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# Empirical Power Comparism of Three Correlation Coefficients

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**Abstract:** A Comparison of Pearson's moment (r), Kendall's ($\tau$) and the Spearman's rank ($r_2$) correlations was made to find out when they may be suitable for use, particularly when the assumptions that support their use are violated. Bi-variate Samples of size n = 5, 10, 15, 20, 25, 30, 40, 50 and 100 from the normal and exponential distributions with population correlation values of $\rho$ = 0, 0.25, 0.5, 0.75 and 0.9 (chosen to represent positive correlation between 0 and 1) were used. The power function for $\alpha$ = 0.01 and 0.05 was calculated for the tests. For the normal distribution, the Pearson's moment correlation coefficient was discovered to be the more powerful. However, in the exponential distribution, the power of the Pearson's moment correlation coefficient was lower than those of the non-parametric correlation coefficients, except for small sample sizes i.e, $n \leq 15$.

**Key words:** Empirical, correlation coefficient, power function, power curve, bi-variate normal distribution, bi-variate exponential distribution

## INTRODUCTION

Let $(X_i, Y_i)$ I = 1,2,...,n be a random sample from a bi-variate distribution. Testing the degree or extent of association between X and Y is a very important problem that often arises in real life study. Correlation coefficient is one of the mostly readily test for such verifications. If the under laying distribution is known to be a normal distribution, them the pearson's moment correlation is known to be most applicable (Hoel, 1971).

When the distribution is non-normal, some non-parametric correlations may be considered to be more appropriate. The two classical ones are the test based on Spearman's rank correlation ($r_2$) and the Kendall's s$\tau$ (Bhattacharyya *et al*., 1970).

However, under the normal distribution, at what sample size or population Correlation or the Combination of the two parameters would the non-parametric test perform if not exactly, at least as close as the pearson's moment correlation coefficient. And when the distribution is non-normal, at what parameter (n, $\rho$ or both) values might the Pearson correlation coefficient be used base on its performance (s).

Some works done in this area in recent times includes; Krishmamorthy and Thomsom (2004) who addressed the problem of hypotheses testing about two Poisson means; the usual conditional test (C-test) and a test based on estimated p-value (E-test). The exact properties of the tests are evaluated numerically and the E-test is almost exact and more powerful than the C-test. In addition, Kung-Jong (2005) discussed sample size and power calculation on the basis of exact distribution under Poisson models for testing non-inferiority and equivalence with respect to the mean incidence rate ration. An exact and two asymptotic procedures for calculation of the minimum required number of index subjects for a desired power 1-$\beta$ at a given $\alpha$-level are presented in Lui (1997a) with a table that summarizes the minimum required number of index subjects for powers equal to 0.90 and 0.80 in the application of the proposed exact equivalence test at 0.05 level in a variety of cases. Also, when each subject in two Comparison groups has a fixed number of repeated measurement, Lui (1997b) developed an asymptotic procedure to calculate the number of subjects required per group to achieve a given power for an $\alpha$-level bioequivalence test.

While earlier works includes; Lehmann (1966) who demonstrated mathematically that the Pearson's moment and Kendall's$\tau$ correlation coefficient possesses desirable properties for some specific types of positive dependence. The suitability and applicability of some Non-parametric test for Independence in Bi-variate population and their asymptotic properties was studied by Bhuchongkul (1964). Also, Konijn (1956) and Wood worth (1965) have both studied the performance of the Pearson's moment, Kendall's$\tau$ and the spearman's rank $r_s$ correlation coefficients using their asymptotic relative efficiencies. When we have equal sample size for two

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populations, Gail (1974) discussed power calculation and sample size determination for comparative Poisson trials. Consequent on this, Brown extended the work to a situation where we have unequal sample sizes. Bhattacharyya *et al*. (1970) proposed a simplified version of the normal score test in addition to a Monte Carlo power evaluation of several test of independence given sample sizes below n = 10.

In this study, we investigate the power function of the three correlation coefficients, namely; Pearson's moment (r), Kendall's ($\tau$) and the Spearman's rank ($r_s$) correlations coefficients under the normal and exponential distributions via a Monte Carlo study. The distributions were chosen because of their wide applicability.

## THE MODELS

- The normal distribution is without question the most useful of all distributions for continuous random variables. Pair of continuous random variables X and Y is said to have a bivariate normal distribution if it has a joint probability density function of the form,

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp$$

$$\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

$$-\infty < x < +\infty, -\infty < y < +\infty, \sigma_x, \sigma_y > 0 \text{ and } -1 \le \rho \le +1$$

A special notation for this is given by Bain (1992) as (X, Y) BVN

$$(\mu_x, \mu_y, \sigma_x{}^2, \sigma_y{}^2, \rho)$$

which depends on 5 parameters

$$-\infty < x < +\infty, -\infty < y < +\infty,$$
$$\sigma_x, \sigma_y > 0 \text{ and } -1 \le \rho \le +1$$

- The exponential distribution is used sufficiently in statistical modeling for life testing, arrival and waiting time distributions or the distribution of time interval. The bi-variate exponential distribution have been studied by some authors like (Gumbel, 1961; Marshal and Olkin, 1967).

The bi-variate exponential distribution introduced by Marshall and Olkin (1967) is essentially a three parameters family with the distribution function:

$$f(x,y) = \exp-[\alpha_1 x + \alpha_2 y + \alpha_{12}\max(x,y)]$$

Where:

$$x, y \ge 0, \alpha_1, \alpha_2, \alpha_{12} \ge 0$$

and the distributions of X and Y are Univariate exponential with,

$$\lambda_1 = (\alpha_1 + \alpha_{12})^{-1} \text{ and } \lambda_2 = (\alpha_2 + \alpha_{12})^{-1}$$

respectively. The Correlation Coefficient between X and Y is given as.

**Statistical hypothesis and the principle of power Function:** When we have a simple null hypothesis versus Simple Alternative, the use of critical region for testing hypotheses is a very powerful tool. But, the problem of determining how good a test is requires studying the size of the type II error for composite hypothesis (Hoel, 1971).

For more general classes of alternatives, the size of the type II error ($\beta$) will depend on the particular alternative value of $\theta$ being considered. In order to determine how powerful the chosen test may be compared to competing tests, it is necessary to compare the type II errors for all possible alternative values of $\theta$ rather than for just one alternative value. For this purpose, it is necessary to consider calculation of the size of the type II error as a function of $\theta$.

Now, $\beta$ is the probability of wrong acceptance of the null hypothesis (by the given test) when $\theta$ is the true value of the parameter. It is usually more convenient to work exclusively with the rejection region (critical region); therefore, it is customary to calculate 1-$\beta$, which is the probability that the test rejects the null hypothesis when $\theta$ is the true value of the parameter. The function 1-$\beta$ is called the power function of the test in question and this we have denoted $\psi_\phi(\theta)$, which reads; the power of the test $\phi$ at $\theta$ i.e. when $\theta$ is the true value of the parameter space. Since $\psi_\phi(\theta) = 1-\beta$, seeking for a test that minimizes the type II error $\beta$ is equivalent to seeking for one that maximizes the power,$\psi_\phi(\theta)$. This principle can assist in comparing tests when we have a composite alternative (Hoel, 1971).

## RESULTS AND DISCUSSION

For each combination of sample size and population correlation coefficients, 100 samples were simulated from each of bi-variate normal and exponential distribution. The sample correlation values were computed using the Pearson's moment (r), Kendall's ($\tau$) and the Spearman's rank ($r_s$) correlations coefficients. Sample sizes n = 5, 10, 15, 20, 25, 30, 40, 50, 100 and population correlation coefficient $\rho$ = 0, 0.25, 0.5, 0.75 and 0.9 were used. The

Table 1: Percentage rejection out of 100 trials under the normal distribution

| ρ = | 0 | 0.25 | 0.5 | 0.75 | 0.9 | 0 | 0.25 | 0.5 | 0.75 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Test | α = 0.01 | | | | | α = .05 | | | | |
| r | r | 0 | 24 | 24 | 44 | 0 | 10 | 22 | 55 | 86 |
| τ | 1 | 1 | 4 | 7 | 29 | 4 | 7 | 16 | 41 | 63 |
| $r_s$ | 6 | 1 | 4 | 7 | 29 | 6 | 7 | 16 | 41 | 63 |
| | N = 5 | | | | | | | | | |
| r | 1 | 3 | 15 | 70 | 97 | 5 | 15 | 45 | 87 | 100 |
| τ | 2 | 1 | 9 | 60 | 87 | 5 | 15 | 37 | 84 | 98 |
| $r_s$ | 2 | 1 | 10 | 60 | 92 | 7 | 14 | 36 | 81 | 98 |
| | N = 10 | | | | | | | | | |
| r | 1 | 3 | 30 | 87 | 99 | 4 | 21 | 59 | 98 | 100 |
| τ | 1 | 3 | 18 | 70 | 96 | 5 | 17 | 51 | 95 | 99 |
| $r_s$ | 1 | 3 | 25 | 78 | 98 | 6 | 15 | 52 | 94 | 99 |
| | N = 15 | | | | | | | | | |
| r | 1 | 10 | 45 | 97 | 100 | 4 | 37 | 65 | 98 | 100 |
| τ | 1 | 8 | 36 | 94 | 100 | 6 | 26 | 65 | 97 | 100 |
| $r_s$ | 2 | 7 | 38 | 94 | 100 | 8 | 28 | 63 | 97 | 100 |
| | N = 20 | | | | | | | | | |
| r | 0 | 15 | 57 | 99 | 100 | 2 | 37 | 85 | 100 | 100 |
| τ | 0 | 10 | 47 | 97 | 100 | 4 | 29 | 80 | 100 | 100 |
| $r_s$ | 0 | 9 | 46 | 97 | 100 | 3 | 28 | 80 | 100 | 100 |
| | N = 25 | | | | | | | | | |
| r | 0 | 16 | 66 | 100 | 100 | 3 | 37 | 90 | 100 | 100 |
| τ | 2 | 10 | 61 | 100 | 100 | 4 | 35 | 84 | 100 | 100 |
| $r_s$ | 2 | 10 | 64 | 100 | 100 | 4 | 34 | 86 | 100 | 100 |
| | N = 30 | | | | | | | | | |
| r | 0 | 22 | 81 | 100 | 100 | 3 | 45 | 94 | 100 | 100 |
| τ | 0 | 15 | 78 | 100 | 100 | 2 | 37 | 91 | 100 | 100 |
| $r_s$ | 0 | 16 | 79 | 100 | 100 | 2 | 39 | 90 | 100 | 100 |
| | N = 40 | | | | | | | | | |
| r | 0 | 35 | 95 | 100 | 100 | 5 | 58 | 100 | 100 | 100 |
| τ | 0 | 31 | 89 | 100 | 100 | 6 | 55 | 98 | 100 | 100 |
| $r_s$ | 0 | 31 | 90 | 100 | 100 | 8 | 55 | 98 | 100 | 100 |
| | N = 50 | | | | | | | | | |
| r | 0 | 69 | 100 | 100 | 100 | 4 | 80 | 100 | 100 | 100 |
| τ | 1 | 61 | 100 | 100 | 100 | 4 | 78 | 100 | 100 | 100 |
| $r_s$ | 1 | 60 | 100 | 100 | 100 | 5 | 77 | 100 | 100 | 100 |
| | N = 100 | | | | | | | | | |

Table 2: Percentage rejection out of 100 trials under The exponential distribution

| ρ = | 0 | 0.25 | 0.5 | 0.75 | 0.9 | 0 | 0.25 | 0.5 | 0.75 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Test | α = 0.01 | | | | | α = 0.05 | | | | |
| r | 2 | 6 | 9 | 15 | 41 | 4 | 15 | 27 | 43 | 64 |
| τ | 0 | 1 | 6 | 12 | 23 | 4 | 13 | 17 | 37 | 49 |
| $r_s$ | 0 | 1 | 6 | 12 | 29 | 3 | 13 | 17 | 37 | 49 |
| | N=5 | | | | | | | | | |
| r | 2 | 9 | 19 | 70 | 85 | 6 | 18 | 40 | 73 | 93 |
| τ | 0 | 3 | 16 | 61 | 75 | 1 | 18 | 40 | 79 | 93 |
| $r_s$ | 0 | 7 | 16 | 58 | 73 | 2 | 18 | 38 | 75 | 93 |
| | N = 10 | | | | | | | | | |
| r | 1 | 9 | 38 | 79 | 99 | 7 | 27 | 33 | 91 | 100 |
| τ | 1 | 6 | 31 | 72 | 97 | 5 | 28 | 37 | 91 | 100 |
| $r_s$ | 2 | 8 | 30 | 72 | 97 | 6 | 28 | 38 | 87 | 100 |
| | N = 15 | | | | | | | | | |
| r | 2 | 10 | 53 | 80 | 100 | 6 | 26 | 73 | 98 | 100 |
| | 0 | 11 | 53 | 88 | 100 | 4 | 33 | 74 | 99 | 100 |
| $r_s$ | 0 | 10 | 51 | 86 | 100 | 6 | 30 | 73 | 97 | 100 |
| | N = 20 | | | | | | | | | |
| r | 1 | 17 | 56 | 95 | 100 | 6 | 39 | 70 | 99 | 100 |
| τ | 1 | 16 | 63 | 95 | 100 | 3 | 43 | 88 | 100 | 100 |
| $r_s$ | 1 | 19 | 62 | 94 | 100 | 5 | 41 | 84 | 99 | 100 |
| | N = 25 | | | | | | | | | |
| r | 0 | 16 | 76 | 98 | 100 | 6 | 42 | 86 | 100 | 100 |
| τ | 1 | 23 | 71 | 98 | 100 | 5 | 51 | 87 | 100 | 100 |
| $r_s$ | 2 | 22 | 69 | 96 | 100 | 5 | 52 | 87 | 100 | 100 |
| | N = 30 | | | | | | | | | |
| r | 3 | 22 | 77 | 100 | 100 | 7 | 40 | 92 | 100 | 100 |

Table 2: Continued

| ρ = | 0 | 0.25 | 0.5 | 0.75 | 0.9 | 0 | 0.25 | 0.5 | 0.75 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Test | | | $\alpha = 0.01$ | | | | | $\alpha = 0.05$ | | |
| τ | 1 | 34 | 80 | 100 | 100 | 4 | 57 | 95 | 100 | 100 |
| $r_s$ | 1 | 31 | 78 | 100 | 100 | 5 | 55 | 95 | 100 | 100 |
| | N = 40 | | | | | | | | | |
| r | 1 | 30 | 91 | 100 | 100 | 5 | 54 | 97 | 100 | 100 |
| τ | 1 | 43 | 93 | 100 | 100 | 4 | 71 | 98 | 100 | 100 |
| $r_s$ | 1 | 43 | 92 | 100 | 100 | 4 | 69 | 96 | 100 | 100 |
| | N = 50 | | | | | | | | | |
| r | 1 | 54 | 100 | 100 | 100 | 3 | 74 | 100 | 100 | 100 |
| τ | 1 | 72 | 100 | 100 | 100 | 5 | 93 | 100 | 100 | 100 |
| $r_s$ | 1 | 68 | 100 | 100 | 100 | 5 | 92 | 100 | 100 | 100 |
| | N = 100 | | | | | | | | | |

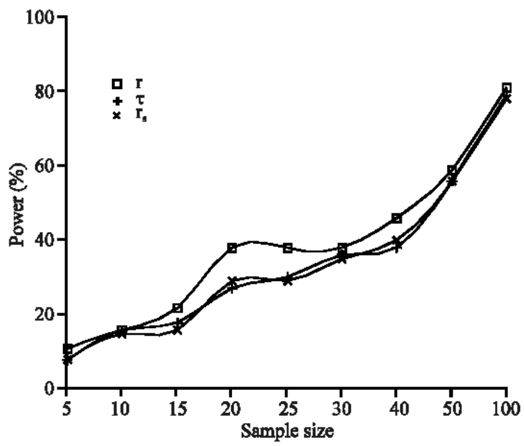


Fig. 1: Power given ρ = 0.25 and α = 0.05 in the normal distribution

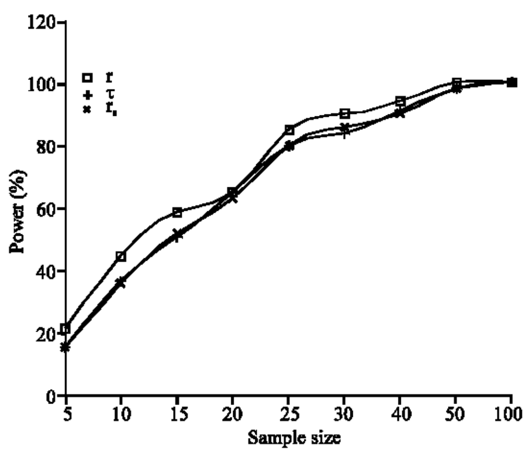


Fig. 2: Power given ρ = 0.5 and α = 0.05 in the normal distribution

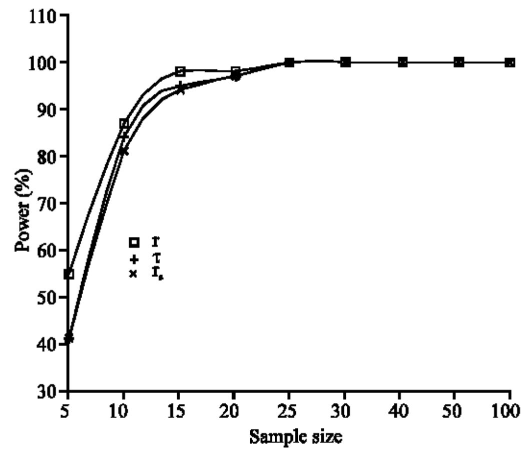


Fig. 3: Power given ρ = 0.75 and α = 0.05 in the normal distribution

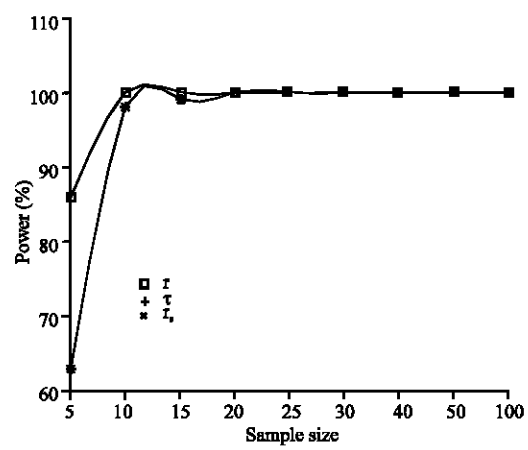


Fig. 4: Power given ρ = 0.9 and α = 0.05 in the normal distribution

number of rejections out of 100 trials for each test given α = 0.01 and α = 0.05 are presented in Table 1 and 2 for the normal exponential distribution, respectively.

In the normal distribution, it is clear that r has the highest power in all the sample sizes considered. This is consistent with the results of Bhattacharyya *et al.* (1970).

With sample size as small as n = 10, r will perform excellently if ρ is high enough (i.e. ρ = 0.9), particularly if α = 0.05.

However, when the sample size is increased to 20, both r and the non-parameter tests are equally effective in measuring independence if the value ρ is high (ρ = 0.9 say), but for lesser value of ρ, r is still preferred.

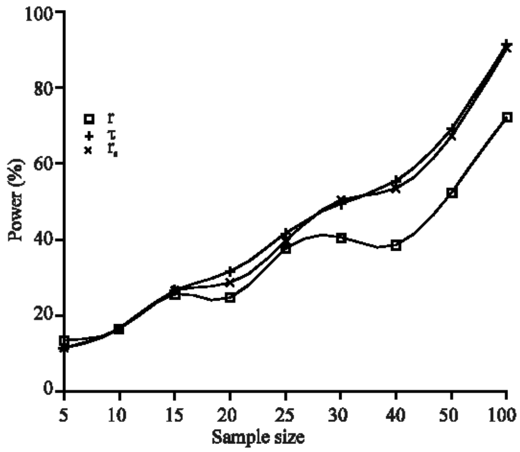


Fig. 5: Power given ρ = 0.25 and α = 0.05 in the Exponential distribution

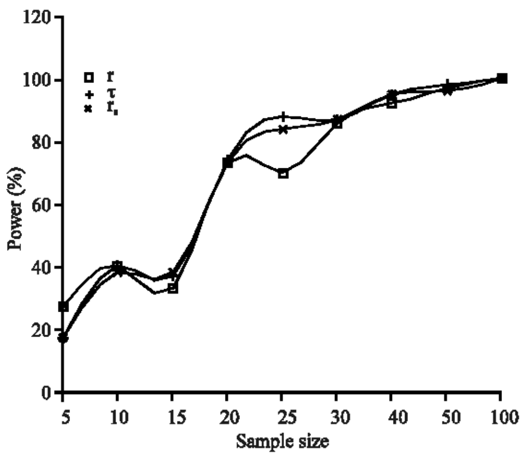


Fig. 6: Power given ρ = 0.5 and α = 0.05 in the Exponential distribution

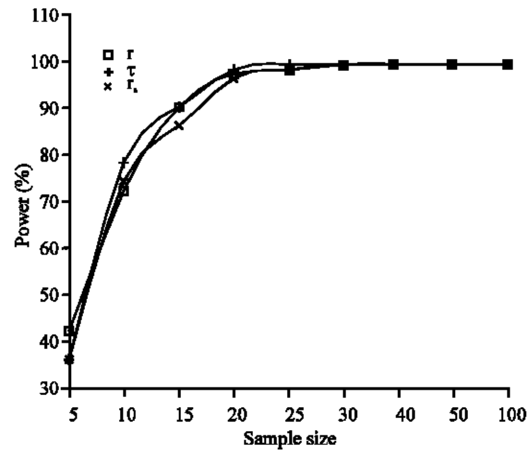


Fig. 7: Power given ρ = 0.75 and α = 0.05 in the Exponential distribution

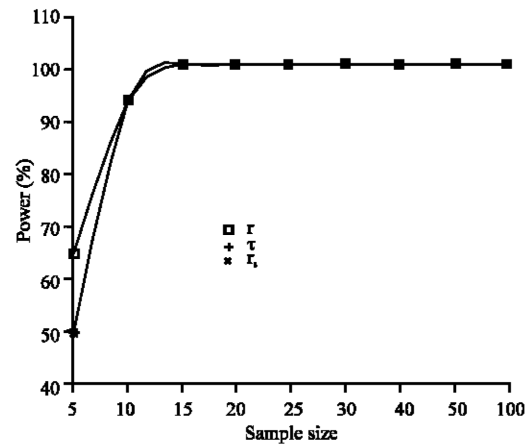


Fig. 8: Power given ρ = 0.9 and α = 0.05 in the Exponential distribution

Furthermore, in samples of size 30, 40 and 50, with ρ = 0.75, the three tests are equally effective, while in the samples of size 100, with ρ = 0.5, the three testes are equally effective (Fig. 1-4).

Under the Exponential distribution, the performance of the non-parametric tests (τand $r_s$) are better than that ofr, except for smaller sample sizes e.g. when n = 5 and n = 10 (α = 0.01).

Also, when the sample size is n = 15 (α = 0.05), 20, 25 and 30 (and α = 0.01) the three tests are equally effective for ρ = 0.9 while for n = 30 (α = 0.05), 40 and 50, with ρ≤ 0.75, the tests are equally effective.

In addition, when n = 100, with ρ≤0.75, the three tests are equally effective (Fig. 5-8).

On the overall, it was discovered that as the sample size increases, the power of the tests improve, irrespective of the distribution, so that with sample size as large as 100, any of the correlation coefficients may be used to test independence if it is confirmed that ρ is at least 0.5. But, the power of Pearson's moment correlation coefficient is consistently higher or equal to those of the non-parametric correlation coefficients under the normal distribution.

Again, as the population correlation (ρ) value increases in all the sample sizes considered, the power of each of the tests also increase in all the sample sizes considered.

Also, under the normal distribution, the non-parametric tests tends to commit type I error (rejecting Independence when ρ = 0) more often than the Pearson's moment correlation coefficient, while in the exponential distribution, the reverse is the case. These are as well consistent with Bhattacharyya *et al.* (1970).

## CONCLUSION

In conclusion, we feel that when both the sample size and ρ are large, any of the correlation Coefficients could be used to test independence, given the two distribution and also, that r is robust for smaller sample sizes under the exponential distribution.

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