Journal of Modern Mathematics and Statistics 1 (1-4): 42-45, 2007
ISSN: 1994-5388

Characterization of Dual Half-Spin Geometry

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Abstract: A point-line geometry of type $D_{5,3}(F)$, F is a finite field , will be constructed to be isomorphic to the classical polar space of type $\Omega^+(10,F)$, moreover we present a characterization of such geometry.

Key words: Half-spin geometry-classical polar space, symplecton, polar space, point line

INTRODUCTION

Hanssenes (1998) presents an axiom system for many point-line geometries associated to finite buildings containing a polar subspace of rank at least three and half-spin geometries. Cooperstein (1977) and Cohen (1883) gave a characterization for $D_{5,5}(F)$ as a group isomorphic to $\Omega^+(10)$ which is called later Half-spin geometry. Buekenhout (1982) generalized the characterization to present a construction of building of type $D_{n,n}$ and $D_{n,1}$ where $n \geq 4$. Cohen (1984) gave description of two buildings of spaces of type of $D_{n,1}$ and $D_{4,2}$. A general technique was presented to generate subspaces such as Grassmannians (1987) $A_{n,3}$ and $A_{n,2}$ that are needed to recognized a point line geometry $D_{5,3}$. Z. Abdelsalam and Mohammad (2005) presented a general case for the class of geometries $D_{n,2}$ ($n \geq 5$), $D_{n,3}$ ($n \geq 6$) and $D_{n,4}$ ($n \geq 7$) that is a point-line geometry $D_{n,k}$ where $k \geq 2$ and $n \geq k+3$, he also gave a theorem which characterized, by axioms on points and lines, the geometry $D_{n,k}$ where $k \geq 2$ and $n \geq k+3$ (Abdel Salam, 2006). We see that $D_{n,k}$ where $k \geq 2$ and $n \geq k+3$ does not contain the case that $n \geq 5$ and $k \geq 3$, so we present a construction and a characterization of a point-line-geometry of type $D_{5,3}$.

First we present some definition of terminology's that will be used. For most of the following definitions (Buekenhout and Shutt, 1974).

Given a set I, a geometry Γ over I is an ordered triple $\Gamma = (X, *, D)$, where X is a set, D is a partition $\{X_i\}$ of X indexed by I, X_i are called components and $*$ is a symmetric and reflexive relation on X called incidence relation such that:

$x * y$ implies that either x and y belong to distinct components of the partition of X or x = y. Elements of X are called objects of the geometry and the objects within one component X_i of the partition are called the objects of type *i*. The subscripts that index the components are called types. The obvious mapping $\tau: X \rightarrow$, which takes each object to the index of the component of the partition containing it is called the type map τ.

A Point-Line geometry (P, L) is simply a geometry for which $|I| = 2$, one of the two types is called points; in this notation the points are the members of P and the other type is called lines. Lines are the members of L. If $p \in P$ and $l \in L$, then $p * l$ if and only if $p \in l$. In point-line geometry (P, L), we say that two points of P are collinear if and only if they are incident with a common line (We use the symbol ~ for collinear).

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there exist a chain of distinct subspaces $\varphi \neq X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$. For example rank(φ)=-1, rank($\{p\}$) = 0 where p is a point and rank(l)=1 where l a line.

$x^\perp$ means the set of all points in P collinear with x, including x itself.

A subspace of a point-line geometry $\Gamma = (P, L)$ is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. <X> means the intersection over all subspaces containing X, where $X \subseteq P$. Lines incident with more than two points are called thick lines, those incident with exactly two points are called thin lines. In a point-line geometry $\Gamma = (P, L)$, a path of length n is a sequence of n+1 $(x_0, x_1, .., x_n)$ where, (x_i, x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point. A geodesic from a point x to a point y is a path of minimal possible length with initial point *x* and end point *y*. We denote this length by $d_\Gamma(x, y)$, the length of the geodesic from x to y is called the distance between x and y. The diameter of the geometry is the maximal distance of points.

A geometry Γ is called connected if and only if for any two of its points there is a path connecting them. A subset X of P is said to be convex if X contains all points of all geodesics connecting two points of X.

A polar space is a point-line geometry $\Gamma = (P, L)$ satisfying the Buekenhout-Shult axiom:

For each point-line pair (p, l) with p not incident with l; p is collinear with one or all points of l, that is $|p^\perp \cap l| = 1$ or else $p^\perp \supset l$. Clearly this axiom is equivalent to saying that $p^\perp$ is a geometric hyperplane of Γ for every point $p \in P$.

A point-line geometry Γ=(P, L) is called a projective plane if and only if it satisfies the following conditions (Cooperstien, 1977):

- Γ is a linear space; every two distinct points x, y in P lie exactly on one line,
- Every two lines intersect in one point,
- There are four points no three of them are on a line.

A point-line geometry Γ = (P, L) is called a projective space if the following conditions are satisfied:

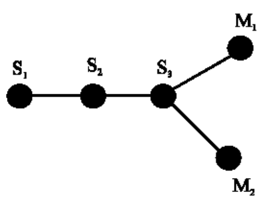
- Every two points lie exactly on one line ,
- If l_1, l_2 are two lines $l_1 \cap l_2 \neq \emptyset$, then $\langle l_1, l_2 \rangle$ is a projective plane. ($\langle l_1, l_2 \rangle$ means the smallest subspace of Γ containing l_1 and l_2.)

A point-line geometry Γ = (P, L) is called a parapolar space if and only if it satisfies the following properties:

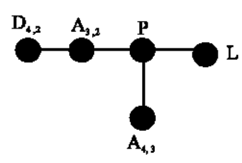
- Γ is a connected gamma space,
- For every line l; $l^\perp$ is not a singular subspace,
- For every pair of non-collinear points x, y; $x^\perp \cap y^\perp$ is either empty, a single point, or a non-degenerate polar space of rank at least 2.

If x, y are distinct points in P and if $|x^\perp \cap y^\perp|=1$, then (x, y) is called a special pair and if $x^\perp \cap y^\perp$ is a polar space, then (x, y) is called a polar pair (or a symplectic pair). A parapolar space is called a strong parapolar space if it has no special pairs (Hanssens, 1986).

Construction of $D_{5,3}$(F):



Polar space Δ



$D_{5,3}$(F)

Consider the classical polar space $\Delta = \Omega^+$ (10, F) that comes from a vector space of dimension 10 over a finite field F = GF(k) with a symmetric hyperbolic bilinear form. The two classes M_1, M_2 consist of maximal totally isotropic 5-dimensional subspaces. Two 5-subspaces fall in the same class if their intersection is of odd dimension.

The geometry of type $D_{5,3}$ (F) is the point-line geometry (P, L), whose set of points P is corresponding to the class S_3 that is: the collection of all totally isotropic 3-dimensional subspaces of the vector space V and whose lines are corresponding to the collection of all 5-dimensional subspaces of the vector space V that are fall in the class M_1. A point C is incident with a line B if and only if C⊂B as a subspaces of V.

To define the collinearity, let C_1 and C_2 be two point (the points are the T.I 3-spaces), then C_1 is collinear to C_2 if and only if the intersection of C_1 and C_2 is a T.I 1-dimensional space. This intersection in addition to the complement of C_1 and C_2 must form a T.I 5-dimensional space. The elements of the class M_2 are geometries of type $A_{4,3}$(F) .

The symplecta of $D_{5,3}$ (F) are the Grassmannians of type $A_{3,2}$(F) that are corresponding to the collection of TI 2 dimensional spaces.

Notation: Let the map Ψ: P→ V defined above, i.e., Ψ(p) is the T.I. 3-dimensional subspace corresponding to the point p. We will use Ψ for the rest of the geometry; for example $\Psi(D_{4,2})$ is the T.I. 1-dimensional subspace corresponding to a geometry of type $D_{4,2}$. The inverse map Ψ^{-1} will be used for the inverse; for example Ψ^{-1}(C) is the point corresponding to the T.I. 3-dimensional subspace C.

THE MAIN RESULTS

To make a characterization for the geometry $D_{5,3}$ we present the theorem.

Main theorem: Let Γ = (P, L) be a point-line geometry of type $D_{5,3}$, then the following are satisfied:

(P_1) Γ is a strong parapolar space of diameter 4.
(P_2) The symlecta of the geometry are of type $A_{3,2}$.
(P_3) If (p, S) is a pair of non-incident point-symplecton, then rank($p^\perp$n S)= -1 ,0, 2.
(P_4) If S_1 and S_2 are two different symplecta of $D_{5,3}$, then rank ($S_1 \cap S_2$) = -1, 0.

The proof of (P_1) is the proofs of the Propositions 1, 2 and 3. (P_3) is the Proposition 4, (P_4) is the Proposition 5 while (P_2) from the above construction and the diagram geometry of $D_{5,3}$.

Propostion: The diameter of the point-line geometry $D_{5,3}$(F) is equal 4.

Proof: We prove that max{d(p, q): p, q∈P} = 4. Let p, q be two points in P and Ψ(p) = <x_1, x_2, x_3>, Ψ(q) = <y_1, y_2, y_3> be their corresponding. Then we have two cases:

- $\Psi(p) \cap \Psi(q)$ = 1-space, say <x>, where x = x_1 = y_1. Then Ψ(p) = <x, x_2, x_3>, Ψ(q) = <y, y_2, y_3>, now we have the following case:

$$x_2^{\perp} \cap \Psi(q) = \Psi(q),$$

Then the two points p and q collinear in a line l such that $\Psi(l) = \langle x, x_2, x_3, y_2, y_3 \rangle$, so $d(p, q) = 1$.

- $\Psi(p) \cap \Psi(q)$ = 2-space, say, $\langle x, y \rangle$ where $x = x_1 = y_1$, $y = x_2 = y_2$ and $x_3^{\perp} \cap \Psi(q) = \Psi(q)$, then the 4-space $\langle x, y, x_3, y_3 \rangle$ is contained in a TI 5-space, $\langle u, x_3, x, y, y_3 \rangle$. Therefore, we find just two points r, s such that $\Psi(r) = \langle u, x, x_3 \rangle$, $\Psi(s) = \langle u, y, y_3 \rangle$ and each of the following are satisfied:
  - $\Psi(r) \cap \Psi(q)$ = 1-space = $\langle x \rangle$. Then r is collinear to q but is not collinear to p,
  - $\Psi(s) \cap \Psi(p)$ = 1-space = $\langle y \rangle$. Then s is collinear to p but is not collinear to q.
    at the same time $\Psi(r) \cap \Psi(s)$ = 1-space = $\langle u \rangle$. Then r is collinear to s in a line *l* where $\Psi(l) = \langle u, x, y, x_3, y_3 \rangle$, so $d(p, q) = 3$.
  - $\Psi(p) \cap \Psi(q)$ = 0-space , we have two cases:

$$\text{i- } y_1^{\perp} \cap \Psi(p) = \langle x_1, x_2 \rangle,$$
$$y_2^{\perp} \cap \Psi(p) = \Psi(p),$$
$$y_3^{\perp} \cap \Psi(p) = \Psi(p)$$

Then $\langle x_1, x_2, y_1, y_2, y_3 \rangle$ is a TI 5-space and r is a point such that $\Psi(r) = \langle x_1, y_1, x_2 \rangle$. Since $\Psi(r) \cap \Psi(q) = \langle y_1 \rangle$, r is collinear to q. Let s be another point such that $\Psi(s) = \langle x_2, y_2, y_3 \rangle$, then we have a TI 5-space $\langle x_1, x_2, y_1, y_2, y_3 \rangle$ and $\Psi(r) \cap \Psi(s) = \langle x_2 \rangle$ and s is collinear to r. we show that s is collinear p since $\Psi(p) \cap \Psi(s) = \langle x_2 \rangle$ and $\langle x_1, x_2, x_3, y_2, y_3 \rangle$ form a TI 5-space, so $d(p, q) = 3$.

$$y_1^{\perp} \cap \Psi(p) = \langle x_1, x_2 \rangle,$$
$$y_2^{\perp} \cap \Psi(p) = \langle x_2, x_3 \rangle$$
$$y_3^{\perp} \cap \Psi(p) = \langle x_1, x_3 \rangle$$

Then $\Psi(q)$ is contained in a Maximal TI 5-space $\langle u, v, y_1, y_2, y_3 \rangle$. Let r , s and t be three points where $\Psi(r) = \langle u, v, y_3 \rangle$, $\Psi(s) = \langle x_2, y_2, y_3 \rangle$ and $\Psi(t) = \langle x_2, u, v \rangle$, then

$$\Psi(r) \cap \Psi(q) = \langle y_3 \rangle,$$
$$\Psi(r) \cap \Psi(s) = \langle y_3 \rangle,$$
$$\Psi(t) \cap \Psi(p) = \langle x_2 \rangle,$$

Then r is collinear to r, r is collinear to s, s is collinear to t and t is collinear to p, so $d(p, q) = 4$, this complete the proof.

The following two propositions proved that the geometry $D_{5,3}(F)$ is strong parapolar.

Proposition: $D_{5,3}(F)$ is a strong geometry.

Proof: We prove that the geometry has no special pair. For any two points p, q where, $\Psi(p) = \langle x_1, x_2, x_3 \rangle$ and $\Psi(q) = \langle y_1, y_2, y_3 \rangle$, there are three cases:

- $\Psi(p) \cap \Psi(q)$ = 1-space = $\langle x \rangle$, $x = x_1 = y_1$ and $\langle x, x_2, x_3, y_2, y_3 \rangle$ is a TI 5-space. Then $d(p,q) = 1$ and $p^{\perp} \cap q^{\perp} = \varphi$, so in this case p and q has no special pair.
- $\Psi(p) \cap \Psi(q)$ = 2-space = $\langle x, y \rangle$, $x = x_2 = y_3$ and $y = x_3 = y_2$. Let $\langle u, x_1, x, y, y_1 \rangle$ be a TI 5-space that contains the 4-space $\langle x_1, x, y, y_1 \rangle$. Then we find the two points r, s such that $\Psi(r) = \langle u, y, y_1 \rangle$ and $\Psi(s) = \langle u, x, x_1 \rangle$. Therefore $r \sim q$ but r is not collinear to p and $s \sim p$ but s is not collinear to q moreover $r \sim s$. Then $d(p, q) = 3$ and $p^{\perp} n q^{\perp} = \varphi$, so in this case p and q has no special pair.
- $\Psi(p) \cap \Psi(q)$ = 0-space. This case leads to $d(p, q) = 4$, see Proposition 1 and then $p^{\perp} n q^{\perp} = \varphi$, so p and q has no special pair. From all cases the geometry is strong.

Proposition: $D_{5,3}(F)$ is a prapolar geometry:

Proof: Firstly we show that $D_{5,3}(F)$ is gamma space, let (p, l) be a non-incidence pair of a point p and a line l where $\Psi(p) = \langle x_1, x_2, x_3 \rangle$ and $\Psi(l) = \langle u_1, u_2, u_3, u_4, u_5 \rangle$ are the corresponding image. Then the intersection $\Psi(p) \cap \Psi(l)$ has three cases:

- $\Psi(p) \cap \Psi(l)$ = 2-space = $\langle x, y \rangle$, $x = x_2 = u_4$, $y = x_3 = u_5$. Then $x_1^{\perp} \cap \Psi(l)$ = 4-space = $\langle x, y, u_2, u_3 \rangle$ and we can find a point r incident to l where $\Psi(r) = \langle x, u_2, u_3 \rangle$. Since $\Psi(r) \subseteq \Psi(l)$ and $\Psi(p) \cap \Psi(r) = \langle x \rangle$, $p \sim r$ i.e., $p^{\perp} \cap l$ is a point.
- $\Psi(p) \cap \Psi(l)$ = 1-space = $\langle x \rangle$, $x = x_3 = u_5$,

$$x_1^{\perp} \cap \Psi(l) = \text{4-space} = \langle x, u_1, u_2, u_3 \rangle,$$
$$x_1^{\perp} \cap \Psi(l) = \text{4-space} = \langle x, u_4, u_2, u_3 \rangle.$$

Then there is a point $\Psi(r) = \langle x, u_2, u_3 \rangle$ where $\Psi(p) \cap \Psi(r)$ = 1-space = $\langle x \rangle$ and $\Psi(l') = \langle x, x_1, x_2, u_2, u_3 \rangle$ is a line incident to the points p and r. Then $r \in l$ and $p \sim r$. i.e., $p^{\perp} n\, l$ is a point.

- $\Psi(p) \cap \Psi(l)$ = 0-space ,

$$x_1^{\perp} \cap \Psi(l) = \text{4-space} = \langle u_1, u_2, u_3, u_4 \rangle,$$
$$x_2^{\perp} \text{ or } x_3^{\perp} \cap \Psi(l) = \text{4-space} = \langle u_2, u_3, u_4, u_5 \rangle.$$

Then there is no any TI 3-space in $\Psi(l)$ intersects $\Psi(p)$ in 1-space. i.,e. there is no any point in l collinear to p. Then $p^{\perp} \cap l = \varphi$. The previous three cases proved that the geometry is a connected gamma space. The remaining part of the proof is to show that for any two non-collinear points p and q, $p^{\perp} \cap q^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2. By Proposition 1 we showed that for any pair of non-collinear points p and q, d(p, q) = 3 or 4, then $p^{\perp}\cap q^{\perp}$ is empty and then for any line l, $l^{\perp}$ is not singular subspace. This completes the proof.

PROPERTIES OF $D_{5,3}(F)$

The most important properties that will be used for characterization are related to the relations between the different varieties of the geometry. The relations between the points and the symplecta and the relation between symplecta themselves will be investigated.

Proposition: Let (p, S) be a non-incidence pair of a point p and a symplecton S. Then $rank(p^{\perp}\cap S) = -1, 0$ or 2.

Proof: Let $\Psi(p) = \langle x_1, x_2, x_3\rangle$ and $\Psi(S) = \langle y_1, y_2\rangle$ be the corresponding TI 3-space and 2-space to the point p and the symplecta S, respectively. Then there are two cases for the intersection $\Psi(p)\cap\Psi(S)$:

Case 1: $\Psi(p)\cap\Psi(S)$ = 1-space = $\langle x\rangle$, where $x = x_1 = y_1$ and $y_2^{\perp}\cap\Psi(p) = \Psi(p)$. Then $\langle x_2, x_3, x, y_2\rangle$ can be extended to a maximal TI 5-space, $\langle x_2, x_3, x, y_2, u\rangle$ and we can find a point r, where $\Psi(r) = \langle u, x, y_2\rangle$. Then

$$\Psi(S)\subseteq\Psi(r),$$
$$\Psi(p)\cap\Psi(S) = 1\text{-space} = \langle x\rangle$$

And $\langle u, x_2, x_3, x, y_2\rangle$ is the corresponding TI 5-space to the line that is incident to the points p and r. Then $r\in S$, $r\sim p$ and $p^{\perp}\cap S$ is a point i.e., $rank(p^{\perp}\cap S) = 0$.

Case 2: $\Psi(p)\cap\Psi(S)$ = 0-space. In this case there are three cases;

- $y_1^{\perp}\cap\Psi(p) = \langle x_1, x_2\rangle$ and $y_2^{\perp}\cap\Psi(p) = \langle x_2, x_3\rangle$. Then we cannot find any TI 5-space corresponding to a line incident to the point p and a point r in S. Then $p^{\perp}\cap S = \varphi$, i.,e. $rank(p^{\perp}\cap S) = -1$.
- $y_1^{\perp}\cap\Psi(p) = \langle x_1, x_2\rangle$ and $y_2^{\perp}\cap\Psi(p) = \Psi(p)$. Since $B(y_1, x_3)\cap 0$, we cannot find a TI 3-space contains $\Psi(S) = \langle y_1, y_2\rangle$ and constitute a TI 5-space includes $\Psi(p)$. Then $p^{\perp}\cap S = \varphi$, i.,e. $rank(p^{\perp}\cap S) = -1$.
- $y_1^{\perp}\cap\Psi(p) = \Psi(p)$ and $y_2^{\perp}\cap\Psi(p) = \Psi(p)$. Then there are many points in S collinear to the point p that are : $\Psi(r) = \langle x_1, y_1, y_2\rangle$, $\Psi(s) = \langle x_2, y_1, y_2\rangle$ and $\Psi(r) = \langle x_3, y_1, y_2\rangle$. Since $\langle y_1, y_2, x_1, x_2, x_3\rangle$ constitute a TI 5-space, all points r, s and t are contained in S and each of them is collinear to p. Then $p^{\perp}\cap S$ is a plane. i.,e. $rank(p^{\perp}\cap S) = 2$. Then $p^{\perp}\cap S$ is either empty a point or a plane i.,e. $rank(p^{\perp}\cap S) = -1, 0$ or 2.

Proposition: Let S_1 and S_2 be two symplecta in $D_{5,3}(F)$. Then $rank(S_1\cap S_2) = -1$ or 0 .

Proof: Let $\Psi(S_1) = \langle x_1, x_2\rangle$ and $\Psi(S_2) = \langle y_1, y_2\rangle$ be the corresponding TI 2-spaces to the symplecta S_1 and S_2 respectively. Then there are two cases for the intersection $\Psi(S_2)\cap\Psi(S_2)$:

- $\Psi(S_2)\cap\Psi(S_2)$ = 1-space = $\langle x\rangle$, where $x = x_2 = y_2$ and $y_1^{\perp}\cap\Psi(S_1) = \Psi(S_1)$. Then $\Psi(p) = \langle x_1, x, y_1\rangle$ is the TI 3-space that corresponds to a point p which contains $\Psi(S_1)$ and $\Psi(S_2)$. i.,e.,

$$\Psi(S_1)\subseteq\langle x_1, x, y_1\rangle,$$
$$\Psi(S_2)\subseteq\langle x_1, x, y_1\rangle$$

Then $S_1 n S_2$ is a point.

- $\Psi(S_2)\cap\Psi(S_2)$ = 0-space. Then there is no any TI 3-space contains each of $\Psi(S_2)$ and $\Psi(S_2)$, which means that $S_1\cap S_2 = \varphi$. By the last two cases, rank $(S_1\cap S_2) = -1$ or 0 .

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