

## Accurate Collocation Multistep Method for Integration of First Order Ordinary Differential Equations

E. A. Areo, R.A. Ademiluyi and P.O. Babatola

Department of Mathematical Sciences, Federal University of Technology, Akure,

P.M.B 704, Akure Ondo State, Nigeria

**Abstract:** In this study we present a collocation multistep method for Integration of first order ordinary differential equations. It is consistent (order seven) zero stable and convergent. When compared with existing multistep method, it is found to be more accurate.

**Key words:** Collocation, multistep, integration, consistent, zero stable and convergent

### INTRODUCTION

The solution of initial value problems of ordinary differential equations of the form

$$y' = f(x,y), y(x_0) = y_0 \quad (1)$$

Where,  $y, f \in \mathbb{R}^n$ ,  $x \in [a,b]$ , has been discussed by various researchers among them are Lie and Norsett, (1989), Onumanyi *et al.* (1994, 1999), Onumanyi and Yusuph (2002), Sirisena (2004), Lambert, (1973) and Gear, (1971). However experience has shown in Lie and Norsett (1989) and Onumanyi *et al.* (1994) that the traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation. These earlier works have focused on the construction of continuous multistep methods by employing the multistep collocation. The continuous multistep methods produce piecewise polynomial solutions over  $k$ -steps  $[x_n, x_{n+k}]$  for the first order systems of Ordinary Differential Equation (ODEs). Sirisena *et al.* (2004) developed a continuous new Butcher type two-step block hybrid multistep method for problem (1). The results obtained showed a class of discrete schemes of order 5 and error constants ranging from  $C_6 = 1.45 \times 10^{-5}$  to  $C_6 = 1.790 \times 10^{-4}$ . In this study, we propose a continuous Butcher type three- step block hybrid method employing multistep collocation approach, which yields a class of 2 discrete schemes of order 7 with error constants.

$$C_8 = -\frac{27}{777420} \text{ and } C_8 = \frac{155525}{1273724928} \text{ for solving problem (1.1).}$$

### DEVELOPMENT OF THE METHODS

In this study we discussed the development of continuous scheme and its discrete schemes using Sirisena (1997) where a  $K$ -step multistep collocation method with  $m$  collocation points was obtained as follows:

$$\bar{y}(x) = \sum_{j=0}^{t-1} \alpha_j(x)y(x_{n+j}) + h \sum_{j=0}^{m-1} \beta_j(x)f(\bar{x}_j, \bar{y}(\bar{x}_j)) \quad (2)$$

Where:

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \quad (3)$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1} x^i \quad (4)$$

are the continuous coefficients of the method and  $x_{n+j}$ ,  $j = 0, 1, \dots, t-1$  in (2) are  $t$  ( $0 < t \leq k$ ) arbitrary chosen interpolation points from  $(x_n, \dots, x_{n+k})$  and,  $\bar{x}_j$ ,  $j = 0, 1, \dots, m-2$  are the  $m$  collocation points belonging to  $\{x_n, \dots, x_{n+k}\}$ .

To determine  $\alpha_j(x)$  and  $\beta_j(x)$ , we use a matrix equation of the form

$$DC = I \quad (5)$$

Where,

$I$  is an identity matrix

While  $D$  and  $C$  are the matrices defined as in Sirisena (1997).

$$D = \begin{bmatrix} 1 & x_n & x_n^2 \dots & x_n^{t+m-2} \\ 1 & x_{n+1} & x_{n+1}^2 \dots & x_{n+1}^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 \dots & x_{n+t-1}^{t+m-2} \\ 0 & 1 & 2\bar{x}_0 \dots & (t+m-2)\bar{x}^{t+m-3} \\ \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots \\ 0 & 1 & 2\bar{x}_{m-1} \dots & (t+m)\bar{x}_{m-1}^{t+m-3} \end{bmatrix} \tag{6}$$

and

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} \dots & \alpha_{t-1,1} & h\beta_{0,1} \dots & h\beta_{n-1,1} \\ \alpha_{0,2} & \alpha_{1,2} \dots & \alpha_{t-1,2} & h\beta_{0,2} \dots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} \dots & \alpha_{t-1,t+m} & h\beta_{0,t+m} \dots & h\beta_{m+1,t+m} \end{bmatrix} \tag{7}$$

The columns of the matrix  $C = D^{-1}$  consists of the continuous coefficients, i.e.,

$$\alpha_j(x); j=0, 1 \dots k-1 \text{ and } \beta_j(x); j=0, 1 \dots k-1.$$

In this study

$$k = t = 3, m = 6, \bar{x}_0 = x_n, \bar{x}_n, \bar{x}_1 = x_{p+1}, \bar{x}_2 = x_{n+2}.$$

Then Eq. 2 becomes

$$\bar{y}(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y + h[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_{5/2}(x)f_{n+5/2}] \tag{8}$$

Thus, the matrix D in (6) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+5/2} & 3x_{n+5/2}^2 & 4x_{n+5/2}^3 & 5x_{n+5/2}^4 & 6x_{n+5/2}^5 & 7x_{n+5/2}^6 \end{bmatrix} \tag{9}$$

We obtained  $C = D^{-1}$  in (9) to determine  $\alpha_i(x); i = 0(1)2$  and  $h\beta_i(x); i = 0, 1, 2, 3, 5/2$  in (8) as follows:

$$\alpha_0(x) = \frac{1}{2468h^7} [-20115h^5(x-x_n)^2 + 39189h^4(x-x_n)^3 - 32910h^3(x-x_n)^4 + 14169h^2(x-x_n)^5 - 3065h(x-x_n)^6 + 264(x-x_n)^7 + 2468h^2] \tag{10}$$

$$\alpha_1(x) = \frac{1}{617h^7} [-180h^5(x-x_n)^2 + 6148h^4(x-x_n)^3 - 10005h^3(x-x_n)^4 + 6156h^2(x-x_n)^5 - 1670h(x-x_n)^6 + 168(x-x_n)^7] \tag{11}$$

$$\alpha_2(x) = \frac{1}{2468h^7} \left[ 20835h^5(x-x_n)^2 - 6378h^4(x-x_n)^3 + 72930h^3(x-x_n)^4 - 38793h^2(x-x_n)^5 + 9745h(x-x_n)^6 - 936(x-x_n)^7 \right] \quad (12)$$

$$h\beta_0(x) = \frac{1}{37020h^6} \left[ 37020h^6(x-x_n) - 136364h^5(x-x_n)^2 + 200531h^4(x-x_n)^3 - 150680h^3(x-x_n)^4 + 61199h^2(x-x_n)^5 - 12782h(x-x_n)^6 + 1076(x-x_n)^7 \right] \quad (13)$$

$$h\beta_1(x) = \frac{1}{7404h^6} \left[ -68220h^5(x-x_n)^2 + 182932h^4(x-x_n)^3 + 1861147h^3(x-x_n)^4 - 90946h^2(x-x_n)^5 - 21483h(x-x_n)^6 - 1972(x-x_n)^7 \right] \quad (14)$$

$$h\beta_2(x) = \frac{1}{2468h^6} \left[ -11250h^5(x-x_n)^2 + 35645h^4(x-x_n)^3 - 42556h^3(x-x_n)^4 + 23805h^2(x-x_n)^5 - 6272h(x-x_n)^6 + 628(x-x_n)^7 \right] \quad (15)$$

$$h\beta_3(x) = \frac{1}{7404h^6} \left[ -980h^5(x-x_n)^2 + 3308h^4(x-x_n)^3 - 4289h^3(x-x_n)^4 + 2666^2(x-x_n)^5 - 797h(x-x_n)^6 + 92(x-x_n)^7 \right] \quad (16)$$

$$h\beta_{\frac{3}{2}}(x) = \frac{1}{9255h^6} \left[ 9216h^5(x-x_n)^2 - 30464h^4(x-x_n)^3 + 38400h^3(x-x_n)^4 - 22976h^2(x-x_n)^5 + 6528h(x-x_n)^6 - 704(x-x_n)^7 \right] \quad (17)$$

Putting Eq. 10-17 into Eq. 8, we obtained a continuous scheme.

$$\begin{aligned} \bar{y}(x) = & \frac{y_n}{2468h^7} \left[ -20115h^5(x-x_n)^2 + 39189h^4(x-x_n)^3 - 32910h^3(x-x_n)^4 + 14169h^2(x-x_n)^5 - 3065h(x-x_n)^6 + 264(x-x_n)^7 + 2468h^7 \right] \\ & + \frac{y_{n+1}}{617h^7} \left[ -100h^5(x-x_n)^2 + 6148h^4(x-x_n)^3 - 10005h^3(x-x_n)^4 + 6156h^2(x-x_n)^5 - 1670h(x-x_n)^6 + 168(x-x_n)^7 \right] \\ & + \frac{y_{n+2}}{2468h^7} \left[ 20835h^5(x-x_n)^2 - 63781h^4(x-x_n)^3 + 72930h^3(x-x_n)^4 - 38793h^2(x-x_n)^5 + 9745h(x-x_n)^6 + 936(x-x_n)^7 + \frac{f_n}{37020h^6} (37020h^6(x-x_n)) \right] \\ & - 136364h^5(x-x_n)^2 + 200531h^4(x-x_n)^3 - 150680h^3(x-x_n)^4 + 61199h^2(x-x_n)^5 \\ & - 12782h^2(x-x_n)^6 + 1076(x-x_n)^7 + \frac{f_{n+1}}{7404h^6} \left[ -62280h^5(x-x_n)^2 + 182932h^4(x-x_n)^3 \right. \\ & \left. - 186147h^3(x-x_n)^4 + 90946h^2(x-x_n)^5 - 21483h(x-x_n)^6 + 1972(x-x_n)^7 \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{f_{n+2}}{2468h^6} [-11250h^5(x-x_n)^2 + 35645h^4(x-x_n)^3 - 42556h^3(x-x_n)^4 + 23805h^2(x-x_n)^5 \\
 & - 6272h(x-x_n)^6 + 628(x-x_n)^7] + \frac{f_{n+3}}{7404h^6} [-980h^5(x-x_n)^2 \\
 & + 3308h^4(x-x_n)^3 + 4289h^3(x-x_n)^4 + 2666h^2(x-x_n)^5 - 797h(x-x_n)^6 + 92(x-x_n)^7] \\
 & + \frac{f_{n+\frac{5}{2}}}{9255h^6} [921h^5(x-x_n)^2 - 30464h^4(x-x_n)^3 + 38400h^3(x-x_n)^4 - 22976h^2(x-x_n)^5 \\
 & + 6528h(x-x_n)^6 - 704(x-x_n)^7]
 \end{aligned} \tag{18}$$

On evaluating (18) at  $x = x_{n+3}$  and  $x = x_{n+\frac{5}{2}}$ , we obtained the following 2 discrete equations.

$$y_{n+3} - \frac{783}{617}y_{n+2} + \frac{135}{617}y_{n+1} + \frac{31}{617}y_n = \frac{h}{18510} [-234f_n - 2970f_{n+1} - 810f_{n+2} + 2790f_{n+3} + 13824f_{n+\frac{5}{2}}] \tag{19}$$

and

$$y_{n+\frac{5}{2}} - \frac{4077}{157952}y_n - \frac{29000}{157952}y_{n+1} + \frac{124875}{157952}y_{n+2} = \frac{h}{157952} [-990f_{n+1} + 16125f_{n+1} + 67500f_{n+2} - 1125f_{n+3} + 32640f_{n+\frac{5}{2}}] \tag{20}$$

The schemes (18) and (19) has order  $p = 7$ , error constants.

$$C_8 = \frac{15525}{1273724928} \text{ and } C_8 = \frac{-27}{777420}, \text{ respectively}$$

Since the order  $p > 1$ , then the Eq. (19) and (20) are consistent as in Lambert (1973). Equation 19 and 20 are two equations with four unknowns. For the two equations to constitute two member block hybrid method, we need to eliminate two unknowns either by using existing standard one-step method or using the analytical solution or develop two more equations. What we adopted is discussed in the next section.

### STARTING VALUES

We adopted the explicit sixth order Runge-Kutta scheme in (Lambert, 1973) to evaluate  $y_{n+j}$ ;  $j=1$  and  $2$ ;  $n = 0$  i.e.

$$y_{n+j} = y_{n+j-1} + \frac{h}{840} [41k_1 + 216k_3 + 27k_4 + 272k_5 + 27k_6 + 216k_7 + 41k_8] \tag{21}$$

Where,

$$\begin{aligned}
 & k_1 = f(x_n, y_{n+j-1}) \\
 & k_2 = f(x_n + \frac{h}{9}, y_{n+j-1} + \frac{h}{9}k_1) \\
 & k_3 = f(x_n + \frac{h}{6}, y_{n+j-1} + \frac{h}{24}(k_1 + 3k_2)) \\
 & k_4 = f(x_n + \frac{h}{3}, y_{n+j-1} + \frac{h}{6}(k_1 + 3k_2 + 4k_3)) \\
 & k_5 = f(x_n + \frac{h}{2}, y_{n+j-1} + \frac{h}{8}(-5k_1 + 27k_2 - 24k_3 - 6k_4)) \\
 & k_6 = f(x_n + \frac{2h}{3}, y_{n+j-1} + \frac{h}{9}(22k_1 - 98k_2 + 867k_3 - 102k_4 + k_5)) \\
 & k_7 = f(x_n + \frac{5h}{6}, y_{n+j-1} + \frac{h}{48}(-183k_1 + 678k_2 - 472k_3 - 66k_4 + 80k_5 + 3k_6)) \\
 & k_8 = f(x_n + h, y_{n+j-1} + \frac{h}{840}(716k_1 - 2079k_2 + 1002k_3 + 834k_4 - 454k_5 - 9k_6 + 72k_7))
 \end{aligned} \tag{22}$$

Table 1: Comparison of Errors

X	Sirisena <i>et al.</i> (2004)	Proposed scheme
0.1	2.0×10 <sup>9</sup>	2.1×10 <sup>10</sup>
0.2	2.0×10 <sup>9</sup>	2.2×10 <sup>10</sup>
0.3	1.0×10 <sup>9</sup>	6.0×10 <sup>10</sup>
0.4	2.0×10 <sup>9</sup>	1.0×10 <sup>10</sup>
0.5	1.0×10 <sup>9</sup>	4.1×10 <sup>9</sup>
0.6	3.0×10 <sup>9</sup>	7.0×10 <sup>10</sup>
0.7	2.0×10 <sup>9</sup>	1.5×10 <sup>9</sup>
0.8	3.0×10 <sup>9</sup>	7.0×10 <sup>10</sup>
0.9	3.0×10 <sup>9</sup>	1.4×10 <sup>9</sup>
1.0	3.0×10 <sup>9</sup>	8.0×10 <sup>10</sup>

Table 2: Comparison of Errors

X	Sirisena <i>et al.</i> (2004)	Our new method
0.1	2.0×10 <sup>9</sup>	0.0
0.2	2.1×10 <sup>9</sup>	0.0
0.3	1.7×10 <sup>9</sup>	6.0×10 <sup>-10</sup>
0.4	0.0	2.0×10 <sup>-11</sup>
0.5	6.7×10 <sup>9</sup>	7.0×10 <sup>-10</sup>
0.6	0.0	1.0×10 <sup>-10</sup>
0.7	1.0×10 <sup>9</sup>	8.0×10 <sup>-10</sup>
0.8	0.0	2.0×10 <sup>-10</sup>
0.9	0.0	9×10 <sup>-10</sup>
1.0	0.0	4.0×10 <sup>-10</sup>

Table 3: Comparison of Errors.

X	Sirisena <i>et al.</i> (2004)	Our method
0.1	3.6×10 <sup>4</sup>	1.7×10 <sup>-5</sup>
0.2	1.5×10 <sup>4</sup>	1.6×10 <sup>-5</sup>
0.3	5.9×10 <sup>-5</sup>	9.3×10 <sup>-6</sup>
0.4	1.6×10 <sup>-5</sup>	4.6×10 <sup>-6</sup>
0.5	4.3×10 <sup>-5</sup>	1.8×10 <sup>-6</sup>
0.6	2.1×10 <sup>-5</sup>	4.2×10 <sup>-7</sup>
0.7	5.7×10 <sup>-7</sup>	1.8×10 <sup>-7</sup>
0.8	1.6×10 <sup>-6</sup>	2.3×10 <sup>-6</sup>
0.9	5.1×10 <sup>-6</sup>	3.8×10 <sup>-7</sup>
1.0	2.8×10 <sup>-6</sup>	3.2×10 <sup>-7</sup>

**CONVERGENCE AND STABILITY ANALYSIS**

In this study, we discuss the stability and convergence properties of the schemes (19) and (20). Zero stability of (19):

$$P|\xi| = \xi^3 - \frac{783}{617}\xi + \frac{135}{617} + \frac{31}{617} = 0$$

$$\xi_1 = 1, \xi_2 = 0.4285 \text{ and } \xi_3 = 0.1587$$

Since  $\xi_1 \neq \xi_2 \neq \xi_3$  and

$|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq 1$ , then the method is zero stable according to Lambert (1973), Zero stability of (20)

$$\ell(\xi) = \xi^2 - \frac{124875}{15795}\xi^2 - \frac{29000}{157952}\xi^3 - \frac{4077}{157952} = 0$$

$$\xi_1 = 1, \xi_2 = 0.0258 \text{ and } \xi_3 = 0.2094$$

Since,  $\xi_1 \neq \xi_2 \neq \xi_3$

The method is zero stable according to Lambert (1973). According to Lambert (1973) the necessary and sufficient condition for a linear multistep method to be convergent are that it be consistent and zero stable. Therefore, proposal schemes are convergent.

**NUMERICAL EXPERIMENT**

In this study, we use the proposal schemes (19) and (20) with Eq. 21 and 22 as starting values to solve the examples stated below. The errors arising from the computed and theoretical values are compared with Sirisena *et al.* (2004) as shown in Table 1-3.

Example 1

$$y' = -y, y(0) = 1, 0 \leq x \leq 1, h = 0.1$$

$$y(x) = e^{-x}$$

Example 2

$$y' = x-y, y(0) = 0, 0 \leq x \leq 1, h = 0.1$$

$$y(x) = x + e^{-x} - 1$$

Example 3

$$y' = 8(y-x) + 1, y(0) = 2, 0 \leq x \leq 1, h = 0.1$$

$$y(x) = x + 2e^{-8x}$$

From the above presented tables, our new method is more accurate when compared with Sirisena *et al.* (2004). The proposed method uses two difference equations per step while Sirisena *et al.* (2004) used three difference equations per step.

**CONCLUSION**

Our new proposed scheme are consistent and convergent. And it compares favourably with the existing scheme.

**REFERENCES**

Gear, C.W., 1971. Numerical IVPs in ODEs. Prentice Hall. Englewood Cliffs, N.I.  
 Lambert, J.D., 1973. Computational Methods in ODEs. JohnWiley and Sons New York, pp: 30-123.  
 Lie, I. and S.P. Norsett, 1989. Superconvergence for Multistep Collocation. Math. Comp., 52: 65-79.  
 Onumanyi, P.A., D.O. Woyemi, S.N. Jator and U.W. Sirisena, 1994. New Linear. Multistep Methods with Continuous Coefficients for first order initial value problems. J. Nig. Math. Soc., 13: 37-51.

- Onumanyi, P., U.W. Sirisena and S.N. Jatos, 1999. Continuous finite difference approximations for solving differential Eq. *Int. J. Comp. Maths*, 72: 15-27.
- Sirisena, U.W., 1997. A reformation of the continuous general linear multistep method by matrix inversion for first order Initial value problems. Ph.D Thesis (Unpublished) University of Ilorin, Ilorin Nigeria.
- Sirisena, U.W., 1999. An accurate implementation of the Butcher hybrid formula for the IVP in ODEs. *Nig. J. Maths Applications*, 12: 199-206.
- Sirisena, U.W., G.M. Kumleng and Y.A. Yahaya, 2004. A new Butcher type two-step block hybrid Multistep method for accurate and efficient parallel solution of Odes. *Abacus Maths Series*, 31: 1-7.
- Yusuph, Y. and P. Onumanyi, 2002. New Multiple FDMs through Multistep Collocation for  $y'' = f(x,y)$ . *Abacus* 29: 92-100.
- Yahaya, Y.A., 2004. Some theories and application of continuous LMM for ODEs. Ph.D Thesis (unpublished), University of Jos.