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Formulation of Some Linear Multistep Schemes for Solving First Order Initial Value Problems Using Canonical Polynomials as Basis Functions

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Abstract: In this study, we shall present the derivation and application of some linear multistep methods in solving initial value problems for first order Ordinary Differential Equations (ODES) using canonical polynomials as basic functions. By interpolating and collocating at some selected points, some important classes of linear multistep methods were generated. The methods derived shall be compared with well known Adams Moulton method of the same order. Some numerical examples were given to illustrate the effectiveness of these methods.

Key words: Canonical polynomials, collocation, interpolation, multistep methods, Chebyshev polynomials

INTRODUCTION

We shall consider the general system

$$y` = f(x,y(x))\ ;\ a \le x \le b < \infty$$
$$y(a) = y_0$$

where f, y, ° R^n, $\|y\| < \infty$ for a suitable norm $\|.\|$. The numerical solution of ordinary differential equations by collocation methods have been extensively studied (Adeniyi and Alabi, 2007; Fairweather and Meade, 1989; Wright, 1970; Zennaro, 1985). In particular Wright (1970) established some relationships between certain Runge-Kutta methods and one step collocation methods. Sarafyan (1990) provided algorithms for continuous solution by Runge- Kutta methods with computation advantages. Lie and Nosett (1989) have recently developed a multistep collocation method which shows that the BDF methods and the one-leg methods of Dahlquist can be produced from their formulation if collocation is done at one point.

Some Linear Multistep schemes have been developed using some polynomials as basis functions. Such include the use of Chebyshev Polynomials to derive some classes of Linear Multistep methods (Adeniyi and Alabi, 2006; Adeniyi *et al.*, 2006). Others used some other methods as reported in the literature (Fatokun *et al.*, 2005; Sirisena *et al.*, 2001).

In this study, the canonical polynomials was used to derive some multistep methods. The recursive generation of the canonical polynomials with a wider scope of applications was proposed by Ortiz (1969). We briefly restate here the basic approach in Ortiz (1969).

Let y (x) be a known function which satisfies,

$$L^y(x) = \sum_{i=0}^{m} P_i(x) y^i(x) = f(x) \qquad (1)$$

and let

$$f(x) = f_0 + f_1 x + f_2 x^2 + \ldots + f_n x^n$$

Associated with the differential operator L in Eq. 1 is a sequence $\{Q_k(x)\}$, k °N_0-S of canonical polynomials $Q_k(x)$ such that

$$L\ Q_k(x) = x^k \qquad (2)$$

Where S is a small finite or empty set of indices with cardinality S ($S \le m + h$), h is the maximum difference between the exponent k of x and the leading exponent of the generating polynomial Lx^k for k ° N_0, $N_0 = \{0, 1, 2, \ldots\}$. The construction of $Q_k(x)$ using Lx^k is described exhaustively in Ortiz (1969).

$$\text{Let } y` = f(x,y)\ x \in [x_i, x_{i+n}]\ ;\ y(xi) = y_i$$

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And let

$$Ly = \left(\frac{d}{dx} + 1\right) y = g(x,y)$$

Where g (x, y) = y + f(x, y). Thus $Lx^r = rx^{r-1} + x^r$ and from Eq. 2 we have
$Lx^r = rL\ Q_{r-1}(x) + Lq_r(x)$. By linearising L we have $Lx^r = L\{r\,Q_{r-1}(x) + Q_r(x)\}$ and from the existence of L^{-1} we have
$x^r = r\,Q_{r-1} + Q_r$ which gives

$$Q_r(x) = x^r - rQ_{r-1}(x)\ r \circ N_0 \tag{3}$$

From Eq. 3 we obtain the following fundamental set of canonical polynomials uniquely associated with L

$$Q_0(x) = 1,\ Q_1(x) = x-1,\ Q_2(x) = x^2-2x+2, \text{ etc.}$$

In this study some linear multistep methods shall be derived using

$$y_k(x) = \sum_{r=0}^{k+1} a_r Q_r(x) \tag{4}$$

Where $Q_r(x)$ is as defined in (3)

DERIVATION OF THE SCHEME

In this study, we shall consider the case when n = 2 in Eq. 4, that is:

$$y_2(x) = a_0 + a_1(x-1) + a_2 (x^2 - 2x + 2) + a_3(x^3 - 3x^2 + 6x - 6) \tag{5}$$

Using y` = f(x, y(x)); $a \le x \le b < \infty$ on Eq. 2.1, we have

$$a_1 + 2a_2(x-1) + 3a_3(x^2 - 2x + 2) = f(x, y(x)) \tag{6}$$

Collocating Eq. 6 at $x = x_{n+1}$ and x_{n+2} and interpolating Eq. 5 at $x = x_n$ and x_{n+1}, we have the matrix equation

$$\begin{pmatrix} 1 & x_n - 1 & x_n^2 - 2x_n + 2 & x_n^2 - 3x_n^2 + 6x_n - 6 \\ 1 & x_{n+1} - 1 & x_{n+1}^2 - 2x_{n+1} + 2 & x_{n+1}^2 - 3x_{n+1}^2 + 6x_{n+1} - 6 \\ 0 & 1 & 2(x_{n+1} - 1) & 3(x_{n+1}^2 - 2x_{n+1} + 2) \\ 0 & 1 & 2(x_{n+2} - 1) & 3(x_{n+2}^2 - 2x_{n+2} + 2) \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \tag{7}$$

Solving the system of Eq. 7 for the value of a's, we have

$$\begin{aligned} a_3 &= \frac{1}{5h^3}\{hf_{n+2} - 3hf_{n+1} + 2(y_{n+1} - y_n\} \\ a_2 &= \frac{1}{h^2}\{hf_{n+1} - y_{n+1} + y_n - h^2(2x_{n+1} + x_n - 3)a_3\} \\ a_1 &= \frac{1}{h}(y_{n+1} - y_n) - a_2(x_{n+1} + x_n - 2) - a_3 \\ &(x^2{}_{n+1} + x_{n+1}x_n + x^2{}_n - 3x_{n+1} - 3x_n + 6) \\ a_0 &= y_n - a_1(x-1) - a_2(x^2{}_n - 2x_n + 2) \\ &-a_3(x^3{}_n - 3x^2{}_n + 6x_n - 6) \end{aligned} \tag{8}$$

Substituting (8) into Eq. 5 leads to

$$\begin{aligned} &\frac{y_n}{5h^3}\left\{\begin{matrix} 5h^3 - 5h^2(x - x_n) + 5h(x - x_n) \\ (x - x_{n+1}) - 2(x - x_n)(x - x_{n+1})^2 \end{matrix}\right\} + \\ &\frac{y_{n+1}(x - x_n)}{5h^3}\{5h^2 - 5h(x - x_n) + 2(x - x_{n+1})^2\} + \\ &\frac{f_{n+1}(x - x_n)(x - x_{n+1})}{5h^2}\{5h - 3(x - x_{n+1})\} + \\ &\frac{f_{n+2}(x - x_n)(x - x_{n+1})^2}{5h^2} \end{aligned} \tag{9}$$

Evaluating (9) at $x = x_{n+2,}$ leads to

$$y_{n+2} - \frac{1}{5}(4y_{n+1} + y_n) = \frac{h}{5}(2f_{n+2} + 4f_{n+1}) \tag{10}$$

Equation (9) is the continuous formulation of the discrete method (10) of order 3.

In like manner, when k = 3 in Eq. 4, we have

$$\begin{aligned} y_3(x) &= a_0 + a_1(x-1) + a_2(x^2 - 2x + 2) + \\ &a_3(x^3 - 3x^2 + 6x - 6) + \\ &a_4(x^4 - 4x^3 + 12x^2 - 24x + 24) \end{aligned} \tag{11}$$

Using equation y` = f(x, y(x)); $a \le x \le b < \infty$ on Eq. 11, we have

$$a_1 + 2a_2(x-1) + 3a_3(x_2 - 2x + 2) + 4a_4(x^3 - 3x^2\ 6x - 6) \tag{12}$$

Collocating Eq. 12 at $x = x_{n+1},\ x_{n+2},\ x_{n+3},$ and interpolating Eq. 11 $x = x_n,\ x_{n+2}$, we have the matrix equation

$$\begin{pmatrix} 1 & x_n-1 & x_n^2-2x_n+2 & x_n^3-3x_n^2+6x_n-6 \\ 1 & x_{n+2}-1 & x_{n+2}^2-2x_{n+2}+2 & x_{n+2}^3-3x_{n+2}^2+6x_{n+2}-6 \\ 0 & 1 & 2(x_{n+1}-1) & 3(x_{n+1}^2-2x_{n+1}+2) \\ 0 & 1 & 2(x_{n+2}-1) & 3(x_{n+2}^2-2x_{n+2}+2) \\ 0 & 1 & 2(x_{n+3}-1) & 3(x_{n+3}^2-2x_{n+3}+2) \end{pmatrix.$$

$$\left. \begin{matrix} x_n^4-4x_n^3+12x_n^2-24x_n+24 \\ x_{n+2}^4-4x_{n+2}^3+12x_{n+2}^2-24x_{n+2}+24 \\ 4\left(x_{n+1}^3-3x_{n+1}^2+6x_{n+1}-6\right) \\ 4\left(x_{n+2}^3-3x_{n+2}^2+6x_{n+2}-6\right) \\ 4\left(x_{n+3}^3-3x_{n+3}^2+6x_{n+3}-6\right) \end{matrix} \right)*$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+2} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \qquad (13)$$

Solving the matrix Eq. 13 for the values of a's and substituting them into Eq. (11) and evaluating at $x = x_{n+3}$ we have

$$y_{n+3} - \frac{1}{8}(9y_{n+2} - y_n) = \frac{3h}{8}(f_{n+3} + 2f_{n+2} - f_{n+1}) \qquad (14)$$

Equation (10) is a linear multistep method of order 4.

ORDER AND STABILITY REGION

For any linear multistep method to converge it must be consistent and zero stable (Lambert, 1973). It was discovered that Eq. 10 is of order 3 while Eq. 14 is of order 4.To determine the region of absolute stability of a Linear Multistep method, we use the boundary locus method given as:

$$h(\theta) = \frac{\rho e^{i\theta}}{\sigma e^{i\theta}} \qquad (15)$$

From Eq. (5)

$$\rho(r) = r^2 - \frac{4r}{5} - \frac{1}{5} \text{ and } \sigma(r) = \frac{1}{5}(2r^2 + 4r)$$

Therefore

$$h(\theta) = \frac{5e^{2i\theta} - \frac{4}{5}e^{i\theta} - \frac{1}{5}}{2e^{i\theta} + 4e^{i\theta}} \qquad (16)$$

On simplifying (16) we have

$$h(\theta) = x(\theta) + i\, y(\theta),$$

Where,

$$x(\theta) = \frac{-3 + 4\cos\theta - \cos 2\theta}{10 + 8\cos\theta}$$

and

$$y(\theta) = \frac{16\sin\theta + \sin 2\theta}{10 + 8\cos\theta}$$

Evaluating θ in the interval $[0,\pi]$, we discovered that Eq. 10 has interval of absolute stability between-4 and 0, that is $-4 \le \theta \le 0$ and Eq. 14 has interval of absolute stability of $-2^{2/3}$ and 0, that is $-2^{2/3} \le \theta \le 0$. Table 1 shows the order, interval of absolute stability and error constant of the two methods that is Eq. 10 and 14 together with that of Adams Moulton scheme of order 3 and order 4. That is

$$y_{n+2} - y_{n+1} = \frac{h}{12}(5f_{n+2} + 8f_{n+1} - f_n) \qquad (17)$$

$$y_{n+3} - y_{n+1} = \frac{h}{24}(9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n) \qquad (18)$$

NUMERICAL EXAMPLES

Here we shall solve some first order ordinary differential equation using Eq. 10 which is of order 3 and Eq. 14 which is of order 4 and the result obtained shall be compared with Adams Moulton scheme of order 3 and order 4 that is Eq. 17 and 18, respectively.

Example 1: Solve $y' = x\,y$; $y(0) = 1$ with $h = 0.1$

The analytic solution is: $y = \frac{e^{x^2}}{2}$

Example 2: Solve $y' = x + y$; $y(o) = 1$ with $h = 0.1$
The analytical solution is $y = 2e^x - (x + 1)$,

The numerical results to these problems were given in the Table 2-5.

Table 1: The order interval of absolute stability and error constant

Methods	Order	Absolute stability interval	Error constant
Adams-Moulton (Eq. 17)	3	[-6,0]	-1/24
Method (10)	3	[-4,0]	-1/30
Adams-Moulton (Eq. 18)	4	[-3,0]	-19/720
Method (14)	4	$[-2^{2/3},0]$	-18/720

ANALYSIS OF RESULTS

From the analysis presented in Table 3, it was shown that the new methods derived were of smaller error constant than the known Adams Moulton schemes of order three and order four and which is one of the valuable features of any multistep method (Lambert, 1973). Also it has been established that Canonical polynomials can be used as basis functions for deriving multistep methods.

In Table 2 and 3 in which the new method (10) of order three and Adams Moulton scheme of order three was used, it was discovered that the absolute error II is less them absolute error I, which shows that method (10) gives a better result when compared with Adams Moulton scheme of order three.The difference between the analytic solution and the Adams Moulton scheme and method (10) used give rise to absolute error I and absolute error II, respectively.

Also from Table 4 and 5, we discovered that comparing the result generated using method (14) with the exact solution and with the Adam Moulton scheme of order four, it was discovered that absolute error II, is smaller them absolute error I. This shows that method (14) gives a better approximate solution than Adams Moulton scheme of order 4.

Table 2: Solution to Example 1

X	Y EXACT	AAM	MTD	ABS ERROR I	AB: ERROR II
0.0	1.000000	1.00000000	1.00000000	0.0000000000	0.00000000
0.10	1.00501252	1.00501252	1.00501252	0.00000000	0.00000000
0.20	1.02020134	1.02021439	1.020211811	1.305E-05	1.0471E-05
0.30	1.040602786	1.04605543	1.04604792	2.757E-05	2.006E-05
0.40	1.08327078	1.08331912	1.08331895	3.2042E-05	3.1872E-05
0.50	1.13314845	1.13321404	1.13319485	6.559E-05	4.64E.05
0.60	1.9721736	1.19730948	1.1928223	9.212E-05	6.487E-05
0.70	1.27762131	1.27774777	1.27771019	1.2646E-04	8.888E-05
0.80	1.37712776	1.37729944	1.37724831	1.7168E-04	1.2055E-04
0.90	1.49930250	1.49953442	1.49946533	2.3192E-04	1.6283E-04
1.00	1.64872127	1.64903425	1.64894108	3.1298E-04	2.1981E-04

ABS ERROR I = AAM-Y. Exact; ABS ERROR II = MTD-Y Exact; AAM (Adams Moulton scheme of order 3); MTD (Method 2.10)

Table 3: Solutions to Example 2

X	Y EXACT	AAM	MTD	ABS ERROR I	AB: ERROR II
0.0	1.0000000000	1.0000000000	1.0000000000	0.000000000	0.00000000
0.10	1.1103418362	1.1103418362	1.1103418362	0.00000000	0.00000000
0.20	1.2428055162	1.2428152611	1.2428133498	9.745E-06	7.833E-06
0.30	1.3997176152	1.13997392312	1.3997334532	2.1616E-05	1.5838E-05
0.40	1.5836493953	1.5836852724	1.5836751133	3.5877E-05	2.5718E-05
0.50	1.7974425414	1.7974954403	1.7974799899	5.2899E-05	2.5718E-05
0.60	2.0442376008	2.0443107051	2.0442889727	7.3105E-05	3.7448E –05
0.70	2.3275054149	2.3276023897	2.327573229	9.6975E-05	6.7808E-05
0.80	2.6510818570	2.6512069146	2.6511689903	1.25057E-04	8.7133E-05
0.90	3.0192062223	3.0193641972	3.0193159959	1.57975E-04	1.09773E-04
1.00	3.4365636569	3.4367600898	3.4366998692	1.96433E-04	1.36213E-04

ABS ERROR I = AMM - Y Exact; ABS ERROR II = MTD-Y Exact; MTD (Method 2.6); AAM (Adams Moulton scheme of order 3)

Table 4: Solutions to Example I

X	Y –EXACT	AAM	MTD	ABS ERROR I	AB: ERROR II
0.00	1.0000000000	1.0000000000	1.0000000000	0.000000000	0.00000000
0.10	1.0050112509	1.0050125209	1.0050125209	0.00000000	0.00000000
0.20	1.0202013400	1.0202013400	1.0202-13400	0.0000000000	0.0000000000
0.30	1.0460278599	1.0460285733	1.0460283728	7.14E-07	5.13E-07
0.40	1.0832870677	1.083290032	1.0832881990	1.936E-06	1.132E-06
0.50	1.13314531	1.1331522489	1.1331506174	3.795E-06	2.164E-06
0.60	1.1972173631	1.1972238562	1.1972209453	6.493E-06	3.582E-06
0.70	1.2776213132	1.2776316321	1.2776269292	1.03193E-06	5.616E-06
0.80	1.3771277643	1.3771434578	1.3771362097	1.5693E-05	1.2411E-05
0.90	1.4993025001	1.4993257181	1.4993149117	2.3218E-05	1.2411E-05
1.00	1.6487212707	1.6487550235	1.6487392269	3.3753E-05	1.7956E-05

ABS ERROR I =AAM-Y Exact; ABS ERRORII = MTD-Y Exact; AAM (Adams Moulton scheme of order 4); MTD (Method 2.14)

Table 5: Solutions to Example 2

X	Y –EXACT	AAM	MTD	ABS ERROR I	AB: ERROR II
0.0	1.0000000000	1.0000000000	1.0000000000	0.000000000	0.00000000
0.10	1.1103418362	1.1103418362	1.1104318362	0.00000000	0.00000000
0.20	1.2428055163	1.2428055163	1.2428055163	0.0000000000	0.0000000000
0.30	1.3997176152	1.3997182664	1.3997180550	6.51E-07	4.40E-07

Table 5: Continue

X	Y –EXACT	AAM	MTD	ABS ERROR I	AB: ERROR II
0.40	1.5836493953	1.5836508453	1.5836501710	1.45E-06	7.76E-07
0.50	1.7974425414	1.7974449485	1.7974438308	2.407E-06	1.289E-06
0.60	2.0442376998	2.0442411503	2.0442394425	3.55E-06	1.842E-06
0.70	2.3275054149	2.3275103204	2.0442394425	3.55E-06	1.842 E-06
0.80	2.6510818570	2.6510883645	2.65108517	6.507E-06	4.265E-06
0.90	3.019206223	3.01922146145	3.0192104877	8.392E-06	4.265E-06
1.00	3.436563656	3.4365742582	3.4365690208	1.0602E-05	5.364E-06

ABS ERROR I = AAM-Y Exact; ABS ERROR II = MTD-Y Exact; AAM (Adams Moulton scheme of order 4); MTD (Method 2.10)

CONCLUSION

In light of the observations enumerated above, we concluded that methods (10) and (14) gives a better result than Adams Moulton schemes of order three and order four. This equally shows that the smaller the absolute error of a method, the better the result.

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