y_0^3 y_1,$$
$$A_2 = 4y_0^3 y_2 + 6y_0^2 y_1^2,$$
$$A_3 = 4y_0^3 y_3 + 4y_1^3 y_0 + 12y_0^2 y_1 + y_2$$

Case 4: The first four Adomian polynomials for F(y) = sin y are given by:

$$A_0 = \sin y_0,$$
$$A_1 = y_1 \cos y_0,$$
$$A_2 = y_2 \cos y_0 - \frac{1}{2!} y_1^2 \sin y_0,$$
$$A_3 = y_3 \cos y_0 - y_1 y_2 \sin y_0 - \frac{1}{3!} y_1^3 \cos y_0$$

Case 5: The first four Adomian polynomials for F(y) = cos y are given by:

$$A_0 = \cos y_0,$$
$$A_1 = -y_1 \sin y_0,$$
$$A_2 = -y_2 \sin y_0 - \frac{1}{2!} y_1^2 \cos y_0,$$
$$A_3 = -y_3 \sin y_0 - y_1 y_2 \cos y_0 + \frac{1}{3!} y_1^3 \sin y_0$$

Case 6: The first four Adomian polynomials for F(y) = exp(y) are given by:

$$A_0 = \exp(y_0),$$
$$A_1 = y_1 \exp(y_0),$$
$$A_2 = \left(y_2 + \frac{1}{2!} y_1^2\right) \exp(y_0),$$
$$A_3 = \left(y_3 + y_1 y_2 + \frac{1}{3!} y_1^3\right) \exp(y_0)$$

Numerical examples

Example 1: Consider the standard integro-differential equation:

$$y(x) = 1 + \sinh(x) - \cosh(x) + \int_0^x y(t)\,dt$$

By the Standard Adomian Decomposition Method (SADM): Let:

$$y_0 = 1 + \sinh(x) - \cosh(x) \quad (27)$$

Then:

$$y_0(x) = 1 + \sinh(x) - \cosh(x)$$
$$y_1(x) = \int_0^x y_0(t)\,dt = x - 1 + \cosh(x) - \sinh(x)$$
$$y_2(x) = \int_0^x y_1(t)\,dt = 1 + \frac{1}{2}x^2 - x + \sinh(x) - \cosh(x)$$
$$y_3(x) = \int_0^x y_2(t)\,dt = -1 + x + \frac{1}{6}x^3 - \frac{1}{2}x^2 + \cosh(x) - \sinh(x)$$
$$y_4(x) = \int_0^x y_3(t)\,dt = 1 - x + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6}x^3 + \sinh(x) - \cosh(x)$$
$$y_5(x) = \int_0^x y_4(t)\,dt = -1 + x - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{120}x^5 - \sinh(x) + \cosh(x) - \sinh(x)$$

Thus:

$$y(x) = \sum_{j=1}^{5} y_j(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + , ...,$$

This obviously implies the solution tends to the exact solution:

$$y(x) = \sinh(x)$$

By the Modified Adomian Decomposition Method (MADM): The source term in equation (27) can be expanded in Taylor series in the form:

$$x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{5040}x^7 - \frac{1}{40320}x^8 + \frac{1}{362880}x^9 - o(x^{10})$$

Take:

$$y_0(x) = x$$
$$y_1(x) = -\frac{1}{2}x^2 + \frac{1}{6}x^3 + \int_0^x y_0(t)\,dt = -\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{2}x^2 = \frac{1}{6}x^3$$
$$y_2(x) = -\frac{1}{24}x^4 + \frac{1}{120}x^5 + \int_0^x y_1(t)\,dt = \frac{1}{120}x^5$$

Thus:

$$y(x) = \sum_{j=1}^{2} y_j(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + , ...,$$

The solution tends to the exact solution. It was observed that the modified method, NADM of just two iterations produces the result of SADM of five iterations. The NADM can be said to have a faster convergence rate when compared with SADM in example 1.

Example 2: Consider the linear equation:

$$y'^{(x)} = 1 - \frac{1}{3}x\int_0^1 xty(t)\,dt, \; y(0) = 0$$

Exact solution is y(x) = x. Applying a one fold integral linear operator defined by:

$$L^{-1} = \int_0^x (.)dx$$

the differential equation is transformed to:

$$y(x) = x-\frac{1}{6}x^2+L^{-1}\left(\int_0^1 xty(t)dt\right)dx$$

Using the SADM:

$$y_0(x) = x-\frac{1}{6}x^2$$
$$y_1(x) = \frac{7}{48}x^2$$
$$y_2(x) = \frac{7}{384}x^2$$
$$y_3(x) = \frac{7}{3072}x^2$$
$$y_4(x) = \frac{7}{24576}x^2$$

we have:

$$y(x) = \sum_{j=0}^{4} y_j(x)+, ...,$$

which obviously tends to the exact solution:

$$y(x) = x$$

Using the Midified ADM, we expand source term:

$$x-\frac{1}{6}x^2$$

we have:

$$x-\frac{1}{6}x^2+, ...,$$

In the light of the new modification:

$$y_0(x) = x,$$
$$y_1(x) = 0$$

Clearly, every value of y_j, $j \geq 1 = 0$, thus:

$$y(x) = \sum_{j=1}^{2} y_j(x) = x$$

It could be observed how the NADM produces its solution in just one iteration in comparison with the SADM producing her result at the fourth iteration.

Table 1: Table of Absolute errors for example 3

x	Exact	Wazwaz and El-Sayed[9]	NADM
0.0	2.000000000	2.000000000	2.000000000
0.1	2.105170918	2.5893-03	2.1729-03
0.2	2.221402759	1.0355-02	8.6915-03
0.3	2.349858807	2.3283-02	1.9556-02
0.4	2.491824697	4.1339-02	3.4766-03
0.5	2.648721265	6.4452-02	5.4321-03
0.6	2.822118771	9.2500-02	7.8224-02
0.7	3.013752588	1.2530-01	1.0647-02
0.8	3.225540527	1.6258-01	1.3906-01
0.9	3.459601938	2.0398-01	1.7600-01
1.0	3.718278771	2.4900-01	2.1729-01

Example 3: Consider the following problem:

$$y^{'(x)} = e^x+\frac{1}{16}(3+e)x+\frac{1}{4}$$
$$\int_0^1 xt\left(1+y(t)-y^2(t)\right)dt;\;\; y(0) = 2$$

Applying a one fold integral linear operator defined by:

$$L^{-1} = \int_0^x (.)dx$$

the differential equation is transformed to:

$$y(x) = 1+e^x+\frac{1}{32}(3+e)x^2+\frac{1}{4}L^{-1}\left(\int_0^1 xt\left(1+y(t)-y^2(t)\right)\right)dx$$

or:

$$y(x) = 1+e^x+0.324658003x^2+\frac{1}{4}L^{-1}\left(\int_0^1 xt\left(1+y(t)-y^2(t)\right)\right)dx$$

Using the relation as obtained by Wazwaz and El-Sayed (2001) and NADM, we have the following table of results at j = 3 (Table 1).

Example 4: Consider the following problem:

$$y^{''}(x) = \frac{1}{2}e^x+\frac{1}{2}\int_0^1 e^{x-2t}y^2(t)dt;\;\; y(0) = y^{'}(0) = 1$$

Applying two folf integral linear operator defined by:

$$L^{-1} = \int_0^x\int_0^x (.)dxdx$$

the differential equation is transformed to:

$$y(x) = \frac{1}{2}+\frac{1}{2}x+\frac{1}{2}e^x+\frac{1}{2}L^{-1}\left(\int_0^1 e^{x-2t}y^2(t)dt\right)dxdx$$

Using the relation as obtained in SADM and NADM, we have the following table of results at j = 3 (Table 2).

Table 2: Table of absolute errors for example 4

x	Exact	Wazwaz and El-Sayed[9]	NADM
0.0	1.000000000	1.000000000	1.000000000
0.1	1.102585459	2.7560-03	2.700-03
0.2	1.210701379	1.1811-02	1.1155-02
0.3	1.324929404	2.8443-02	2.7824-02
0.4	1.445912349	5.4153-02	5.2977-02
0.5	1.574360636	9.0667-02	8.8870-02
0.6	1.711059400	1.3997-02	1.3670-01
0.7	1.856876354	2.0436-01	1.9998-01
0.8	2.012770464	2.8644-01	2.8038-01
0.9	2.179801556	3.8925-01	3.8117-01
1.0	2.359140914	5.1621-01	5.0581-01

RESULTS AND DISCUSSION

This work introduced the new modified Adomian Decomposition Method (MADM) for Integro-Differential Equation. This new method converges faster and could be observed that the expansion of the source term needs to be as long as possible. The slight increase in selection of the terms of decomposed source terms of decomposed source term is to improve its convergence.

CONCLUSION

The increase in terms used in the integral sign is to improve the accuracy and subsequently the Adomian polynomials.

Simple summary: Adomain decomposition method is a very important tool for solving problems in differential equations. Some authors have improved on the method lately and has been applied to solutions of real life problems. We have made a new modification to the method and our aim is to solve some integral problems. The results obtained have proven effective in the solution of these problems. We therefore recommend the method for solution of scientific problems which may arise in real life situations.

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REFERENCES

01. Miura, M.R., 1978. Backlund transformation. 1989, Springer, Berlin, Germany, Pages: 295.
02. Hirota, R., 1980. Direct Methods in Solition Theory. In: Solitions. Bullogh, R.K., and P.J. Caudrey, (Eds.), Springer, Berlin, Germany, pp: 157–176.
03. Gu, C.H., H.S. Hu, Z.X. Zhou, 1999. Darboux Transformation in Solitons Theory and Geometry Application. 1999 Edn., Shanghai Science Technology Publication House, Shanghai, China,.
04. Olver, P.J., 1986. Application of Lie Groups to Differential Equations. Springer, New York, USA.,.
05. Adomian, G., 1994. Solving Frontier Problems of Physics: The Decomposition Method. 1st Edn., Kluwer Academic, Boston, Pages: 352.
06. Adomian, G. and R. Rach, 1986. On the solution of nonlinear differential equations with convolution product nonlinearities. J. Math. Anal. Appl., 114: 171-175.
07. Keskin, A.U., 2019. Adomoian Decomposition Method (ADM). In: Boundary Value Problems for Engineers. Keskin, A.U., Springer, Berlin, Germany, pp: 311-359.
08. Wazwaz, A.M., 1999. A reliable modification of adomian decomposition method. Applied Math. Comput., 102: 77-86.
09. Wazwaz, A.M. and S.M. El-Sayed, 2001. A new modification of the adomian decomposition method for linear and nonlinear operators. Applied Math. Comput., 122: 393-405.
10. Xie, L.J., 2013. A new modification of Adomian decomposition method for Volterra integral equations of the second kind. J. Appl. Math., 2013: 1-7.