Journal of Economic Theory 5 (5-6): 106-109, 2011
ISSN: 1994-8212

Analyze the Duration of Busy Period in $M^{[X]}/G/1/K$ Systems with Vacation Time and under the Partial Batch Acceptance Strategy

Mohammad Taremi and Mohammad Reza Salehi Rad
Department of Statistics, Faculty of Economics, Allameh Tabatabaei University, Tehran, Iran

Abstract: One of the main characteristics in optimization and performance evaluation in a Queueing Model is its busy period. Finding the exact distribution of this variables in Queueing Models which service distribution and or inter-arrival distribution is general is so complicated and there is usually no closed formula for that. Traditionally in these situations, Laplase-Stieltjes transform this variable computed and then by using that the moments it determined.

Key words: $M^{[X]}/G/1/K$ type queue, Laplase-Stieltjes transform, moments, vacation time, acceptance strategy, delay busy period

INTRODUCTION

In the Queueing Models, there are some features, so that through them, researchers can analyze the model. These characteristics are known as the effective sizes. On the other hand, optimization of queueing systems is done with the help of cost functions that are themselves functions of effective sizes. Also since, the arrival and service processes in queueing systems are random Therefore, effective measures are random. Hence, one of the important objectives in the analysis of queueing systems is obtained the probability distribution of this variable and or determined the characters of their distribution such as mean, variance and etc.

One of the important effective sizes in the Queueing Models, the busy period is. Finding the exact distribution of this variables in Queueing Models which service distribution and or inter-arrival distribution is general is so complicated and there is usually no closed formula for that. Since that, the moments determine the main feature of probability distribution for a random variable, so this study deals with the busy period moment in $M^{[X]}/G/1/K$ system with vacation times and under the partial batch acceptance strategy.

The batch-poisson arrival $M^{[X]}/G/1/K$ finite capacity queue with server vacation is now common in telecommunications. For example, a processor (server) has secondary jobs (customers) to be performed aside from primary jobs. The processing time for a secondary job corresponds to a vacation time in queueing terminology. Another example is a buffer (queue) under the Time Division Multiple Access (TDMA) environment (Stuck and Arthurs, 1985). An arriving packet (customer) who finds the system idle cannot be transmitted (served) immediately and it has to wait until the slotted boundary comes. A constant slotted time period corresponds to a vacation time. Performances issues in these examples then necessitate the queue with vacation time.

Traditionally, busy periods have been characterized through their Laplace-Stieltjes transforms (Casella and Berger, 1976; Harris, 1971; Miller, 1975; Perry *et al*., 2000). However, the Laplace-Stieltjes approach has some important practical limitations that are not shared by procedure developed in this study.

Researchers analyze the duration of busy periods in $M^{[X]}/G/1/K$ systems by conditioning on the number of customers that arrive to the system during the first vacation time while concurrently taking full advantage of the Markov-regenerative structure of the number of customers in $M^{[X]}/G/1/K$ systems (Kulkarni, 1995) for the definition and properties of Markov regenerative processes). By busy period it is (usually) meant the period of time that starts when a customer arrives to an empty system and ends at the first subsequent time at which the system becomes empty (Abramov, 1997; Righter, 1999). In this study, we obtain results for the slightly generalized case of delay busy periods (Conway *et al*., 1967) where a busy period is initiated by some task other than the processing of an ordinary job. Researchers end this introduction with a brief outline of the study.

Corresponding Author: Mohammad Taremi, Department of Statistics, Faculty of Economics, Allameh Tabatabaei University, Tehran, Iran

The MODEL AND SOME NOTATION

Researchers consider an $M^{|X|}/G/1/K$ queue where K equals the number of waiting places in the queue including the space for the customer that may be in service.

Researchers assume that the arrival epochs of the batches form a homogeneous poisson process with intensity λ and the service times form a sequence of i.i.d., random variables with general distribution function G(.). Service times are independent of the arrival process and are not affected by the discipline and customers accepted by the system are served by a single server exhaustively, i.e., the server serves the queue continuously until the queue is empty.

As regards the customer acceptance policy, researchers consider what is known as partial blocking (Vijaya Laxmi and Gupta, 2000) in which if at arrival of a batch of l customers there are only m, m<1, free position available in the system then m customers of the batch enter the system and the remaining l-m customers of the batch are blocked.

In the case of delay busy periods at a time when there are no customers in the system, the server is made unavailable for a time V having distribution function V(.). At the end of the delay (vacation) the server begins work with a backlog of however many customers (possibly none) have arrived during the delay and continues until there are none left. Researchers let $(f_i)_{i\in N}$ denote the batch size probability function where, $N_+ = \{1, 2, 3, ...\}$ and $f_i^{(r)}$ denotes the probability that the total number of customers in r customer batches is equal to i. Note that $f_a^{(0)} = \delta_{0a}$ and:

$$f_a^{(r)} = \sum_{i=r-1}^{a-1} f_{a-i} f_i^{(r-1)}$$

for r∊N and a = r, r+1,.... where δ_{ia} is the Kronecker delta function, i.e., $\delta_{ia} = 1$ if i = a and $\delta_{ia} = 0$ otherwise.

In addition, researchers let P_a, a∊N = {0, 1, 2, ...}, denote the probability that a customers arrive during the vacation time. Then by conditioning on the number of batches arriving during the vacation time, researchers have:

$$p_a = \sum_{r=0}^{0} f_a^{(r)} \alpha_r \tag{1}$$

where, α_r is r-th mixed-poisson probability with arrival rate λ and mixing distribution G(.) (German, 2000; Kwiatkowska *et al*., 2002):

$$\alpha_r(S) = \int_0^{\infty} \frac{e^{-\lambda s}(\lambda s)^r}{r!} dG_S(s) \tag{2}$$

MOMENTS OF THE DURATION OF A BUSY PERIOD

Let $B_{i,K}$ be the duration of a busy period that starts with a backlog of i jobs when the queue has a total capacity of K jobs waiting. In this notation, $B_{1,K}$ corresponds to the ordinary busy period. $B_{i,K}$ is independent of the discipline, so that the busy period is equivalent to the sum of i busy periods, the j-th one having length $B_{1,K-i+j}$ because i - j spaces are occupied by original jobs. Thus:

$$B_{i,K} = \bigoplus_{j=K+1-i}^{K} B_{1,j} \tag{3}$$

where, ⊕ denotes convolution of random variables, i.e., the sum of independent random variables.

Now, considering the random variable representing the duration of the delay busy period with K waiting spaces by D_K, its Laplase-Stieltjes transform by $\delta_K(t)$ and distribution function and Laplase-Stieltjes transform of $B_{1,K}$, respectively by $B_{1,j}(.)$ and $\beta_{1,j}(t)$, researchers derive the following results.

Theorem 1: The Laplase-Stieltjes transform for the distribution of the duration of a delay busy period for an $M^{|X|}/G/1/K$ queue is:

$$\delta_K(t) = v_0(t) + \sum_{a=1}^{K-1} \omega_a(t) \prod_{j=K-a+1}^{K} \beta_{1,j}(t) + \sum_{a\geq K} \omega_a(t) \prod_{j=1}^{K} \beta_{1,j}(t) \tag{4}$$

Where:

$$\omega_a(t) = \int_0^{\infty} \sum_{r=0}^{a} f_a^{(r)} \frac{(\lambda v)^r}{r!} e^{-(t+\lambda)v} dV(v)$$

Proof: Let A be the number of arrivals during the vacation time. Then:

$$E\left[\exp(-tD_K) \middle| V = v, A = a, B_{\min(a,K),K} = x\right] = \exp[-t(v+x)] \tag{5}$$

where, x = 0 if a = 0. For 1 ≤a≤K, $B_{a,K}$ can be replaced by its decomposition, given by Eq. 3. Then integrate with respect to $B_{1,j}(.)$, K+1 - a≤j≤K and apply the convolution theorem for Laplase transform so that, exp(-tx) is replaced with products of the $\beta_{1,j}(t)$. Finally, weight each term by the probability associated with the number of arrivals during the vacation time and integrate with respect to V(.).

However, moments of the model may be obtained by differentiation of Eq. 4 but this procedure for computing the higher moments, especially when increasing capacity of the system, the complicated computation is needed. In the following theorem, researchers computation this moments by another way.

Theorem 2: The integer moments of the duration of delay busy periods for an $M^{[X]}/G/1/K$ queue is:

$$E(D_K^m) = E(V^m) + \sum_{a=1}^{K-1} p_a E(B_{a,K}^m) + \sum_{a=1}^{K-1} p_a \sum_{j=1}^{m-1} \binom{m}{j} E(\overline{V}_a^j) E(B_{a,K}^{m-j}) + \sum_{a\geq K} p_a E(B_{K,K}^m) + \sum_{a\geq K} p_a \sum_{j=1}^{m-1} \binom{m}{j} E(\overline{V}_a^j) E(B_{K,K}^{m-j}) \quad (6)$$

where, $m \in N$.

Proof: Researchers let $\overline{V}_a$ denote the duration of the vacation time of the first delay busy period given that exactly a customers arrive to the system during his delay.

If no customers arrive to the system during the vacation time of the first delay busy period, the delay busy period ends with terminate the vacation time.

Otherwise, the customers that arrive to the system during the vacation time and are not blocked initiate at end the vacation time, a multiple-busy period that is part of the first delay busy period under consideration and adds to the duration of the vacation time. Namely:

$$(D_K | A = a) \stackrel{d}{=} \overline{V}_c \oplus B_{\min(a,K),K}$$

where, $\stackrel{d}{=}$ denotes equality in distribution. Taking into account that has probability function P_a, Eq. 1 leads to:

$$E(D_K^m) = \sum_{a=0}^{\infty} p_a E(\overline{V}_a \oplus B_{\min(a,K),K})^m$$

with $B_{0,K}$ denoting a random variable that is null with probability one. By using Newton's binomial formula in previous equation, we have:

$$E(D_K^m) = p_0 E(\overline{V}_0^m) + \sum_{a=1}^{K-1} p_a \sum_{j=0}^{m} \binom{m}{j} E(\overline{V}_a^j) E(B_{a,K}^{m-j}) + \sum_{a\geq K} p_a \sum_{j=0}^{m} \binom{m}{j} E(\overline{V}_a^j) E(B_{K,K}^{m-j})$$

By separating the terms for which j = 0 and j = m from the remaining terms in the previous equation taking into account, researchers conclude following equation:

$$E(V^m) = \sum_{a\geq 0} p_a E(\overline{V}_a^m)$$

As it mentioned, Pacheco and Riberio (2008) has addressed a recursive algorithm to compute moments $E(B^m_{a,K})$, $1 \leq a \leq K$. In following theorem, researchers provide a simpler form for the computation the coefficients:

$$\left(p_a E[\overline{V}_a^l]\right)_{0\leq a\leq K-2,\ 0\leq l\leq m}$$

Theorem 3: The absolute moment of order m, $m \in N_=$, of conditional random variable $\overline{V}_a$, verifies:

$$p_a E[\overline{V}_a^m] = \sum_{j=0}^{a} \frac{(m+j)}{\lambda^m j!} \alpha_{m+j} f_a^{(j)} \quad (7)$$

for $a \in N$ and moreover:

$$\sum_{a\geq K} p_a E(\overline{V}_a^m) = E(V^m) - \sum_{a=0}^{K-1} \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \alpha_{m+j} f_a^{(j)} \quad (8)$$

Proof: For $m \in N_+$ and $a \in N$:

$$\begin{aligned} p_a E[\overline{V}_a^m] &= E\left[V^m 1_{\{A=a\}}\right] \\ &= \int_0^{\infty} u^m \sum_{j=0}^{a} e^{-\lambda u} \frac{(\lambda u)^j}{j!} f_a^{(j)} dV(u) \\ &= \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \int_0^{\infty} e^{-\lambda u} \frac{(\lambda u)^{m+j}}{(m+j)!} dV(u) f_a^{(j)} \\ &= \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \alpha_{m+j} f_a^{(j)} \end{aligned}$$

Finally, Eq. 8 follows from Eq. 7 since:

$$E(V^m) = \sum_{a\geq 0} p_a E(\overline{V}_a^m)$$

thus, implying that:

$$\sum_{a\geq K} p_a E(\overline{V}_a^m) = E(V^m) - \sum_{a=0}^{K-1} p_a E(\overline{V}_a^m)$$

The most immediate application of theorem 2 is for the computation of the expected value of the duration of the delay busy period of M/G/1/K systems in which case, researchers conclude that:

Table 1: Expected value, coefficient of variation, skewness and kurtosis of the duration of delay busy period in systems with arrival rate, unit service rate and exponential vacation time distribution with unit mean

	M/M(1)/1/K			
K	EV_K	CV_K	SV_K	KV_K
0	1.0000	1.0000	2.0000	9.0000
1	2.0000	2.0000	4.0000	18.0000
2	2.9500	2.2246	4.3238	19.7353
3	3.8525	2.4525	4.7064	22.2209
4	4.7099	2.6573	5.0727	25.0158
5	5.5244	2.8420	5.4149	27.9797
6	6.2982	3.0108	5.7354	31.0580
11	9.6240	3.6995	7.1128	47.5952
16	12.1975	4.2334	8.2691	65.6381
21	14.1888	4.6739	9.3009	85.0173
26	15.7296	5.0475	10.2491	105.6351
31	16.9219	5.3684	11.1339	127.3832
36	17.8444	5.6456	11.9655	150.1300
41	18.5583	5.8856	12.7498	173.7177
46	19.1106	6.0931	13.4895	197.9643
51	20.5380	6.2721	14.1859	222.6676
∞	21.0000	7.2450	20.7309	596.7692

$$E(V_K) = E(V) + \sum_{j=1}^{K} E(B_{1,j}) \sum_{a=K-j+1}^{\infty} \alpha_a$$

This leads to equation derived by Miller (1975) to compute the duration of a delay busy period of an M/G/1/K system.

NUMERICAL ILLUSTRATION

To provide a numerical illustration of how the duration of delay busy period depends on system capacity, researchers computed the data in Table 1 for the M/M/1/K system with arrival rate $\lambda = 0.95$ unit service rate and exponential vacation time distribution with unit mean.

In the example, we let EV_K(CV_K, SV_K and KV_K) denote the expected value or mean (coefficient of variation, skewness and kurtosis) of the duration of delay busy period (Casella and Berger, 2002).

CONCLUSION

In this study, researchers address using the Laplase-Stieltjes approach to compute moments of the duration of the busy period in $M^{[X]}$/G/1/K systems with vacation times and under the partial batch acceptance strategy is associated with some important practical limitations, we provide other procedure to calculate the moments of this Queueing Model.

REFERENCES

Abramov, V.M., 1997. On a property of a refusals stream. J. Applied Probability, 34: 800-805.

Casella, G. and R.L. Berger, 1976. On the relationship between the distribution of maximal queue length in the M/G/1 queue and the mean busy period in the M/G/1/n queue. J. Applied Probability, 13: 195-199.

Casella, G. and R.L. Berger, 2002. Statistical Inference. 2nd Edn., Thomson Learning, UK., ISBN: 9780534243128, Pages: 660.

Conway, R.W., W.L. Maxwell and L.W. Miller, 1967. Theory of Scheduling. Addison-Wesley Pub. Co., New York, Pages: 294.

German, R., 2000. Performance Analysis of Communication Systems: Modeling with Non-Markovian Stochastic Petri Nets. 1st Edn., Willy, Chichester, UK., ISBN: 978-0471492580, Pages: 456.

Harris, T.J., 1971. The remaining busy period of a finite queue. Operations Res., 19: 219-223.

Kulkarni, V.G., 1995. Modeling and analysis of stochastic systems. Chapmanand Hall, London,?.

Kwiatkowska, M., G. Norman and A. Pacheco, 2002. Model checking CSL until ???formulae with random time bounds. Herman, H. and R. Segala, (Eds.). Process algebra and probabilistic methods performance modeling and verification, Berlin, Springer,.

Miller, L.W., 1975. A note on the busy period of an M/G/1 finite queue. Operations Res., 23: 1179-1182.

Pacheco, A. and H. Riberio, 2008. Moments of the duration of busy periods of systems. Probability Eng. Inf. Sci., 22: 347-354.

Perry, D., W. Stadje and S. Zacks, 2000. Busy period analysis for M/G/1 and G/M/1 type queues with restricted accessibility. Operations Res. Lett., 27: 163-174.

Righter, R., 1999. A note onl osses in M=GI=1=n queues. J. Applied Probability, 36: 1240-1243.

Stuck, B.W. and E. Arthurs, 1985. A Computer and Communications Network Performance Primer. Prentice-Hall, New Jersey.

Vijaya Laxmi, P. and U.C. Gupta, 2000. Analysis of finite-buffer multi-server queues with Group arrivals: Queueing Sys., 36: 125-140.