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Some Properties of Quasi- *Paranormal Operators

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Abstract: In this study, we will investigate some properties of quasi-*paranormal operators and some relations between normal operators and quasi-*paranormal operators. Also we study sufficient conditions for normal operators and if T is any quasi-*paranormal operator in H. 2000 AMS subject classification: 47B20 and 47B38.

Key words: Hilbert space, *-paranormal, quasi-* paranormal, riesz operators

INTRODUCTION

Let H be an arbitrary complex Hilbert space and let B(H) denote the algebra of all bounded operators on H.We say that a bounded linear operator T on H is quasi -*paranormal $\|T^*Tx\|^2 \leq \|T^3 x\| \, \|Tx\|$ if for each vector x in H (Arora and Thukral, 1990). In this study, we consider normality conditions on a quasi - *paranormal operator T such that T = Q + C for a compact operator C and a quasi -nilpotent operator Q. An operator means a bounded linear operator on a Hilbert space. For an operator T ϵ B(H), let $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ denote the spectrum, point spectrum ,continuous spectrum and the residual spectrum respectively. Let $N_T(\mu)$ be the μ-space of a quasi -*paranormal operator T, that is $N_T(\mu) = \{x \epsilon H: Tx = \mu x\}$. A vector $x \epsilon N_T(\mu)$ is said to be a proper vector for T. A scalar μ is said to be a proper value for the operator T if there exists a vector $x \neq 0$ such that $Tx = \mu x$. Then by some properties of quasi -*paranormal operators,it is easy to verify that $\{N_T(\mu): \mu \epsilon \sigma_p(T)\}$ is a family of mutually orthogonal reducing subspaces of H (Arora and Thukral, 1990). In this study, we study some properties of quasi-*paranormal operators and obtain some sufficient conditions for normal operators.

PRELIMINARIES

Definitions 1: An operator T on a Hilbert space H is said to be

- Normal if $T^*T = TT^*$;
- Quasi normal if T commutes with $T^* T$;
- Hyponormal if $T^*T - TT^* \geq 0$;
- *-Paranormal if $\|T^*x\|^2 \leq \|T^2 x\| \, \|x\|$ for all x ϵ H.

Definition 2: (Arora and Thukral, 1990).

An operator T is said to be quasi-*paranormal if $\|T^* Tx\|^2 \leq \|T^2 x\| \, \|x\|$ for each vector x in H

Arora and Thukral (1990) has proved by some properties of quasi-*paranormal operators that $\{N^T(\mu): \mu \epsilon \sigma_p(T)\}$ is a family of mutually orthogonal reducing subspaces of H.

Lemma 1: (Arora and Thukral, 1990).

Let T be a quasi- *paranormal operator. Then

- For any scalar μ, $N_T(\mu): \subset N_{T^*}(\overline{\mu})$
- For a fixed scalar μ and let $N = N_T(\mu)$ then N reduces T and T/n is normal .
- $N_T(\mu) \perp N_T$ whenever $\mu \neq \gamma$.
- If the μ -space of T are a total family, then T is normal.

Patel (1974) has proved the following theorem.

Theorem 1: Let A be a hyponormal operator and let B be *- paranormal operator. If A and B are doubly commutative (i.e. AB = BA and AB* =B*A), then AB is a *-paranormal operator.

In the following theorem we show that if a quasi-*paranormal operator doubly commutes with an isometric operator then the product is quasi-*paranormal.

Theorem 2: Let T be a quasi-*paranormal operator such that T doubly commutes with an isometric operator S. Then TS is quasi-*paranormal.

Proof: For a unit vector x in H

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$$\|(TS)^*(TS)x\|^2 = \|S^*T^*TSx\|^2 \leq \|ST^*TSx\|^2$$
$$\|T^*TSx\|^2 = \|T^*Tx\|^2 \leq \|T^3x\|\|Tx\|$$
$$\|ST^3x\|\|STx\| = \|S^3T^3x\|\|STx\|$$
$$\|(TS)^3x\|\|(TS)x\|.$$

Hence TS is a quasi-*paranormal operator.

MAIN RESULTS

Theorem: 3: If T is quasi-*paranormal then T can be expressed uniquely as the direct sum $T = T_1 \oplus T_2$ efined on the product space $H = H_1 \oplus H_2$ with the following properties.

- H_1 is spanned by the proper vectors of T
- T_1 is normal
- T_2 is quasi-* paranormal and $\sigma_p(T_2) = \phi$
- T is normal if and only if T_2 is normal.

Proof:

- Let $H_1 = \Sigma\oplus_{\mu\epsilon\sigma p(T)}\{NT(\mu)\}$. Then H_1 is spanned by the proper vectors of T. Since $N_T(\mu)$ is a closed linear subspace, H_1 is a closed linear subspace implies that $H = H_1 \oplus H_1^{\perp} = H_1 \oplus H_2$, where $H_2 = H_1^{\perp}$. Let $T_1 = T/H_1$ and $T_2 = T/H_2$ then $T = T_1 \oplus T_2$ uniquely.
- Let $H_1 = N_T(\mu) \oplus N_T(\alpha) \oplus \ldots$ and let $x \epsilon H_1$ then $x = x_\mu + x_\alpha + \ldots$ for any $x_\mu \epsilon N_T(\mu)$, $x_\alpha \epsilon N_T(\alpha), \ldots$ By Lemma 2,

$$\begin{aligned} T_1{}^*T_1(x) &= T_1{}^*T_1\,x_\mu + x_\alpha + \cdots) \\ &= \mu T_1{}^*x_\mu + \alpha T_1{}^*x_\alpha + \cdots \\ &= \mu\bar{\mu}x_\mu + \alpha\bar{\alpha}x_\alpha + \cdots \\ T_1{}^*T_1(x) &= T_1{}^*T_1x_\mu + x_\alpha + \cdots) \\ &= \bar{\mu}T_1x_\mu + \bar{\alpha}T_1x_\alpha + \cdots \\ &= \bar{\mu}\mu x_\mu + \bar{\alpha}\alpha x_\alpha + \cdots \end{aligned}$$

Thus $T_1{}^*T_1\,x = T_1T_1{}^*\,x$. Hence T_1 is normal.

- Let $x \epsilon H_2$ then $x = 0 + x \epsilon H = H_1 \oplus H_2$. Since T is quasi-*paranormal,

$$\|T_2{}^*T_2x\|^2 = \|T^*T(0+x)\|^2 \leq$$
$$\|T^3(0+x)\|\|T(0+x)\| = \|T_2{}^3x\|\|T_2x\|$$

for every unit vector x in H. Hence T_2 is quasi-*paranormal. We will claim that $\sigma p(T_2) = \phi$. Suppose that $\sigma_p(T_2) \neq \phi$. Let $\gamma \epsilon \sigma_p(T_2)$. Then there exists $x \neq 0 \epsilon H_2$ such that $(T_2 - \gamma)x = 0$.

$$T(0+x) = T_2x = \gamma(x) =$$
$$\gamma(0+x), \gamma(0+x), 0+x \in H$$

Thus $x = 0 + x \epsilon N_T(\gamma)$. Hence x is in H_1. This is a contradiction to $H_2 = H_1^{\perp}$ and $x \neq 0$.

- $(\Leftarrow)$ Since T_1 and T_2 are normal. $T = T_1 \oplus T_2$ is normal.

$$(\Rightarrow)T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \in H_1 \oplus H_1{}^{\perp}$$

Since $T = T_1 \oplus T_2$ and T_1 are normal

$$T^*T = \begin{pmatrix} T_1{}^*T_1 & 0 \\ 0 & T_2{}^*T_2 \end{pmatrix}$$

$$TT^* = \begin{pmatrix} T_1T_1{}^* & 0 \\ 0 & T_2T_2{}^* \end{pmatrix}$$

Thus $T_2{}^*T_2 = T_2T_2{}^*$. Hence T_2 is normal.

Theorem 4: If H is finite dimensional, every quasi-*paranormal operator T is normal.

Proof : By induction on the dimension n of H, for n = 1, every operator is normal. Assume the theorem holds for dimension <n. It can be shown that every linear mapping in a finite dimensional space has atleast one proper value (Berberian, 1961). By Lamma 2, let μ be a proper value for T and let $N = N_T(\mu)$, implies N reduces T and $T/_N$ is normal. But $N^{\perp}$ also reduces and $T/_{N_\perp}$ is normal. By the inductive assumption T is normal.

Definition 3: (Istratescu, 1981)

A point $\mu \epsilon \sigma(T)$ is called a Riesz point of T if (N(T - μ), R(T-μ)) reduces T and the following conditions hold.

- Dim $N(T - \mu) < \infty$
- $T/N(T - \mu)$ is nilpotent (i.e.: $\sigma(.) = \{0\}$)
- $T/R(T - \mu)$ is a homeomorphism.

Definition 4: (Istratescu, 1981)

An operator $T \epsilon B(H)$ is called a Riesz operator if every point $\mu \epsilon \sigma(T)$, $\mu \neq 0$ is a Riesz point.

Theorem 5: (West, 1966)

Every Riesz operator T on a Hilbert space can be decomposed into T = C + Q where C is a compact operator and Q is a quasi-nilpotent operator.

Let C(H) be the algebra of all compact operators on H. On H, let $\pi(T)$ be the image of T ϵ B(H) under the canonical mapping

$$\pi : B(H) \rightarrow B(H)/C(H)$$

Lemma 2: (Yoshino, 1 968)

Let T be an operator of the form T = A + C with a compact operator C and let M be an invariant subspace of T. Then $\mu \in \sigma_{ap}(T/_M)$ and $\mu \notin \sigma(A)$ imply $\mu \in \sigma_p(T/_M)$ so, $\sigma_p(T/_M) \subset \sigma(A)$.

Theorem 6: Let T be an operator of the form T = C + Q where C is a compact operator and Q is a quasi-nilpotent operator. If π (T) is quasi-*paranormal. Then T is a compact operator.

Proof: First we note that $A \geq 0$ implies that $\pi(A) \geq 0$ and $\sigma(A) = \{0\}$ implies $\sigma(\pi(A)) = \{0\}$ for each operator A ϵ B(H). Let T = C + Q with a compact operator C and a quasi-nilpotent operator Q. Then $\pi(T) = \pi(Q)$ so that $\sigma(\pi(T)) = \sigma(\pi(Q)) = \{0\}$. By hypothesis $\pi(T)$ is quasi-*paranormal operator (by considering B(H)/C(H) as an algebra of operator on some Hilbert space). Hence we have $\|\pi(T)=$ suo $\{|\mu|: \mu \in \sigma(\pi(T))\} = 0$, since $\pi(T)$ is normaloid. This shows that $\pi(T) = 0$ and T is a compact operator.

Lemma 3: (Arora and Thukral, 1990).

Let T be a quasi-*paranormal operator such that

$$T^{*p_1}\, T^{q_1} \ldots\ldots T^{*p_m}\, T^{q_m}$$

is completely continuous for some non-negative integers

$$p_1, q_1, p_2, q_2, \ldots\ldots p_m, q_m$$

Then T is normal.

Theorem 7: Let T be a quasi*paranormal operator of the form T = C + Q where C is a compact operator and Q is a quasi nilpotent operator. Then T is a normal operator.

Proof: Let T = C + Q with a compact operator C and a quasi-nilpotent operator Q. Since T is normaloid, there exists an element $\mu_0 \in \sigma(T)$ such that $\|T\| = |\mu_0|$ we may assume that $\mu_0 \neq 0$. Since $\mu_0 \in \sigma_p(T)$ and $\mu_0 \notin \sigma(Q)$, $\mu_0 \in \sigma(T)$. By lemma 3 $\|T\| = |\mu_0|$ and $\mu_0 \in \sigma_p(T)$ imply $\bar{\mu}_0 \in \sigma_p(T)$. Thus

$$\{x \in H : Tx = \mu_0 x\} \cap$$
$$\{x \in H : T^* x = \bar{\mu}_0 x\} \neq \{0\}$$

To prove the theorem, it suffices to follow the proof of Lemma 3.

Corollary 1: If T is a quasi-*paranormal operator and $T^{*p} T^p$ is a Riesz operator for non-negative integer p, then T is a normal operator.

Proof: Since $T^{*p} T^p$ is a self adjoint Riesz operator, it follow that $T^{*p} T^p$ is compact and the result follows from Lemma 3.

Corollary 2: If T is a quasi-*paranormal Riesz operator on Hilbert space, then its compact part $C \neq 0$.

Proof: Since T is quasi-*paranormal, T is normaloid. If there exists a decomposition with its compact part C = 0, then T is a quasi-nilpotent normaloid operator, it follows that T = 0.

Theorem 7: Let T = H + iJ be quasi-*paranormal operator. If

- A real element of $\sigma(T)$ must be zero.
- J is compact, then T is normal.

Proof: Since T is normaloid, $\|T\| = |\mu|$ for some $\mu \in \sigma(T)$. Because of condition (1), $\mu$is not real; $\mu - \bar{\mu} \neq 0$, since $\mu \in \sigma(T)$ and $\|T\| = |\mu|$, $\mu \in \sigma_{ap}(T)$. Let $\{x_n\}$ be a sequence of unit vectors such that $\|Tx_n - Mx_n\| \rightarrow 0$. Now for $\beta = \mathrm{Im}(\mu)$, $\beta \neq 0$

$$\text{Let } Tx_n = (H + iJ) = Hx_n + iJx_n$$

$$\begin{aligned} 0 \leftarrow \|Tx_n - \mu x_n\| &= \|(Hx_n + iJx_n) + (\mathrm{Re}(\mu)x_n + i\,\mathrm{Im}(\mu)x_n\| \\ &= \|(Hx_n - \mathrm{Re}(\mu)x_n) + i(Jx_n - \beta x_n)\| \\ &= \geq \|Jx_n - \beta x_n\| \geq 0, n \rightarrow \infty. \end{aligned}$$

Hence

$$Jx_n - \beta x_n \rightarrow 0, \text{as} n \rightarrow \infty$$

since J is a compact operator there exists a vector x such that $Jx_n \rightarrow x$. Let $x_0 = x/\beta$ then $x_n \rightarrow x$. and $x \neq 0$.
For,

$$\begin{aligned} \|\beta x_n - x\| &= \|\beta x_n - Jx_n + Jx_n - x\| \\ &\leq \|\beta x_n - Jx_n\| + \|Jx_n - x\| \rightarrow 0 \end{aligned}$$

Therefore as $\beta x_n \rightarrow x$ as $n \rightarrow \infty$.
Thus $x_0 \neq 0$, Now,

$$\begin{aligned}\|Tx_0 - \mu x_0\| &= \|(T - \mu I)(x_0 - x_n) + (Tx_n - \mu x_n)\| \\ &\leq \|(T - \mu I)(x_0 - x_n)\| + \|Tx_n - \mu x_n\| \\ &\leq \|(T - \mu I)\|\|(x_0 - x_n)\| + \|Tx_n - \mu x_n\| \to 0 \text{ as } n \to \infty\end{aligned}$$

Therefore we have $Tx_0 = \mu x_0$.

So that the family $F = \{N_T(\mu): \mu \epsilon \sigma(T)\}$ is a mutually orthogonal reducing subspace of T.

Let

$$H_0 = \sum_{\mu \in \sigma_p(T)} \oplus \{N_T(\mu)\}$$

Then H_0 reduces T and the restriction $T_0 = T/_{H0}$ is normal operator .It is sufficient to show that $T_0 = T/_{H0^\perp} = 0$.To prove the thorem it is sufficient to follow the proof of Lemma 3.

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