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Application of Rational Function Approximations to Polynomial Functions

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Abstract: This study deals, with approximating polynomial function using rational functions. This situation arises in experimental measured dependencies, which is useful in non-linear analogue function block designs and the approximation of inverse functions. The dependencies are usually given in analytical form or measured data form. We examine these dependencies in this study and show the application with a simple example. The results showed that rational function method is a more general method of approximation than the polynomial functions.

Key words: Polynomial functions, rational function approximations, block design, dependencies

INTRODUCTION

A class of approximants that possess remarkable analytical and numerical properties exist but are not widely known and used as they should be. The starting point was when Pade (1951), considered the representation of functions by rational function in his thesis. These days, this research plays an important role in several branches of theoretical and applied researches (Shank, 1955). Pade' method also makes it possible to obtain from a power series (converging or diverging) for a function a table of rational approximations to it (Borwen and Zhou, 1993).

The rational function representations of a power series or expansion are now known as Pade' approximants (Horgan and Saccomandi, 2006). We will briefly present the central results of Pade' research in modern notation (Baker, 1975). The rational function approximations play a key role in nonlinear elasticity, detail of which is not the subject of this study.

REPRESENTATION IN RATIONAL FUNCTIONS

If the approximated function f (x) is given in analytical form for approximation by rational function, we can use the Pade's approximation in the following form:

$$f(x) \cong R_n(x) = \frac{P_r(x)}{Q_s(x)} = \frac{a_0 + a_1 + a_2x^2 + ... + a_rx^r}{1 + b_1 + b_2x^2 + ... + b_sx^s} \quad (1)$$

Where, P_r (x) and Q_s (x) are polynomial functions and (R_n x) is a rational function for n = r + s.

The most useful of the Pade' approximations are those with the degree of numerator equal to, or one more than, the degree of denominator. The coefficient calculation is based on Maclaurin's expansion of f (x) to make f (x) and R_n (x) agree at x = 0 and to make the first N derivatives agree at x = 0. From these conditions, we have the expressions:

$$f(x) = c_0 + c_1x + c_2x^2 + ... + c_Nx^N = R_n(x) \quad (2)$$

Where,

$$C_i = f^{(i)}(0)/i\,!$$

and

$$R_n(x).\, Q_s(x) - P_r(x) = 0 \quad (3)$$

For nonlinear elasticity, an analogous representation of Eq. 1 is:

$$f(x) = R_N(x) = \frac{C_{10} + C_{11}x + C_{12}x^2 + ...}{C_{20} + C_{21}x + C_{22}x^2 + ...} \quad (4)$$

which is used to approximate response functions. We can also write Eq. 4 as:

$$f(x) = \cfrac{1}{\cfrac{C_{00}}{C_{10}} + \cfrac{C_{00} + C_{01}x + C_{02}x^2}{C_{10} + C_{11}x + C_{12}x^2} + ... + \cfrac{-C_{00}}{C_{00}}} = \frac{C_{10}}{C_{00} + x\, f_1(x)} \quad (5)$$

Where,

$$f_1(x) = \frac{C_{20} + C_{21}x + C_{22}x^2 + ...}{C_{10} + C_{11}x + C_{12}x^2 + ...} \quad (6)$$

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And

$$C_{2k} = C_{10}C_{0,\,k+1} - C_{00}C_{1,k+1},\ k = 0, 1, 2 \quad (7)$$

In a similar manner, we continue:

$$f_{j(x)} = \frac{C_{j+10} + C_{j+11}x + C_{j+12}x^2 + ...}{C_{j0} + C_{j1}x + C_{j2}x^2 + ...} \quad (8)$$

with

$$C_{j+1,k} = C_{j,0}C_{j-1,k+1} - C_{j-2,0}C_{j,k+1},\ k = 0, 1, 2, \ldots \quad (9)$$

Where, Eq. (9) can be rewritten in the form:

$$C_{j,k} = -\begin{vmatrix} C_{j-2,0} & C_{j-2,k+1} \\ C_{j-1,0} & C_{j-1,k+1} \end{vmatrix}, \qquad j \geq 2 \quad (10)$$

The continued fraction expansion expression now becomes:

$$f(x) = \left[0, \frac{C_{10}}{C_{00}}, \frac{C_{20}x}{C_{10}}, \frac{C_{30}x}{C_{20}}, \ldots\ldots, \frac{C_{j0}x}{C_{j-1,0}} \ldots\ldots \right] \quad (11)$$

If the expressions in the numerator and denominator of f (x) are polynomials, the process will be terminating continued fraction and yield lower order rational functions as approximation to f (x).

Remark 1: We assume for convenience and without loss of generality, that $C_{00} = 1$, so that in Eq. (1), $b_0 = C_{00} = 1$ (Horgan and Saccomandi, 2006).

MATRIX REPRESENTATION

Following the form of Eq. (2) and (3), the resultant matrix expressions are:

$$\begin{bmatrix} C_r & C_{r-1} & \ldots & C_{r-S+1} \\ C_{r+1} & C_r & \ldots & C_{r-s+2} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ C_{r+s-2} & C_{r+s-3} & \ldots & C_{r-1} \\ C_{r+s-1} & C_{r+s-2} & \ldots & C_r \end{bmatrix} \begin{bmatrix} b_1 \\ b_1 \\ . \\ . \\ b_{s-1} \\ b_s \end{bmatrix} = \begin{bmatrix} -C_{r+1} \\ -C_{r+2} \\ . \\ . \\ -C_{r+S-1} \\ -C_{r+s} \end{bmatrix} \quad (12)$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ a_r \end{bmatrix} = \begin{bmatrix} C_0 & 0 & 0 \ldots & 0 \\ C_1 & C_0 & 0 \ldots & 0 \\ C_2 & C_1 & C_0 \ldots & 0 \\ . & . & \ldots \ldots & . \\ . & . & \ldots \ldots & . \\ C_r & C_{r-1} & C_{r-2} \ldots & C_{r-s} \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ . \\ . \\ b_s \end{bmatrix} \quad (13)$$

From the matrix Eq. (12) and (13), we can calculate the required coefficients $a_0, a_1, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ by the following matrix expressions:

$$[A_2]\,.\,[b] = [C] \Rightarrow [b] = [A_2]^{-1}\,[C] \quad (14)$$

And

$$.[a] = [A_1]\,[b^*] \quad (15)$$

Where:

A_2 = The matrix of C_r in Eq. 12

A_1, b^* = From Eq. 14

The dimension of matrices A_2, A_1 are s.s and (r + 1), (s + 1), respectively. In nonlinear aspects, we shall try to extend the values of r and s to be more than five, which is the usual restriction in technical applications.

If the function f (x) is given by measured data pair, then the rational function approximation can be made by the following manner. For every data pair, x_i, y_i, i = 0, 1, 2, 3, …, N, the valid expression is 1 and we can write it in the following manner:

$$y_i = \frac{a_0 + a_1x_i + a_2x_i^2 + \ldots + a_rx_i^r}{1 + b_1x_i + b_2x_i^2 + \ldots + b_sx_i^s} \quad (16)$$

Where:

x_i, y_i = Known (measured) data.

a_j, b_k = Required unknown coefficients.

The expression (16) can also be written in the following form as well:

$$\begin{aligned} y_i \ = \ & a_0 + a_1x_i + a_2x_i^2 + \ldots\ldots\ldots a_rx_i^r \\ & - b_1x_1y_i - b_2x_i^2y_i \ldots\ldots - \ldots\ldots - b_sx_i^sy_i \end{aligned} \quad (17)$$

From Eq. (19) for i = 0, 1, 2, …, N, we can have N + 1 equations written in matrix form:

$$[y] = [x]\,.\,[a] \quad (18)$$

Where,

$$[y] = [y_0, y_1, y_2, \ldots, y_N]^T \quad (19)$$

$$[a] = [a_0, a_1, a_2, \ldots, a_r, b_1, B_2, \ldots, b_s]^T \quad (20)$$

$$[X] = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^r & -y_0x_0 & -y_0x_0^2 & \ldots & -y_0x_0^s \\ 1 & x_1 & x_1^2 & \ldots & x_1^r & -y_1x_1 & -y_1x_1^2 & \ldots & -y_1x_1^s \\ . & . & . & \ldots & . & . & . & \ldots & . \\ 1 & x_N & x_N^2 & \ldots & x_N^r & -y_Nx_N & -y_Nx_N^2 & \ldots & -y_Nx_N^s \end{bmatrix} \quad (21)$$

Remark 2:

- If there is a term N = r + s in the system of the equations, then the required coefficients a_0, a_1..., a_r, b_1, b_2, ..., b_s can be calculated by the following expressions:

$$[a] = [X]^{-1} . [y] \quad (22)$$

- If in the system of Eq. (18), there is N > r + s terms, then we can calculate the required coefficients by the following expression (least-square method)

$$[a] = \left[[X]^T.[X]\right]^{-1}.[X]^T[y] \quad (23)$$

APPLICATION TO INVERSE FUNCTIONS

The approximation by rational function is useful on the calculation of inverse function. This is done through linearization of measurements in characteristically nonlinear materials. The problem of measured data pair usually associated with polynomial functions approximation are easily and conveniently overcome using rational functions approximations. We illustrate this with theoretical example using inverse function.

Consider a non-linear material with a measured data pair, x_i, y_i, the rational function approximation is given by the expression:

$$y \cong f(x) = a_0 + a_1x + a_2x^2 + ... + a_Nx^N \quad (24)$$

and

$$y \cong f(x) = \frac{a_0 + a_1x + a_2x^2 + ... + a_rx^r}{1 + b_1x + b_2x^2 + ... + b_sx^s} \quad (25)$$

In linearization process of the above Eq. (24-25) we exploited the property of inverse functions φ(x), which we can express as:

$$\varphi(y) = \varphi(f(x)) = f^{-1}(f(x)) = x \quad (26)$$

For the rational form of the Inverse function, it will be valid if:

$$x_i = \frac{A_0 + A_1y_i + A_2y_i^2 + ... + A_ry_i^r}{1 + B_1y_1 + B_2y_2^2 + ... + B_sy_i^s} \quad (27)$$

For i = 0,1, 2, .., N by Eq. (27) we can write r + s + 1 equations for unknown coefficient A_u, A_1, A_2,...,A_r, B_1, B_2, .., B_s and the value of N must be N≥R + s. Then the solution of these equations can be made by the same method used in the solution of Eq. 18. The inverse functions will then have the following rational form:

$$\varphi(x) = \frac{A_0 + A_1x + A_2x^2 + ... + A_rx^r}{1 + B_1x + B_2x^2 + ... + B_sx^s} \quad (28)$$

THEORETICAL RESULTS

Theorem: Consider an arbitrary function f∊$C_{[-1, 1]}$. Let, P_n be the space of polynomials of degree n and let $R_{n,n}$ stand for rational functions such that:

$$\left\{ g/h : g, h \in P_n; h > 0 \right\} \quad (29)$$

Let, P_n* be the best approximation to f from P_n then zero is the unique best approximation from $R_{n,n}$ to f -P^*_n.

Proof: The function f-P^*_n equioscilates, i.e, there are points ξ_1, ξ_2, ξ_3, ...ξ_{n+2} ∊ [-1, 1] such that:

$$(f - p_n^*)(\xi_j)\ \lambda(-1)^j \left\|f - p_n^*\right\|$$

Where,

$$\lambda \pm 1 \ (\text{say, } \lambda = -1)$$

Now if,

$$\left\| f - p_n^* - g/h \right\| \le \left\|f - p_n^*\right\|$$

then,

$$g/h(\xi_j) \ge 0 \text{ for j even and}$$
$$g/h(\xi_j) \le 0 \text{ for j odd}$$

Since, h is strictly positive, the function g ∊ p_n should satisfy the same condition:

$$g(\xi_j) \ge 0 \text{ for j even}$$
$$g(\xi_j) \le 0 \text{ for j odd}$$

That forces g to have n+1 zeros and hence, g = 0. Hence, the theorem.

DISCUSSION

In this study, Pade approximation on polynomial functions is presented and modification made using rational approximation. Approximating with rational functions yield lower order functions. Equation 22 shows how the required coefficients can be calculated and Eq. 23 accounts for N>r + s terms, which can be calculated by the method of the least-square.

Explicit theoretical illustration of with inverse square function is shown and a theoretical result with a theorem is given. The result shows that rational approximations have a unique best approximation than the Pade polynomial functions.

CONCLUSION

The approximation by rational function method can be used to approximate functions such as sinx, cosx, tanx and others. The described methods of the approximations by rational functions enable the calculation of the required parameters by solving linear system of equations and the application of the least-square method in the calculations.

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