Journal of Modern Mathematics and Statistics 2 (3): 102-108, 2008
ISSN: 1994-5388

Two (*s*, *S*) Inventories with Perishable Units

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Abstract: Two inventories A and B with maximum capacity S and with perishable items are considered. Inter occurrence times of demands and the lead time for order are random variables. Whenever the inventory level falls to s from S, order is made for *S* - *s* units to fill up inventories A or B or both as the case may be. Inventory level probabilities are presented for exponential distribution. Numerical examples are also presented.

Key words: Inventory systems, inventories in series, perishable units, renewal theory

INTRODUCTION

Single (s, S) inventory systems have been discussed by several research workers. Thangaraj and Ramanarayanan (1983) have treated such a system with two ordering levels. Two ordering levels inventory system with rest time to the server is discussed by Chenniappan and Ramanarayanan (1994). System of inventories in series is discussed by Chenniappan and Ramanarayanan (1996). Ghare and Schrader (1963) discussed exponentially decaying inventory, Goyal and Giri (2001) discussed deteriorating inventory and Nahamias (1982) discussed perishable inventory. So far models in which two inventories with perishable units have not been discussed. In most of the cases inventory level decreases only by demand or sales. The influence of any other phenomena has not been considered. But in real life situations, there are number of inventories in which stock level goes down not only by demand but also by other factors such as spoilage, physical depletion and deterioration. Such inventories are known as decaying inventories.

In practice, perishable units such as batteries required for computers and cars, medicines with limited life time are sold with non perishable mechanical items and with long time medicines and medical equipments. In this study a model in which perishable and non perishable items stored is treated with exponential inter occurrence times of demands.

We treat this Markovian case by identifying its infinitesimal generator. We calculate the inventory level probabilities. Numerical examples are also presented.

MARKOVIAN MODEL

Main assumptions of the model are given:

- There are two inventories A and B each with capacity S and the reordering level is s (S-s > s) for both A and B.
- The inter-occurrence times of demands are i.i.d random variables. Generally when a demand occurs one unit of A and one unit B are both sold. Demands occur in accordance with Poisson process with parameter λ.
- The lead time distribution of units A is exponential with parameter μ_1 and that for units B is exponential with parameter μ_2.
- When a demand occurs and if both units of A and B are available, they are supplied. If unit B is not available and A is available A is alone sold. If unit B is available but unit A is not available, the demand which occurs is lost. If both A and B are not available, the demand that occurs is lost.
- Units of A are perishable each with rate α.

We may define the state of the system as {(i, j) for $0 \le i, j \le S$}. The system is in state (i, j) if i units of A are available and j units of B are available, respectively for $0 \le I, j \le S$.

Here we obtain the steady state probabilities of the system. The infinitesimal generator Q with order $(S+1)^2$ of the above continuous time Markovian chain may be block partitioned as follows:

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$$
Q = \begin{array}{c|ccccccccccccc|}
 & \underline{S} & \underline{S-1} & \underline{S-2} & \cdots & \underline{S-s} & \cdots & \cdots & \underline{s} & \underline{s-1} & \cdots & \cdots & \underline{1} & \underline{0} \\
\hline
\underline{S} & A_S & B_S & & & & & & & & & & & \\
\underline{S-1} & & A_{S-1} & B_{S-1} & & & & & & & & & & \\
\underline{S-2} & & & A_{S-2} & \ddots & & & & & & & & & \\
\vdots & & & & \ddots & B_{S-s+1} & & & & & & & & \\
\underline{S-s} & & & & & A_{S-s} & B_{S-s} & & & & & & & \\
\vdots & & & & & & \ddots & \ddots & & & & & & \\
\vdots & & & & & & & \ddots & \ddots & & & & & \\
\underline{s} & \mu_1 I & & & & & & & A_s & B_s & & & & \\
\underline{s-1} & & \mu_1 I & & & & & & & A_{s-1} & B_{s-1} & & & \\
\vdots & & & \ddots & & & & & & & \ddots & \ddots & & \\
\vdots & & & & \ddots & & & & & & & \ddots & \ddots & \\
\underline{1} & & & & & \mu_1 I & & & & & & & A_1 & B_1 \\
\underline{0} & & & & & & \mu_1 I & & & & & & & A_0 \\
\hline
\end{array} \qquad (1)
$$

The sub matrices of Q are given as follows:

$$
A_j = \begin{array}{c|cccccccc|}
 & (j,S) & (j,S-1) & (j,S-2) & \cdots & (j,S-s) & \cdots & (j,s) & \cdots \; (j,0) \\
\hline
(j,S) & -j\alpha-\lambda & & & & & & & \\
(J,S-1) & & -j\alpha-\lambda & & & & & & \\
(j,S-2) & & & -j\alpha-\lambda & & & & & \\
\vdots & & & & \ddots & & & & \\
(j,S-s) & & & & & -j\alpha-\lambda & & & \\
\vdots & & & & & & \ddots & & \\
(j,s) & \mu_2 & & & & & & -j\alpha-\lambda-\mu_2 & \\
\vdots & & \ddots & & & & & & \ddots \\
(j,0) & & & & \mu_2 & & & & -j\alpha-\lambda-\mu_2 \\
\hline
\end{array} \qquad (2)
$$

for $s+1 \le j \le S$.

$$
A_{j=} \begin{array}{c|ccccccc|}
 & (j,S) & (j,S-1) & \cdots & (j,S-s) & \cdots & (j,s) & \cdots \; (j,0) \\
\hline
(j,S) & -j\alpha-\lambda-\mu_1 & & & & & & \\
(j,S-1) & & -j\alpha-\lambda-\mu_1 & & & & & \\
\vdots & & & \ddots & & & & \\
(j,S-s) & & & & -j\alpha-\lambda-\mu_1 & & & \\
\vdots & & & & & \ddots & & \\
(j,s) & \mu_2 & & & & & -j\alpha-\lambda-\mu_1-\mu_2 & \\
\vdots & & \ddots & & & & & \ddots \\
(j,0) & & & \mu_2 & & & & -j\alpha-\lambda-\mu_1-\mu_2 \\
\hline
\end{array} \qquad (3)
$$

for $s \le j \le 1$.

$$A_0 = \begin{array}{c} (0,S) \\ (0,S-1) \\ \vdots \\ (0,S-s) \\ \vdots \\ (0,s) \\ \vdots \\ (0,0) \end{array} \overset{\begin{array}{ccccccc} (0,S) & (0,S-1) & \dots & (0,S-s) & \dots & (0,s) & \dots & (0,0) \end{array}}{\begin{bmatrix} -\mu_1 & & & & & & & \\ & -\mu_1 & & & & & & \\ & & \ddots & & & & & \\ & & & -\mu_1 & & & & \\ & & & & \ddots & & & \\ -\mu_2 & & & & & -\mu_1-\mu_2 & & \\ & \ddots & & & & & \ddots & \\ & & -\mu_2 & & & & & -\mu_1-\mu_2 \end{bmatrix}} \quad (4)$$

for $1 \le j \le S$.

$$B_j = \begin{array}{c} (j,S) \\ (j,S-1) \\ \dots \\ \dots \\ (j,1) \\ (j,0) \end{array} \overset{\begin{array}{cccccc} (j,S) & (j,S-1) & \dots & \dots & (j,1) & (j,0) \end{array}}{\begin{bmatrix} j\alpha & \lambda & & & & \\ & j\alpha & \lambda & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & j\alpha & \lambda \\ & & & & & j\alpha+\lambda \end{bmatrix}} \quad (5)$$

for $1 \le j \le S$.

The above infinitesimal generator is suitable for numerical solutions for evaluating the steady state probability vector. Let

$$\underline{\Pi} = (\underline{\Pi}_S, \underline{\Pi}_{S-1}, \dots, \underline{\Pi}_1, \underline{\Pi}_0)$$

be the partitioned invariant probability vector satisfying the following equations

$$\underline{\Pi}Q = 0 \text{ and } \underline{\Pi}\,\underline{e} = 1 \quad (6)$$

Where:

$$\underline{\Pi}_i = (\underline{\Pi}_{i,S}, \underline{\Pi}_{i,S-1}, \dots \underline{\Pi}_{i,j}, \dots, \underline{\Pi}_{i,1}, \underline{\Pi}_{i,0}),\ i = 0,1,\dots, S, \quad (7)$$

$\underline{\Pi}_{i,j}$ = P (the inventory level in i, j for $0 \le i \le S$ and $0 \le j \le S$) and $\underline{e} = (1, 1, 1, \dots, 1)^t$.

We now present the equation satisfied by the various partitioned components of the vector $\underline{\Pi}$ in terms of the vector $\underline{\Pi}_s$. Using Eq. 6, we note

$$\underline{\Pi}_s B_s + \underline{\Pi}_{s-1} A_{s-1} = 0$$

which gives

$$\underline{\Pi}_{s-1} = \underline{\Pi}_s B_s (-A_{s-1})^{-1} \quad (8)$$

Similarly, we note that

$$\underline{\Pi}_{s-j} = \underline{\Pi}_s \prod_{k=0}^{k=j-1} B_{s-k}(-A_{s-1-k})^{-1} \quad \text{for} \quad j = 1,2,\dots,s \quad (9)$$

Noting again

$$\underline{\Pi}_{s+1} B_{s+1} + \underline{\Pi}_s A_s = 0$$

we get

$$\underline{\Pi}_{s+1} = \underline{\Pi}_s (-A_s)(B_{s+1})^{-1} \quad (10)$$

and similarly we may write

$$\underline{\Pi}_{s+j} = \underline{\Pi}_s \prod_{k=0}^{k=j-1} (-A_{s+k})(B_{s+k+1})^{-1} \quad (11)$$

$$\text{for } j = 1,2,\dots S-2s-1$$

Using the first column of Q we find

$$\underline{\Pi}_S = \underline{\Pi}_s \mu (-A_S)^{-1} \quad (12)$$

We note that the following equations are satisfied by

$$\underline{\Pi}_{S-j}, \text{ for } 1 \le j \le s$$

$$\underline{\Pi}_{S-j} = \underline{\Pi}_{S-j+1} B_{S-j+1} (-A_{S-j})^{-1} + \mu \underline{\Pi}_{s-j} (-A_{S-j})^{-1} \quad (13)$$

for $1 \le j \le s$. Noting the recurrence relation step by step, we may derive $\underline{\Pi}_{S-1}, \underline{\Pi}_{S-2}$ and so on as given:

$$\begin{aligned} \underline{\Pi}_{S-j} = \underline{\Pi}_s \Bigg[& (\mu I)(-A_S)^{-1} \prod_{k=0}^{k=j-1} B_{S-k}(-A_{S-1-k})^{-1} \\ & + \sum_{m=0}^{j-1} \prod_{k=0}^{k=m} B_{s-k}(-A_{s-1-k})^{-1} (\mu I)(-A_{S-m-1}) \\ & \prod_{k'=m+1}^{k'=j-1} B_{S-k'}(-A_{S-1-k'})^{-1} \\ & + \prod_{k=0}^{k=j-1} B_{s-k}(-A_{s-1-k})^{-1} (\mu I)(-A_{S-j})^{-1} \Bigg] \end{aligned} \quad (14)$$

for j = 1,2,..., s.

Now $\underline{\Pi}_s$ is to be calculated using total law of probability. Using Eq. 6, 9, 11, 12 and 14, we note that

$$\left[\Pi_S + \sum_{j=1}^{s}\Pi_{S-j} + \sum_{j=1}^{S-2s-1}\underline{\Pi}_{s+j} + \underline{\Pi}_s + \sum_{j=1}^{s}\underline{\Pi}_{s-j}\right]\underline{e}=1$$

gives

$$\begin{aligned}\underline{\Pi}_s\Bigg[&(\mu I)(-A_S)^{-1} + \sum_{j=1}^{s}\Bigg\{(\mu I)(-A_S)^{-1}\prod_{k=0}^{k=j-1}B_{S-k}(-A_{S-1-k})^{-1}\\ &+\sum_{m=0}^{j-1}\left[\prod_{k=0}^{k=m}B_{s-k}(-A_{s-1-k})^{-1}(\mu I)(-A_{S-m-1})\prod_{k'=m+1}^{k'=j-1}B_{S-k'}(-A_{S-1-k'})^{-1}\right]\\ &+\prod_{k=0}^{k=j-1}B_{s-k}(-A_{s-1-k})^{-1}(\mu I)(-A_{S-j})^{-1}\Bigg\}+\sum_{j=1}^{S-2s-1}\prod_{k=0}^{k=j-1}(-A_{s+k})(B_{s+k+1})^{-1}\\ &+1+\sum_{j=1}^{s}\prod_{k=0}^{k=j-1}B_{s-k}(-A_{s-1-k})^{-1}\Bigg]\underline{e} = 1\end{aligned} \tag{15}$$

where, $\underline{e}$ = (1,1,1,...,1). We note

$$\underline{\Pi}_s = \frac{a^t}{|\underline{a}|^2} \tag{16}$$

Where:

$$\begin{aligned}\underline{a} = \Bigg[&(\mu I)(-A_S)^{-1} + \sum_{j=1}^{s}\Bigg\{(\mu I)(-A_S)^{-1}\prod_{k=0}^{k=j-1}B_{S-k}(-A_{S-1-k})^{-1}\\ &+\sum_{m=0}^{j-1}\left[\prod_{k=0}^{k=m}B_{s-k}(-A_{s-1-k})^{-1}(\mu I)(-A_{S-m-1})\prod_{k'=m+1}^{k'=j-1}B_{S-k'}(-A_{S-1-k'})^{-1}\right]\\ &+\prod_{k=0}^{k=j-1}B_{s-k}(-A_{s-1-k})^{-1}(\mu I)(-A_{S-j})^{-1}\Bigg\}+\sum_{j=1}^{S-2s-1}\prod_{k=0}^{k=j-1}(-A_{s+k})(B_{s+k+1})^{-1}\\ &+1+\sum_{j=1}^{s}\prod_{k=0}^{k=j-1}B_{s-k}(-A_{s-1-k})^{-1}\Bigg]\underline{e}.\end{aligned}$$

Combining (11), (12), (14), (15) and (16) we get the steady state probability vector.

NUMERICAL EXAMPLE

Let the maximum capacity of the inventories of A and B be 3(S = 3) and the reorder level is 1(s = 1). The infinitesimal generator (order is 16) of the finite state space continuous time Markov chain is as follows:

$$\begin{array}{c} \\ \underline{3} \\ \underline{2} \\ \underline{1} \\ \underline{0} \end{array}\begin{array}{c} \begin{matrix} \underline{3} & \underline{2} & \underline{1} & \underline{0} \end{matrix} \\ \begin{bmatrix} A_3 & B_3 & & \\ & A_2 & B_2 & \\ \mu_1 I & & A_1 & B_1 \\ & \mu_1 I & & A_0 \end{bmatrix}\end{array}$$

where A_0, A_1, A_2, A_3, B_1, B_2, B_3 and μ_1 I are as given:

$$A_0 = \begin{bmatrix} -\mu_1 & & & \\ & -\mu_1 & & \\ \mu_2 & & -\mu_1-\mu_2 & \\ & \mu_2 & & -\mu_1-\mu_2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\alpha-\lambda-\mu_1 & & & \\ & -\alpha-\lambda-\mu_1 & & \\ \mu_2 & & -\alpha-\lambda-\mu_1-\mu_2 & \\ & \mu_2 & & -\alpha-\lambda-\mu_1-\mu_2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2\alpha-\lambda & & & \\ & -2\alpha-\lambda & & \\ \mu_2 & & -2\alpha-\lambda-\mu_2 & \\ & \mu_2 & & -2\alpha-\lambda-\mu_2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3\alpha-\lambda & & & \\ & -3\alpha-\lambda & & \\ \mu_2 & & -3\alpha-\lambda-\mu_2 & \\ & \mu_2 & & -3\alpha-\lambda-\mu_2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \alpha & \lambda & & \\ & \alpha & \lambda & \\ & & \alpha & \lambda \\ & & & \alpha+\lambda \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2\alpha & \lambda & & \\ & 2\alpha & \lambda & \\ & & 2\alpha & \lambda \\ & & & 2\alpha+\lambda \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2\alpha & \lambda & & \\ & 2\alpha & \lambda & \\ & & 2\alpha & \lambda \\ & & & 2\alpha+\lambda \end{bmatrix},$$

and

$$\mu_1 I = \begin{bmatrix} \mu_1 & & & \\ & \mu_1 & & \\ & & \mu_1 & \\ & & & \mu_1 \end{bmatrix}$$

For fixed values of $\lambda = 2$, $\alpha = 1$, $\mu_1 = 3$ and $\mu_2 = 2$, we compute the components of $\underline{\Pi}$. The steady state probability vector

$$\begin{aligned} \underline{\Pi} &= \left(\underline{\Pi}_3, \underline{\Pi}_2, \underline{\Pi}_1, \underline{\Pi}_0\right) \\ &= \left(\underline{\Pi}_{3,3}, \underline{\Pi}_{3,2}, \underline{\Pi}_{3,1}, \underline{\Pi}_{3,0}, \underline{\Pi}_{2,3}, \underline{\Pi}_{2,2}, \underline{\Pi}_{2,1}, \underline{\Pi}_{2,0}, \underline{\Pi}_{1,3}, \underline{\Pi}_{1,2}, \underline{\Pi}_{1,1}, \underline{\Pi}_{1,0}, \underline{\Pi}_{0,3}, \underline{\Pi}_{0,2}, \underline{\Pi}_{0,1}, \underline{\Pi}_{0,0}\right) \end{aligned}$$

is given by

$$\begin{array}{llll} \underline{\Pi}_{3,3} = 0.0405 & \underline{\Pi}_{3,2} = 0.0662 & \underline{\Pi}_{3,1} = 0.0240 & \underline{\Pi}_{3,0} = 0.0156 \\ \underline{\Pi}_{2,3} = 0.0987 & \underline{\Pi}_{2,2} = 0.1644 & \underline{\Pi}_{2,1} = 0.0596 & \underline{\Pi}_{2,0} = 0.0432 \\ \underline{\Pi}_{1,3} = 0.0516 & \underline{\Pi}_{1,2} = 0.0998 & \underline{\Pi}_{1,1} = 0.0560 & \underline{\Pi}_{1,0} = 0.0365 \\ \underline{\Pi}_{0,3} = 0.0513 & \underline{\Pi}_{0,2} = 0.0972 & \underline{\Pi}_{0,1} = 0.0511 & \underline{\Pi}_{0,0} = 0.0443 \end{array}$$

The sum of steady state probabilities is found to be 1.0000.

Table 1: The expected inventory levels (e.i.l) of A and B for α

	e.i.l	
α	A	B
1	0.8824	2.2287
2	0.6127	2.2983
3	0.4623	2.3398
4	0.3802	2.3674
5	0.3196	2.3869

Table 2: The fixed values of (e.i.l) of A and B for λ

	e.i.l	
λ	A	B
1	0.8824	2.2287
2	0.6923	2.0676
3	0.5676	1.9648
4	0.48	1.894
5	0.4154	1.8418

Table 3: The fixed values of (e.i.l) of A and B for μ_1

	e.i.l	
μ_1	A	B
1	0.6923	2.0676
2	1.1325	1.8706
3	1.4146	1.772
4	1.6047	1.7175
5	1.7391	1.6847

Case 1: (e.i.l for increasing perishable rate)

For the fixed values of $\lambda = 1$, $\mu_{1=1}$ and $\mu_2 = 2$, we compute the expected inventory levels (e.i.l) of A and B for $\alpha = 1, 2, 3, 4$ and 5 (Table 1).

It is clear from Table 1, when the perishable rates increase, the e.i.l of A decreases and the e.i.l of B increases. To each value of the e.i.l of A are less than that of B.

Case 2: (e.i.l for increasing demand rate).

For the fixed values of $\alpha = 1$, $\mu_1 = 1$ and $\mu_2 = 2$, we compute the expected inventory levels (e.i.l) of A and B for $\lambda = 1, 2, 3, 4$ and 5 (Table 2).

It is clear from Table 2, when the demand rates increase, the e.i.l of A and B decrease. To each value of λ the e.i.l of A are less than that of B.

Case 3: (e.i.l for increasing lead times of units A).

For the fixed values of $\alpha = 1$, $\lambda = 2$ and $\mu_2 = 2$, we compute the (e.i.l) of A and B for $\mu_1 = 1, 2, 3, 4$ and 5 (Table 3).

It is clear from Table 3, when the lead times of A increase, the e.i.l of A increase and e.i.l B decrease. To each value of μ_1 the e.i.l of A are less than that of B except $\mu_1 = 5$ at $\mu_1 = 5$, the e.i.l A is slightly more than that of B.

Table 4:The fixed values of (e.i.l) of A and B for μ_2

	e.i.l	
μ_2	A	B
1	0.6923	1.7028
2	0.6923	2.0676
3	0.6923	2.2086
4	0.6923	2.2815
5	0.6923	2.3257

Table 5: The fixed values of (e.i.l) of compute A and B for λ

	e.i.l.	
λ	A	B
1	1.6	1.7118
2	1.1	1.4063
3	0.8235	1.2769
4	0.6538	1.2097
5	0.5405	1.1692

Table 6: The fixed values of (e.i.l) of Inventories A and B for λ

	e.i.l.	
λ	A	B
1	1.6	2.0667
2	1.1	1.8449
3	0.8235	1.7282
4	0.6538	1.6576
5	0.5405	1.6102

Case 4: (e.i.l for increasing lead times of units B).

For the fixed values of $\alpha = 1$, $\lambda = 2$ and $\mu_1 = 1$, we compute the (e.i.l) of A and B for $\mu_2 = 1, 2, 3, 4$ and 5 (Table 4).

It is clear from Table 4, when the lead times of B increase, the e.i.l of A remains the same. But e.i.l B increase with the lead times of B. To each value of μ_2 the e.i.l of A are less than that of B.

Case 5: (e.i.l for increasing demand rates when the units are non perishable and same lead times of the inventories A and B).

For the fixed values of $\alpha = 0$, $\mu_1 = 1$ and $\mu_2 = 1$, we compute the (e.i.l) of A and B for $\lambda = 1, 2, 3, 4$ and 5 (Table 5).

In the case of non-perishable units in A, when demand rate increases, the e.i.l of A and B both decrease. To each value of λ, the e.i.l of A are less than that of B.

Case 6: (e.i.l for increasing demand rates when the units are non perishable and different lead times of the inventories A and B).

For the fixed values of $\alpha = 0$, $\mu_1 = 1$ and $\mu_2 = 2$, we compute the (e.i.l) of inventories A and B for $\lambda = 1, 2, 3, 4$ and 5 (Table 6).

In the case of non-perishable units in A, when demand rate decreases, the e.i.l of A and B both decrease. To each value of λ, the e.i.l of A are less than that of B.

REFERENCES

Chennianppan, P.K. and R. Ramanarayanan, 1994. Inventory system with 2 ordering levels and rest time to the server. Int. J. Inform. Manage. Sci., 6: 47-55.

Chennianppan, P.K. and R. Ramanarayanan, 1996. Inventories in series. Int. J. Inform. Manage. Sci., 7: 45-52.

Ghare, P.M. and G.F. Schrader, 1963. A Model for an exponentially decreasing inventory. J. Indus. Eng., 14: 238-243.

Goyal, S.K. and B.C. Giri, Recent trends in Modeling of deteriorating inventory. Eur. J. Operations Res., 134: 1-16.

Nahamias, S., 1982. Perishable Inventory theory: A review. Opsearch, 30: 680-708.

Thangaraj, V. and R. Ramanarayanan, 1983. An operating policy in inventory systems with random lead times and unit demands. Math. Oper. Stat. Series Optimization, 14: 111-124.