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Limit Analysis of Oscillating Batch Arrival $M^{[x]}/G/1$ Systems with Finite Capacity: EMC Approach

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Abstract: An oscillating M/G/1 Systems with finite capacity and batch arrivals is considered under partial batch acceptance strategy. Applying the Theory of Markov Regenerative Processes and resorting to Markov chain embedding the limit distributions of the number of customers in the system is obtained where no assumption on the batch size distribution is made.

Key words: $M^{[x]}/G/1$ Systems, finite capacity, oscillating systems, acceptance strategy, batch arrivals, Iran

INTRODUCTION

One way to increase server utilization while keeping customer waiting times under control is to consider queueing systems whose characteristics (such as service rate or input rate) depend on the evolution of the state of the system (such as the number of customers in the system or the workload). For instance, Choi *et al.* (2001,1999), Harris (1967, 1970), Larsen and Agrawala (1983), Ramalhoto (1991), Rhee and Sivazlian (1990) and Welch (1964) studied queueing systems whose service characteristics depend on the evolution of the number of customers in the system. Chydzinski (2003) and Takagi (1985) studied queueing systems with input rate depending on the evolution of the number of customers in the system. Ivnitskiy (1975), Li (1989) and Lu and Serfozo (1984) studied queueing systems with input rates and service rates depending on the evolution of the number of customers in the system. Altman and Jean-Marie (1998) studied queueing systems with workload dependent service times and Bekker *et al.* (2004) and Golubchik and Lui (2002) studied queueing systems in which the arrival rate and service rate depends on the workload in the system. In this study, the researchers investigate the limit distribution of the number of customers in oscillating batch arrival $M^{[x]}/G/1$ Systems with finite capacity. The researchers use the term (service) oscillating systems in the sense used by Bratiychuk and Chydzinski (2003) and Chydzinski (2002, 2004), i.e., as a queueing systems that oscillates between two operating phases 1 and 2 which impact the service rates or service characteristics as described.

The limit distribution of the number of customers in a queueing system is an important characteristic of the system as it provides information about the evolution of its congestion level over time. Federgruen and Tijms (1980) compute the limit distribution of the queue length in oscillating M/G/1 Systems recursively by using the Theory of Markov Regenerative Processes (MRGP). Bratiychuk and Chydzinski (2003) and Chydzinski (2002) have addressed the limit analysis of the number of customers in oscillating systems with infinite capacity and Chydzinski (2004) has studied steady state characteristics of oscillating systems with single arrivals and finite capacity by means of the potential method.

In general terms when an oscillating systems is in phase 1 the number of customers moves between 0 and b-1 and when it is in phase 2 the number of customers moves between a+1 and n, $0 \le a < b \le n$ with the integers a and b denoting the lower barrier and the upper barrier of the system, respectively. More precisely if at time t the system is operating in phase 1, so that the number of customers in the system is smaller than the upper barrier b then the system remains in phase 1 until the first subsequent epoch at which the number of customers in the system becomes greater or equal to the upper barrier b. At this epoch, the system changes to phase 2 and remains in this phase until the first subsequent epoch at which the number of customers in the system becomes (smaller or) equal to the lower barrier a at which time the system changes again to phase 1 and so on.

The researchers consider two types of oscillating systems, I and II that are characterized in terms of two distribution functions A_1 and A_2 as follows; in a type I system, a customer service time that is initiated in phase

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j has distribution A_j and j = 1, 2 is independent of the customer arrival process and of previous customer service times.

In a type II system, a customer service initiated in phase 2 has customer service time distribution A_2 and is independent of the customer arrival process and of previous customer service times. Conversely, even though a customer service initiated in phase 1 is started with service time distribution A_1, independent of the customer arrival process and of previous customer service times if before this time has elapsed the system moves to phase 2 (due to the number of customer in the system becoming greater or equal to the upper barrier b) then a reset of the service is done at the instant the system changes phases and an additional time with distribution function A_2 is added to the customer service time with this time being independent of the customer arrival process and of previous customer service times.

Type I oscillating systems have been addressed by several researchers including Bahary and Kolesar (1972), Choi and Choi (1996), Sriram *et al.* (1991), Loris-Teghem (1981), Bratiychuk and Chydzinski (2003), Choi *et al.* (1999) and Federgruen and Tijms (1980). In particular, type I oscillating systems propose (Choi and Choi, 1996; Choi *et al.*, 1999; Sriram *et al.*, 1991) the analysis of cell-discarding schemes for voice packets in ATM networks by allowing dropping of low-priority (less significant) bits of information during congestion periods. It is noted that Li (1989) uses similar models for overload control in message storage buffers such that both the input and service rates or characteristics may depend on the phase of the system.

In addition, type II oscillating systems coincide with the queueing systems defined in Chydzinski (2002, 2004).

In this study, the researchers address oscillating batch arrival $M^{[x]}/G/1$ Systems with finite capacity n. These are queueing systems with a single server at which customers arrive in batches with independent and identically distributed (iid) sizes, according to a poisson process.

The sequences of batch sizes and batch interarrival times are independent and the system has finite capacity n including the customer in service if any.

As regards the customer acceptance policy, it is considered what is known as partial blocking (Vijaya Laxmi and Gupta, 2000) in which if at arrival of a batch of l customers there are only m, m<l , free positions available in the system then m customers of the batch enter the system and the remaining l-m customers of the batch are blocked.

The approach to investigate the limit distribution of the state of the system based on the fact that the state process in these systems constitutes a MRGP associated with appropriate Markov renewal sequences by means of the imbedded Markov Chain (EMC) (Kendall, 1951, 1953). Specifically, the information on the state of the system in continuous time is obtained from the analysis of the embedded Discrete Time Markov Chains (DTMCs) associated with the sequence post-customer departure instants.

The researchers remark that other researchers have used the Theory of MRGP to derive recursive relations in M/G/1 Systems, e.g., Fakinos and Economou (2001) and Federgruen and Tijms (1980).

NOTATION

It is denoted that the oscillating batch arrival $M^{[x]}/G/1$ Systems with finite capacity n and with lower barrier a and upper barrier b as $M^{[x]}/G_1$-$G_2/1/n/(a, b)$ Systems, the service times oscillate between two forms according to evolution of the number of customers in the system as described in the introduction.

Let λ denote the batch arrival rate and $(f_i)_{i\in N_+}$ denote the batch size probability function where $N_+ = \{1, 2, 3,...\}$ and $f_i^{(r)}$ denotes the probability that the total number of customers in r customer batches is equal to i. Note that $f_j^{(0)} = \delta_{0j}$ and:

$$f_j^{(r)} = \sum_{i=r-1}^{j-1} f_{j-i} f_i^{(r-1)} \tag{1}$$

for $r\in N_+$ and j = r, r+1, ... where δ_{ij} is the Kronecker delta function, i.e., $\delta_{ij} = 1$ if i = j and $\delta_{ij} = 0$ otherwise. As mentioned before, let A_1 and A_2 denote the distribution function associated with operating phases 1 and 2, respectively, in $M^{[x]}/G_1$-$G_2/1/n/(a, b)$ Systems. Moreover, let $1/\mu_1$ and $1/\mu_2$ denote the expected values of the distributions A_1 and A_2, respectively. In addition, the researchers let $r_j(A_i)$, $j\in N = \{0, 1, 2,...\}$ denote the probability that j customers arrive during a customer service time with distribution A_i. Then, by conditioning on the number of batches arriving during a customer service time with distribution A_i, the researchers have:

$$r_j(A_i) = \sum_{l=0}^{j} f_j^{(l)} \alpha_l(A_i), \qquad i = 1 \text{ or } 2 \tag{2}$$

Where, α_l (A_i) is lth mixed-poisson probability with arrival rate λ and mixing distribution A_i (Kwiatkowska *et al.*, 2002; Willmot, 1993):

$$\alpha_l(A_i) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^l}{l!} A_i(dt) \quad (3)$$

Let Y = {Y(t) = $Y_1(t)$, $Y_2(t)$, t≥0} denote the (continuous time) state process in $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ System where $Y_1(t)$ denoting the number of customers in the system at time t and $Y_2(t)$ denoting the phase of the system at time t and Y has state space:

$$E^{(n,a,b)} = \bar{E}^{(n,a,b)} \cup \{(0,1)\}$$

With:

$$\bar{E}^{(n,a,b)} = \{(c_1,1): 1 \le c_1 \le b-1\} \cup \{(c_1,2): a+1 \le c_1 \le n\}$$

and let $(T_m)_{m\in N+}$ denote the time sequence of customer service completion epochs, i.e., T_m is the instant at which the mth service completion takes place. In addition, let:

$$Y^p = \{Y_m^p = (Y_{m1}^p, Y_{m2}^p), m \in N\}$$

denote the post-customer departure state process in this system where:

$$Y_{m1}^p = Y_1(T_m^+)$$

denoting the number of customers that stay in the system after the m-th service completion and:

$$Y_{m2}^p = Y_2(T_m^+)$$

denoting the phase of the system after the mth service completion.

POST-CUSTOMER DEPARTURE STATE IN OSCILLATING SYSTEMS

In this study, it is presented that the derivation of the limit distribution of the post-customer departure state in type I and II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems, i.e., the limit distribution of Y^p.

Note that Y^p is a DTMC whose transitions depend on the number of customers that arrive to the system during the successive customer service times. Thus to characterize Y^p, it is useful to first characterize the probability that l customers arrive to the system during a customer service initiated in state c which has been denoted by $r_{cl}^{(b)}$ for l∈N and c∈$E^{(n,a,b)}$. The next result shows how the probabilities $r_{cl}^{(b)}$ may be computed in type I and II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems.

Lemma 1: In type I $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems the $r_{(i,j)l}^{(b)}$ probabilities are such that:

$$r_{(i,j)l}^{(b)} = r_l(A_j) \quad (4)$$

for (i, j)∈$E^{(n,a,b)}$ and l∈N where as defined before Eq. 2, $r_l(A_j)$ is the probability that customers arrive during customer service time with distribution A_j. In type II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems, the $r_{(i,j)l}^{(b)}$ probabilities are such that:

$$r_{(i,j)l}^{(b)} = \begin{cases} r_l(A_1) & j=1 \text{ and } 0 \le l \le b-1-i \\ r_{b-i,l}^{*}(A_1,A_2) & j=1 \text{ and } b-i \le l \\ r_l(A_2) & j=2 \end{cases} \quad (5)$$

Where:

$$r_{m,l}^{*}(A,B) = \sum_{u=m}^{l} q_{mu}(A) r_{l-u}(B), 1 \le m \le l$$

for distribution functions A and B of non-negative random variables with q_{mu} (A), 1≤m≤l denoting the probability that during a customer service with distribution A, m or more customer arrivals take place and exactly u customers arrive until the first moment at which m or more customer arrivals have occurred. Moreover:

$$q_{mu}(A) = \lambda \sum_{s=0}^{m-1}\sum_{v=s}^{m-1} f_v^{(s)} f_{u-v} \bar{\alpha}_s(A) \quad (7)$$

With:

$$\bar{\alpha}_s(A) = \int_0^\infty \frac{e^{-\lambda s}(\lambda t)^s}{s!} \int_{(t,\infty)} A(u)du\,dt \quad (8)$$

denoting the sth mixed-poisson expected value with rate λ and mixing distribution A, satisfying:

$$\bar{\alpha}_0(A) = \frac{1}{\lambda}\left(1-\alpha_0(A)\right) \quad (9)$$

$$\bar{\alpha}_s(A) = \bar{\alpha}_{s-1}(A) - \frac{1}{\lambda}\alpha_s(A),\ s \ge 1 \quad (10)$$

where, $\alpha_s(A)$ the sth mixed-poisson probability with rate λ and mixing distribution A is as defined in Eq. 3.

Proof: In a type I $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ System, researchers get Eq. 4 by considering the number of batches arriving during a customer service as described in Eq. 2, since the customer service time distribution is A_1 if the service starts with the system in phase 1 and is A_2 if the service starts with the system in phase 2. Similarly, the $r_{(i,2)l}^{(b)}$ probabilities in Eq. 5 for a type II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ System in follow by conditioning on the number of batches arriving during a customer service that starts with the system in phase 2 which has distribution

A_2. In the same way, the $r_{(i,1)l}^{(b)}$ probabilities, $l \le b-i-1$ in Eq. 5 for a type II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ System follow by conditioning on the number of batches arriving during a customer service that starts with i customers in the system and the system being in phase 1 which has distribution A_1 if fewer than b-i customers arrive during the service time.

The researchers now address the computation of $r_{(i,1)l}^{(b)}$ probabilities, $l \ge b-i$ for type II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems.

These probabilities are associated to customer services initiated with the system in phase 1 such that the system changes from phase 1-2 during the customer service. For that let C_{ml} denote the event that during a random time with distribution A_1, independent of the customer arrival process, m or more customer arrivals take place and exactly l customers arrive until the first moment at which m or more customer arrivals have occurred, whose probability is $q_{ml}(A_1)$.

Moreover, let D_u denote the event that during a random time with distribution A_2, independent of the customer arrival process and of the events $(C_{ml})_{1\le m\le l}$, u customer arrivals take place whose probability is $r_u(A_2)$. Then for $1 \le i \le b-1$ and $l \ge b-i$, we have:

$$
\begin{aligned}
r_{(i,1)l}^{(b)} &= \sum_{u=b-i}^{l} P(C_{b-i,u} \cap D_{l-u}) = \sum_{u=b-i}^{l} P(C_{b-i,u})P(D_{l-u}) \\
&= \sum_{u=b-i}^{l} q_{b-i,u}(A_1) r_{l-u}(A_2) = r^{*}_{b-i,l}(A_1, A_2)
\end{aligned}
$$

Thus to conclude the proof, it remains to show Eq. 7, 9 and 10. Equation 7 follows since by conditioning on the value of the product of the time it takes to observe m or more customer arrivals by the indicator function of this time being smaller than an independent random variable with distribution A, we conclude that, for $1 \le m \le l$:

$$
\begin{aligned}
q_{ml}(A) &= \int_0^{\infty}\int_{(0,u)} \sum_{s=0}^{m-1} e^{-\lambda t}\frac{(\lambda t)^s}{s!}\sum_{v=s}^{m-1} f_v^{(s)}\lambda f_{l-v}\,dt\,A(du) \\
&= \lambda\sum_{s=0}^{m-1}\sum_{v=s}^{m-1} f_v^{(s)} f_{l-v}\bar{\alpha}_s(A)
\end{aligned}
$$

Finally, Eq. 9 and 10 follow since from (Kwiatkowska *et al.*, 2002). Theorem 2:

$$
\bar{\alpha}_s(A) = \frac{1}{\lambda}\sum_{j=s+1}^{\infty}\alpha_j(A) \qquad (11)
$$

Researchers are now able to characterize the post-customer departure state process Y^p. We first note that Y^p is a DTMC with state space:

$$
\tilde{E}^{(n,a,b)} = \hat{E}^{(n,a,b)} \cup \{(0,1)\}
$$

With:

$$
\hat{E}^{(n,a,b)} = \begin{cases} \{(c_1,1): 1 \le c_1 \le b-2\} \cup \begin{Bmatrix}(c_1,2): a+ \\ 1 \le c_1 \le n-1\end{Bmatrix}, & a < b-1 \\ \{(c_1,1): 1 \le c_1 \le b-1\} \cup \begin{Bmatrix}(c_1,2): a+ \\ 1 \le c_1 \le n-1\end{Bmatrix}, & a = b-1 \end{cases}
$$

Moreover, a careful inspection leads to the following result for the transition probability matrix of Y^p.

Theorem 1: The (one step) transition probability matrix $P = (p_{cd})$ c, d, $\in \tilde{E}^{(n,a,b)}$ of the DTMC Y^p is such that for $c \ne (0, 1)$:

$$
p_{cd} = \begin{cases} r^{(b)}_{c,d_1-c_1+1} & c_2 = d_2 \text{ and } c_1 - 1 \le d_1 \le b-2 \\ r^{(b)}_{c,d_1-c_1+1} & c_2 \le d_2 \text{ and } \max(b-1, c_1-1) \le d_1 \le n-2 \\ r^{(b)}_{c0} & (c,d) = ((a+1,2),\ (a,1)) \\ \sum_{l \ge n-c_1} r^{(b)}_{cl} & d_1 = n-1 \\ 0 & \text{otherwise} \end{cases} \qquad (12)
$$

with $r_{cl}^{(b)}$ given in Lemma 1. Moreover:

$$
p_{(0,1)(d_1,1)} = \begin{cases} \sum_{l=1}^{d_1+1} f_l r^{(b)}_{(l,1)d_1-l+1} & d_1 < b-1 \\ \sum_{l=1}^{b-1} f_l r^{(b)}_{(l,1)b-l} + f_b r^{(b)}_{(b,2)0} & d_1 = b-1 = a \end{cases} \qquad (13)
$$

And:

$$
p_{(0,1)(d_1,1)} = \begin{cases} 0 & a+1 \le d_1 < b-1 \\ \sum_{l=1}^{b-1} f_l r^{(b)}_{(l,1)d_1-l+1} + \sum_{l=b}^{d_1+1} f_l r^{(b)}_{(l,2)d_1-l+1} & b-1 \le d_1 < n-1 \\ \sum_{l=1}^{b-1} f_l \sum_{j \ge n-l} r^{(b)}_{(l,1)j} + \sum_{l \ge b} f_l \sum_{j \ge (n-l)^+} r^{(b)}_{(l\wedge n,2)j} & d_1 = n-1 \end{cases}
$$

with $x^+ = \max(x, 0)$ and $x \wedge y = \min(x, y)$.

Continuous time state in oscillating systems: In this study, researchers characterize the limit distribution of the continuous time state process for the type I and II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems, i.e., the limit distribution Y.

To start, let $S_l(A)$ denotes a random variable whose distribution is the distribution of the duration of a customer service time, S with distribution function A given that l customers arrive to the system during this customer service time and let $\tilde{S}_{j,l}, 1 \le j \le l$, denoting a random variable with the same distribution as the accumulated service time until the first epoch at which j or more customer arrivals take place given that exactly l customers arrive until the first moment at which j or more arrivals have occurred in a service period with distribution function A_1.

The following lemma shows how absolute moments of the conditional random variables $\bar{S}_l(A), \tilde{S}_{jl}^k$ may be computed.

Lemma 2: The absolute moment of order k, k∈N_+ of conditional random variables $\bar{S}_{l}(A_l)$ and $\tilde{S}_{jl}^k$, verifies:

$$r_l(A)E[\bar{S}_l^k(A)] = \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j}(A) f_l^{(j)} \tag{15}$$

for l∈N and;

$$\sum_{l \ge n-1} r_l(A)E[\bar{S}_l^k(A)] = E[S^k(A)] - \sum_{l=0}^{n-2}\sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j}(A) f_l^{(j)} \tag{16}$$

Moreover:

$$q_{jl}(A_1)E[\tilde{S}_{jl}^k] = \lambda \sum_{m=0}^{j-1} \frac{(m+k)!}{\lambda^k m!} \bar{\alpha}_{m+k}(A_1) \sum_{s=m}^{j-1} f_s^{(m)} f_{l-s} \tag{17}$$

Proof: Let G denotes the number of customer arrivals in duration of a customer service time. For k∈N_+ and l∈N:

$$\begin{aligned} r_l(A)E[\bar{S}_l^k(A)] &= E\left[S^k(A) 1_{\{G=l\}}\right] \\ &= \int_0^\infty u^k \sum_{j=0}^{l} e^{-\lambda u} \frac{(\lambda u)^j}{j!} f_l^{(j)} A(du) \\ &= \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{k+j}}{(k+j)!} A(du) f_l^{(j)} \\ &= \sum_{j=0}^{l} \frac{(k+j)!}{\lambda^k j!} \alpha_{k+j}(A) f_l^{(j)} \end{aligned}$$

Finally, Eq. 16 follows from Eq. 15 since :

$$\sum_{l \ge n-1} r_l(A)E[\bar{S}_l^k(A)] = E[S^k(A)] - \sum_{l=0}^{n-2} r_l(A)E[\bar{S}_l^k(A)]$$

taking into account that:

$$E[S^m(A)] = \sum_{l \ge 0} r_l(A)E[\bar{S}_l^k(A)]$$

Researchers now address the computation of $E[\tilde{S}_{jl}^k]$ which goes as follows:

$$\begin{aligned} q_{jl}(A_1)E[\tilde{S}_{jl}^k] &= \int_0^\infty u^k \sum_{m=0}^{l} e^{-\lambda u} \frac{(\lambda u)^m}{m!} \sum_{s=m}^{j-1} f_s^{(m)} \lambda f_{l-s} \bar{A}_1(u) du \\ &= \lambda \sum_{m=0}^{j-1} \frac{(m+k)!}{\lambda^k m!} \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{m+k}}{(m+k)!} \bar{A}_1(u) du \\ &\sum_{s=m}^{j-1} f_s^{(m)} f_{l-s} \end{aligned}$$

which leads to Eq. 17 in view of Eq. 2. We using the above lemma to address the limit distribution Y. We first note that the state process Y is a MRGP with state space $E^{(n,a,b)}$ associated with the time sequence $(T_k)_{k \in N+}$ of post-customer departure epochs, therefore using Kulkarni (1995) (Theorem 3), we conclude that the limit probability vector of Y, $p = (p_d)_{d \in E^{(n,a,b)}}$ given by the following function of the limit post-customer departure state probability vector:

$$\pi = (\pi_c)_{c \in \tilde{E}^{(n,a,b)}}$$

$$p_d = \frac{\sum_{c \in \tilde{E}^{(n,a,b)}} \pi_c \phi_{cd}}{\sum_{c \in \tilde{E}^{(n,a,b)}} \pi_c \varphi_c} \tag{18}$$

$$d \in E^{(n,a,b)}$$

Where, φ_c denotes the mean time elapsed between two consecutive service completions conditioned on the state of the system after the first of these service completions being c, i.e.:

$$\varphi_c = E[T_{k+1} - T_k \mid Y(T_k^+) = c]$$

For:

$$c \in \tilde{E}^{(n,a,b)}$$

φ_{cd} denotes the expected sojourn time of Y in state d in-between two consecutive service completions conditioned on the state of the system after the first of these service completions being c, i.e.:

$$\phi_{cd} = E\left[\int_{T_k}^{T_{k+1}} 1_{\{Y(t)=d\}} dt \,\middle|\, Y(T_k^+) = c\right]$$

for;

$$c \in \tilde{E}^{(n,a,b)}, d \in E^{(n,a,b)}$$

A careful analysis leads to the following result on the computation of the mean time elapsed between two consecutive service completions conditioned on the state of the system after the first of these service completions being c, φ_c.

Theorem 2: In type I $M^{[x]}/G_1$-$G_2/1/n/(a, b)$ Systems:

$$\varphi_c = \begin{cases} \frac{1}{\lambda} + \frac{1}{\mu_2} & c = (0,1) \text{ and } b = 1 \\ \frac{1}{\lambda} + \frac{1}{\mu_1} & c = (0,1) \text{ and } b > 1 \\ \frac{1}{\mu_1} & c_2 = 1 \text{ and } c_1 > 0 \\ \frac{1}{\mu_2} & c_2 = 2 \end{cases} \quad (19)$$

and in type II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems:

$$\varphi_c = \begin{cases} \frac{1}{\mu_2} & c_2 = 2 \\ \frac{1}{\mu_2}\sum_{l\geq b-c_1} r_l(A_1) + \\ \frac{1}{\lambda}\sum_{l=0}^{b-c_1-1}\sum_{j=1}^{l+1} j\alpha_j(A_1) f_l^{(j-1)} \\ \frac{1}{\lambda}\sum_{l=1}^{b-c_1} l\sum_{s>l}\alpha_s(A_1) \\ \sum_{u=l-1}^{b-c_1-1} f_u^{(l-1)}\sum_{m\geq b-c_1-u} f_m & c_2 = 1 \text{ and } c_1 \neq 0 \\ \frac{1}{\lambda} + \sum_{l=1}^{b-1} f_l\varphi_{(l,1)} + \frac{1}{\mu_2}\sum_{l\geq b} f_l & c = (0,1) \end{cases} \quad (20)$$

Proof: We first note that Eq. 19 for type I oscillating systems follows similarly to the case of regular systems taking into account that the duration of a service initiated with the system in phase i has expected value $1/\mu_2$, i = 1, 2.

Suppose now that the oscillating system is of type II. The first branch of Eq. 20 follows from the fact that a service time initiated with the system in phase 2 has distribution function A_2 with mean $1/\mu_2$. In addition if $c = (c_1, 1)$, $c_1>0$ then φ_c is the mean duration of a service time initiated with the system in state c, for which conditioning on the number of customers that arrive to the system during the first service time, we obtain:

$$\varphi_c = \sum_{l=0}^{b-c_1-1} r_l(A_1)E[\bar{S}_l(A_1)] + \sum_{l\geq b-c_1} q_{b-c_1,l}(A_1)\left(E[\tilde{S}_{b-c_1,l}] + \frac{1}{\mu_2}\right) \quad (21)$$

Now, taking into account Lemma 2, we have:

$$\sum_{l=0}^{b-c_1-1} r_l(A_1)E[\bar{S}_l(A_1)] = \sum_{l=0}^{b-c_1-1}\sum_{j=0}^{l}\frac{1}{\lambda}(j+1)\alpha_{j+1}(A_1)f_l^{(j)} = \sum_{l=0}^{b-c_1-1}\sum_{j=1}^{l+1}\frac{1}{\lambda}j\alpha_j(A_1)f_l^{(j-1)} \quad (22)$$

Similarly, taking into account Eq. 17, in Lemma 2 and Eq. 11, we have:

$$q_{b-c_1,m}(A_1)E[\tilde{S}_{b-c_1,m}] = \frac{1}{\lambda}\sum_{l=1}^{b-c_1} l\sum_{s>l}\alpha_s(A_1)\sum_{u=l-1}^{b-c_1-1} f_u^{(l-1)} f_{m-u}$$

for $m \geq b-c_1$ so that:

$$\sum_{m\geq b-c_1} q_{b-c_1,m}(A_1)E[\tilde{S}_{b-c_1,m}] = \frac{1}{\lambda}\sum_{l=1}^{b-c_1} l\sum_{s>l}\alpha_s(A_1)\sum_{u=l-1}^{b-c_1-1} f_u^{(l-1)}\sum_{m\geq b-c_1} f_{m-u} \quad (23)$$

Now, the validation of the second branch of Eq. 20 follows from Eq. 21, by taking into account Eq. 22 and 23 and the fact that:

$$\sum_{l\geq b-c_1} q_{b-c_1,m}(A_1) = \sum_{l\geq b-c_1} r_l(A_1)$$

Finally, the third branch of Eq. 20 follows from the facts already established by conditioning on the size of the first batch arriving after a service completion that leaves the system empty, taking into account that the mean waiting time for this batch to arrive to the system is equal to $1/\lambda$. By conditioning on the number of customer arrivals in the first service that takes place after a service completion that leaves the system in state c, we conclude the following result.

Theorem 3: In $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems, $\phi_{cd} = 0$ if $d_1 \leq c_1$. In type I and II Systems:

$$\phi_{(c_1,2)(d_1,2)} = \begin{cases} \bar{r}_{d_1-c_1}(A_2) & c_1 \leq d_1 \leq n-1 \\ \sum_{l\geq n-c_1}\bar{r}_l(A_2) & d_1 = n \end{cases} \quad (24)$$

In turn, in type I systems:

$$\phi_{(c_1,1)d} = \begin{cases} \bar{r}_{d_1-c_1}(A_1) & (d_1 \geq c_1 \text{ and } d_2 = 1) \\ & \text{or } (b \leq d_1 < n \text{ and } d_2 = 2) \\ 0 & d_1 < b \text{ and } d_2 = 2 \\ \sum_{l\geq n-c_1}\bar{r}_l(A_1) & d_1 = n \end{cases} \quad (25)$$

and in type II:

$$\phi_{(c_1,1)d} = \begin{cases} \bar{r}_{d_1-c_1}(A_1) & d_1 \geq c_1 \text{ and } d_2 = 1 \\ 0 & d_1 < b \text{ and } d_2 = 2 \\ \sum_{l=b-c_1}^{d_1-c_1} q_{b-c_1,l}(A_1) & b \leq d_1 < n \text{ and } d_2 = 2 \\ \bar{r}_{d_1-c_1-l}(A_2) & \\ \sum_{l\geq b-c_1} q_{b-c_1,l}(A_1) & \\ \sum_{j\geq(n-c_1-l)}{}^{+}\bar{r}_j(A_2) & d_1 = n \end{cases} \quad (26)$$

if $c_1>0$. Moreover:

$$\phi_{(0,1)d} = \begin{cases} 1/\lambda & d_1 = 0 \\ \sum_{l=1}^{d_1} f_l(A)\phi_{(l,1)d} & d_1 > 0 \text{ and } d_2 = 1 \\ 0 & d_1 < b \text{ and } d_2 = 2 \\ \sum_{l=1}^{b-1} f_l \phi_{(l,1)d} + & b \le d_1 < n \\ \sum_{l=b}^{d_1} f_l \phi_{(l,2)d} & \\ \sum_{l=1}^{b-1} f_l \phi_{(l,1)(n,2)} & d_1 = n \\ \sum_{l\ge b} f_l \phi_{(l\wedge n,2)(n,2)} & \end{cases} \tag{27}$$

in type I and II Systems.

Algorithmic analysis: Researchers summarize the above results as a procedure to calculate the limit distributions of the post-customer departure state and the continuous time state in type I and II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ Systems as Fig. 1. This algorithm requires as input the mixed-poisson probabilities $(\alpha_l(A_1))_{0\le l\le n-2}$ and $(\alpha_l(A_2))_{0\le l\le n-2}$ along with the batch size probabilities $(f_l)_{1\le l\le n-2}$. The algorithm consists of eleven steps, with the first six steps including the computation of auxiliary quantities that are used in steps 7-11. The computation of the limit probability vector of the post-customer departure state $\pi = (\pi_c)$ is done in step 8 where 1 denotes a vector of ones.

The computation of the limit probability vector of the continuous time state p (p_c) is done in step 11 and requires the quantities computed in steps 8-10. We note that in considered the oscillating systems, the lower and upper barriers (respectively, a and b) are smaller or equal to n.

Algorithm
Input: N, A, B, λ, μ_1, μ_2, $(f_l)_{1\le l\le n-2}$, $(\alpha_l(A_1), \alpha_l(A_2))_{0\le l\le n-2})$
[Step 1] Compute $(f_l^{(j)})_{0\le j\le l\le n-2}$ using Eq. 1
[Step 2] Compute $(\bar{\alpha}_l(A_1), \bar{\alpha}_l(A_2))_{0\le l\le n-2}$ using Eq. 9 and 10
[Step 3] Compute $(q_{ml}(A))_{1\le m\le b-1,\, m\le l\le n-2}$ using Eq. 7 if the system is of type II
[Step 4] Compute $(r_l(A_1), r_l(A_2)_{0\le l\le n-2}$ using Eq. 2
[Step 5] Compute $\left(r_{(c_1,l)1}^{(b)}\right)_{1\le c_1\le b-1,\, b-c_1\le l\le n-2}$ using Eq.5 if the system is of type II
[Step 6] Compute $\left(\bar{r}_l(A_1), \bar{r}_l(A_2)\right)_{0\le l\le n-2}$ using
$\bar{r}_l(A_i) = \sum_{j=0}^{l} \bar{\alpha}_j(A_i) f_l^{(j)}$
[Step 7] Compute $P = (p_{ij})_{i,j\in \tilde{E}^{(n,a,b)}}$ using Eq. 12-14
[Step 8] Compute π such that πP and $\pi 1 = 1$
[Step 9] Compute $(\varphi_c)_{c\in \tilde{E}^{(n,a,b)}}$ using Eq. 19 if the system of type I and using Eq. 20 if the system is of type II
[Step 10] Compute $(\phi_{cd})_{c\in \tilde{E}^{(n,a,b)},\, d\in E^{(n,a,b)}}$ using Eq. 24-27
[Step 11] Compute $p = (p_c)_{c\in E^{(n,a,b)}}$ using Eq. 18
Output: $\pi = (\pi_c)_{c\in \tilde{E}^{(n,a,b)}}$ and $p = (p_c)_{c\in E^{(n,a,b)}}$

Fig. 1: Algorithm to compute the limit distributions of the post-customer departure state and the continuous time state in type I and II $M^{[x]}/G_1\text{-}G_2/1/n/(a, b)$ System

NUMERICAL RESULTS

As shown in Table 1, the values limit probability vectors of the number of customers at post-customer departures epochs and in continuous time in type I and II $M^{Geo(1/4)}/D(2)\text{-}D(5/4)/1/20/(7, 11)$ Systems with customer batch arrival rate $\lambda = 1/4$, with D(a) denoting the deterministic distribution with value a and Geo(1/4) the geometric distribution with parameter 1/4.

The result in Table1 show that the limit probability vectors of the number of customer at post-customer departure and in continuous time are similar for the compared type I and II $M^{Geo(1/4)}/D(2)\text{-}D(5/4)/1/20/(7, 11)$ systems.

Table 1: Limit probability vectors of the number of customers at post-customer departures epochs and in continuous time in type I and II $M^{Geo(1/4)}/D(2)\text{-}D(5/4)/1/20/(7, 11)$ Systems with batch arrival rate $\lambda = 1/4$

	Type I		Type II	
K	π'_k	p'_k	π'_k	p'_k
0	4.6378×10^{-3}	1.3358×10^{-2}	4.5711×10^{-3}	1.3185×10^{-2}
1	6.4870×10^{-3}	8.6658×10^{-3}	6.3937×10^{-3}	$8.5534\times10\text{-}^{3}$
2	8.8699×10^{-3}	$1\times1534\times10^{-2}$	8.7423×10^{-3}	$1\times1385\times10^{-2}$
3	$1\times1978\times10^{-2}$	1.5338×10^{-2}	$1\times1805\times10^{-2}$	1.5140×10^{-2}
4	1.6061×10^{-2}	2.0386×10^{-2}	1.5830×10^{-2}	2.0122×10^{-2}
5	$2\times1450\times10^{-2}$	2.7087×10^{-2}	$2\times1142\times10^{-2}$	$2.\ 6736\times10^{-2}$
6	2.8581×10^{-2}	3.5984×10^{-2}	2.8170×10^{-2}	3.5518×10^{-2}
7	3.8031×10^{-2}	4.7799×10^{-2}	3.7484×10^{-2}	4.7179×10^{-2}
8	4.3690×10^{-2}	4.3685×10^{-2}	4.3062×10^{-2}	4.3119×10^{-2}
9	4.8890×10^{-2}	4.6437×10^{-2}	4.8187×10^{-2}	4.5835×10^{-2}
10	5.3317×10^{-2}	4.7955×10^{-2}	5.2550×10^{-2}	$4.\ 7334\times10^{-2}$
11	5.7881×10^{-2}	5.2511×10^{-2}	5.7504×10^{-2}	5.2182×10^{-2}
12	6.2644×10^{-2}	5.6159×10^{-2}	6.2572×10^{-2}	5.6086×10^{-2}
13	6.7651×10^{-2}	6.0126×10^{-2}	6.7823×10^{-2}	6.0267×10^{-2}
14	7.2945×10^{-2}	6.4426×10^{-2}	7.3315×10^{-2}	6.4748×10^{-2}
15	7.8565×10^{-2}	6.9074×10^{-2}	7.9100×10^{-2}	6.9554×10^{-2}
16	8.4549×10^{-2}	7.4090×10^{-2}	8.5226×10^{-2}	7.4710×10^{-2}
17	9.0936×10^{-2}	7.9496×10^{-2}	$9\times1739\times10^{-2}$	8.0243×10^{-2}
18	9.7764×10^{-2}	8.5317×10^{-2}	9.8683×10^{-2}	8.6182×10^{-2}
19	1.050710^{-1}	$9\times1580\times10^{-2}$	1.0610×10^{-1}	9.2559×10^{-2}
20	-	4.8992×10^{-2}	-	4.9365×10^{-2}
Mean	13. 1145	12.8558	13.1568	12.9020
SD	4.57138	5.0917	4.5603	5.0822

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