Journal of Modern Mathmatics and Statistics 2 (2): 73-77, 2008
ISSN: 1994-5388

Coefficient Estimates for α-λ Sprial-Like Functions of Order β

[1]S. Rajasekaran and [2]K. Subathra
[1]Department of Mathematics, Crescent Engineering College, Chennai, Tamilnadu, India
[2]Sri Sai Ram Engineering College, Chennai, Tamilnadu, India

Abstract: Let $\alpha \geq 0$, $0 \leq \beta < 1$, $|\lambda| < \pi/2$ and suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be α-λ sprial-like of order β if it satisfies Re $\{(e^{i\lambda}-\alpha)\, zf'(z)/f(z) + \alpha\,(1+zf''(z)/f'(z))$. In this study we determine sharp bounds for the moduli $|a_n|$ of the coefficients.

Key words: Univalent functions, α-star like functions, λ-spiral-like functions, Bazilevic functions

INTRODUCTION

Let A denote the class of functions normalized by

$$f(z)= z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $U = \{z: |z| < 1\}$. For $0 \leq \beta < 1$, we will let $S^*(\beta)$ represent the class of functions contained in A which are univalent and star like of order β.

A function f(z) ε A is said to be spiral-like if there exists a $\lambda(|\lambda| < \pi/2)$ such that Re $e^{i\lambda}\, zf'(z)/f(z) > 0$.

Libera (1967) extended this definition to function spiral-like of order β. We say that f(z) εA is λ-spiral-like of order $\beta(0 \leq \beta, 1, |\lambda| < \pi/2)$ if Re $e^{i\lambda}\, zf'(z)/f(z) > \beta \cos\lambda$.

A function f(z) εA satisfyng f(z). $f'(z) \neq 0$ $(0 < |z| < 1.)$ is said to be α-star like of order B $(\alpha \geq 0, 0 \leq \beta < 1)$ if Re $\{(1-\alpha)\, zf'(z)/f(z) + \alpha\,(1+zf''(z)/f'(z))\} > \beta$.

A function f(z) εA satisfyng f(z). $f'(z) \neq 0$ $(0 < |z| < 1.)$ is said to be α-λ sprial-like of order β if

$$\text{Re}\left\{\begin{array}{l}(e^{i\lambda} - a)\, zf'(z)/f(z) \\ +\alpha\,(1 + z f''(z)/f'(z))\end{array}\right\} > \beta\cos\lambda \quad (1.1)$$

Where $\alpha \geq 0$, $0 \leq \beta < 1$, $|\lambda| < \pi/2$. It is denoted by $S(\alpha, \lambda, \beta)$

The class of α-λ-spiral-like functions of order β was first introduced by Silvia (1974).
The class of α-λ-spiral-like functions of order β.

Remarks:

- For α = 0, S(0, λ, β) is the class of λ-spiral-like functions of order β.
- For λ = 0, S(α, 0, β) is the class of α-convex functions of order β.
- For λ = β, S(0, 0, β) is the class of α-star-like functions of order β.

Definition: Due to the results by Eenigenburg *et al*. (1972). A function

$$f(z)= z + \sum_{n=2}^{\infty} a_n z^n$$

is said to be a Bazilevic function of type $e^{i\lambda}/\alpha$ and order β if

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha}\int_0^z \sigma(\xi)^{\frac{\cos\lambda}{\alpha}}\, \xi^{-1+\frac{i\sin\lambda}{\alpha}}\, d\xi\right]^{\alpha e^{-i\lambda}}$$

for some $\sigma(\xi)$ ε $S^*(\beta)$. We denote this by f(z) ε

$$B\left(\frac{e^{i\lambda}}{a}, \beta\right)$$

In this study we give sharp coefficient estimates for a_n, n = 2, 3.....

INTEGRAL REPRESENTATION

Lemma 1:

$$\text{Let } g(z) = f(z).\left[\frac{z f'(z)}{f(z)}\right]^{a e^{-i\lambda}}$$

zεU. If we choose the branch of

Corresponding Author: K. Subathra, Department of Mathematics, Crescent Engineering College, Chennai, Tamilnadu, India

$$\left[\frac{zf'(z)}{f(z)}\right]^{ae^{-i\lambda}}$$

Which is 1 at z = 0 for α > 0 and |λ| < π/2, then g ε S(0, λ, β) if and only if f(z) ε S(α, λ, β).

Proof:

$$\text{Since, } g(z) = f(z).\left[\frac{z\,f'(z)}{f(z)}\right]^{\alpha e^{-i\lambda}}$$

$$e^{i\lambda}\frac{zg'(z)}{g(z)} = \left(e^{i\lambda} - a\right)\frac{z\,f'(z)}{f(z)} + \alpha\left(1 + \frac{z\,f''(z)}{f(z)}\right)$$

Hence g ε S(0, λ, β) ⇔ f(z) ε S(α, λ, β).
The following resultss are due to Silvia (1974)

Lemma 2: A function f(z) ε

$$\varepsilon\, B\left(\frac{e^{i\lambda}}{\alpha}, \beta\right)$$

If and only if there exists a function g(ξ) ε S(0, λ, β) such that

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha}\int_0^z g(\xi)^{\frac{e^{i\lambda}}{\alpha}}\xi^{-1}d\xi\right]^{\alpha e^{-i\lambda}} \qquad (2.1)$$

Where the powers are meant as principal values.

Lemma 3:

$$B\left(\frac{e^{i\lambda}}{\alpha}, \beta\right) = S(\alpha, \lambda, \beta) \quad \text{for } a > 0, 0 \le \beta < 1, |\lambda| < \frac{\pi}{2}$$

Lemma 4: A neccessary and sufficient for f(z) to be in S(α, λ, β) is that f(z) have the integral representation,

$$f(z) = \left[\frac{e^{i\lambda}}{\alpha}\int_0^z g(\xi)^{\frac{e^{i\lambda}}{a}}\xi^{-1}d\xi\right]^{\alpha e^{-i\lambda}} \qquad (2.2)$$

for some g(ξ) ε S(0, λ, β), where the powers are assumed to be principal values.

If g(z) ε S(0, λ, β), then the Herglotz representation Singh (1969) for g(z) is

$$\log\frac{g(z)}{z} = -2(1-\beta)\cos\lambda \; e^{-i\lambda}\int_{-\pi}^{\pi}\log\left(1 - e^{-it}z\right)d\mu(t) \qquad (2.3)$$

By lemmal, for f(z) ε S(α, λ, β) this becomes

$$\frac{1}{(1-\beta)\cos\lambda\; e^{-i\lambda}}\log\left[\frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right]^{\alpha e^{-i\lambda}}\right] = 2\int_{-\pi}^{\pi}\log\left(1 - e^{-it}z\right)^{-1}d\mu(t) \qquad (2.4)$$

As the arguments given in Silvia (1974), the relation (2.4) shows that the principal values of

$$\frac{1}{(1-\beta)\cos\lambda\; e^{-i\lambda}}\log\left[\frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right]^{\alpha e^{-i\lambda}}\right]$$

lie for each zεE, in a closed convex domain boundry by the curve

$$\Gamma_\rho = \left\{\begin{matrix}\log\left(1 - e^{-it}z\right)^{-1}; \\ -\pi \le t \le \pi, 0 < \rho < 1\end{matrix}\right\}$$

Under the mapping

$$\log(1 - \varepsilon z)^{-2}, |\varepsilon| = 1$$

Let log (f(z)/z). log (zf ‘ (z)/f(z)) and log (1-z) be regular in U and possess branches which have the value zero at z = 0. If we assume that the measure μ(t) has value zero every here on Γ_p except one point where it has value 1, then we get from (2.4)

$$\frac{1}{(1-\beta)\cos\lambda\; e^{-i\lambda}}\log\left[\frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right]^{\alpha e^{-i\lambda}}\right] = 2\log(1 - \varepsilon z)^{-1}$$

$$\log\left[\frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right]^{\alpha e^{-i\lambda}}\right] = \log(1 - \varepsilon z)^{-2(1-\beta)\cos\lambda\; e^{-i\lambda}} \qquad (2.5)$$

setting w = log f(z)/z in (2.5) and noting that

$$\frac{zf'(z)}{f(z)} = 1 + z\frac{dw}{dz}$$

The Eq. 2.5 becomes

$$w + \alpha e^{-i\lambda}\log\left(1 + z\frac{dw}{dz}\right) = \log(1 - ez)^{-2(1-\beta)\cos\lambda\; e^{-i\lambda}}$$

A solution of this differential equation with the initial condition f(0) = 0

$$f(z)=\left[\frac{e^{i\lambda}}{\alpha}\int_0^z \xi^{\frac{e^{i\lambda}}{\alpha}-1}(1-\varepsilon\xi)^{\frac{-2(1-\beta)\cos\lambda}{\alpha}}d\xi\right]^{\alpha e^{-i\lambda}} \quad |\varepsilon|=1$$

The functionn (2.7) gives an integral representation for member of the class = S(α, λ, β).
Since, the function

$$z(1\text{-}z)^{-2(1-\beta)\cos\lambda e^{-i\lambda}}$$

is extremal in S(0, λ, β), it appears that the function

$$f^*(z)=\left[\frac{e^{i\lambda}}{\alpha}\int_0^z \xi^{\frac{e^{i\lambda}}{\alpha}-1}(1-\xi)^{\frac{-2(1-\beta)\cos\lambda}{\alpha}}d\xi\right]^{\alpha e^{-i\lambda}} \tag{2.8}$$

is the extremal in the class S (α, λ, β).

RESULTS

Coefficient estimates: Considering the principal values of the power in (2.8). We can write

$$f^*(z) = zH(z) \tag{3.1}$$

where

$$H(z)=\left[1+\sum_{n=1}^{\infty}c_n z^n\right]^{\alpha e^{-i\lambda}} \tag{3.2}$$

and

$$c_n=\frac{1}{n!\left(1+n\,\alpha e^{-i\lambda}\right)}\prod_{k=0}^{n-1}\left\{\frac{2(1-\beta)\cos\lambda}{\alpha}+k\right\} \tag{3.3}$$

Theorem: Let

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \;\varepsilon\; S(\alpha,\lambda,\beta)\cdot$$

Let S(n) be the set of all n-tuples $(x_1, x_2, \ldots\ldots x_n)$ of non-negative integers for which

$$x_1+2x_2+3x_3+\ldots\ldots+nx_n=n$$

and for each n tuple define q by

$$x_1+x_2+\ldots\ldots+x_n=q$$

If

$$\tau\left(\alpha e^{-i\lambda},q\right)=\alpha\varepsilon^{-i\lambda}\left(\alpha e^{-i\lambda}\text{-}1\right)\ldots\ldots\ldots\left(\alpha e^{-i\lambda}\text{-}q\right)\text{ with }\tau\left(\alpha e^{-i\lambda},0\right)=\alpha e^{-i\lambda},$$

then for n = 1, 2.........

$$|a_{n+1}|\le|b_{n+1}| \tag{3.4}$$

$$|b_{n+1}|=\left|\sum\frac{\tau\left(\alpha e^{-i\lambda},q\text{-}1\right)C_1^{x_1}C_2^{x_2}\ldots\ldots\ldots C_n^{x_n}}{x_1!x_2!\ldots\ldots\ldots x_n!}\right| \tag{3.5}$$

Where the summation is taken over all n-tuples in S(n) and C_n ‘s are given by (3.3).

Proof: By (3.1) and (3.2) , Since

$$\begin{aligned}H(z)&=[h(z)]^{\alpha e^{-i\lambda}}\\&=1+\sum_{n=1}^{\infty}a_{n+1}z^n,\end{aligned} \tag{3.6}$$

Where:

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n$$

on differentiating (3.6) we obtain

$$\therefore H'(z).h(z)=\alpha e^{-i\lambda}h'(z)H(z)$$

$$\begin{aligned}&\text{i.e}\left(\sum_{n=1}^{\infty}n\,a_{n+1}z^{n\text{-}1}\right)\left(1+\sum_{n=1}^{\infty}c_n z^n\right)\\&=\alpha e^{-i\lambda}\left(\sum_{n=1}^{\infty}n\,c_n z^{n\text{-}1}\right)\left(1+\sum_{n=1}^{\infty}a_{n+1}z^n\right)\end{aligned} \tag{3.7}$$

Equating the coefficients of z^{n-1}, we get

$$\sum_{k=0}^{n}\left[k-\alpha e^{-i\lambda}(n-k)\right]c_{n\text{-}k}a_{k+1}=0 \quad c_0=a_1=1 \tag{3.8}$$

From this we get

$$a_{n+1}=-\frac{1}{n}\sum_{k=0}^{n\text{-}1}\left[k-\alpha e^{-i\lambda}(n-k)\right]\;c_{n\text{-}k}a_{k+1} \tag{3.9}$$

We prove the theorem by induction.

For each integer n, the coefficients b_{n+1} defined by (3.5) satisfy Eq. 3.9
For each k =1, 2,.........n-1

$$b_{n+1}=\sum\frac{\tau\left(\alpha e^{-i\lambda},j-1\right)\;c_1^{x_1}c_2^{x_2}\ldots\ldots\ldots c_k^{x_k}}{x_1!\;x_2!\ldots\ldots\ldots\ldots x_n!} \tag{3.10}$$

Where:

$$j = x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n$$

And the sum is taken over S(k), the set of all non negative k-tuples ($x_1, x_2, \ldots\ldots\ldots x_n$) for which

$$\sum_{i=1}^{n} i x_i = k$$

Now if k < n, we can enlarge the k-tuples to an n-tuple by adjoining suitable many zeros.
Then any solution of

$$x_1 + 2x_2 + \ldots\ldots\ldots\ldots + n x_n = k, k < n. \quad (3.11)$$

In non negative integers must give xi = 0 for I = k + 1, k + 2.........n and the inclusion of the factors $c_i^{xi}/x_i!$ in (3.10) does not change the value because these factors are 1 for I = k+ 1, k + 2.........n
Hence (3.10) can be replaced by

$$b_{n+1} = \sum \frac{\tau\left(\alpha e^{-i\lambda}, j\text{-}1\right) c_1^{x_1} c_2^{x_2} \ldots\ldots\ldots\ldots c_n^{x_n}}{x_1! x_2! \ldots\ldots\ldots\ldots x_n!} \quad k \le n \quad (3.12)$$

Where

$$j = \sum_{i=1}^{n} x_i$$

and the sum taken over S(k), the set of all non negative intagers solution of (3.11).
If we use (3.12) in the right hand side of (3.9) we get

$$\frac{-1}{n} \sum_{k=0}^{n-1} \sum_{S(K)} \frac{\left[k - \alpha e^{-i\lambda}(n-k)\right] \tau\left(\alpha e^{-i\lambda}, j\text{-}1\right) c_{n\text{-}k} c_1^{x_1} c_2^{x_2} \ldots\ldots\ldots\ldots c_n^{x_n}}{x_1! x_2! \ldots\ldots\ldots\ldots x_n!} \quad (3.13)$$

Now let ($y_1, y_2, \ldots\ldots y_n$) be any fixed n-tuple in S(n), such that

$$\sum_{i=1}^{n} i y_i = n, \ \sum_{i=1}^{n} y_i = q \quad (3.14)$$

We are to determine the coefficient C of

$$c_1^{y_1} c_2^{y_2} \ldots\ldots\ldots\ldots c_n^{y_n} \text{ in } (3.13)$$

This coefficient may arise from combining several terms from the sum and such terms arise if and only if

$$c_{n\text{-}k} c_1^{x_1} c_2^{x_2} \ldots\ldots\ldots\ldots c_n^{x_n} = c_1^{y_1} c_2^{y_2} \ldots\ldots\ldots\ldots c_n^{y_n}$$

In particular let 'a' be an index for which $y_a \ge 1$ and let $x_i = y_i$ if $i \ne a$ and let $x_a = y_a$-1. For this fixed a, we have

$$j = \sum_{i=1}^{n} x_i = q - 1.$$

In (3.13) we put n-k = a. If A is the set of a for which $y_a \ne 0$ then

$$\begin{aligned}
C &= \frac{-1}{n} \sum_{a \in A} \frac{\left(n - a - a\,\alpha\, e^{-i\lambda}\right) \tau\left(\alpha e^{-i\lambda}, q-2\right)}{x_1! x_2! \ldots\ldots\ldots\ldots x_n!} \\
&= \frac{-1}{n} \sum_{a \in A} \frac{y_a\left(n - a - a\,\alpha\, e^{-i\lambda}\right) \tau\left(\alpha e^{-i\lambda}, q-2\right)}{y_1! y_2! \ldots\ldots\ldots\ldots y_n!} \\
&= \frac{\tau\left(\alpha e^{-i\lambda}, q-2\right)}{n\, y_1! y_2! \ldots\ldots\ldots\ldots y_n!} \sum_{a \in A} \left(a y_a + a\,\alpha\, e^{-i\lambda} y_a - n y_a\right) \\
&= \frac{\tau\left(\alpha e^{i\lambda}, q-2\right)}{n\, y_1! y_2! \ldots\ldots\ldots\ldots y_n!} \sum_{a=1}^{n} \left(a y_a + a\,\alpha\, e^{i\lambda} y_a - n y_a\right) \quad (3.15) \\
&= \frac{\tau\left(\alpha e^{i\lambda}, q-2\right)}{n\, y_1! y_2! \ldots\ldots\ldots\ldots y_n!} \left(n + n\alpha e^{-i\lambda} - nq\right) \\
&= \frac{\tau\left(\alpha e^{i\lambda}, q-2\right)}{n\, y_1! y_2! \ldots\ldots\ldots\ldots y_n!} n\left[\alpha e^{-i\lambda} - (q-1)\right] \\
&= \frac{\tau\left(\alpha e^{i\lambda}, q-1\right)}{y_1! y_2! \ldots\ldots\ldots\ldots y_n!}
\end{aligned}$$

Which is precisely the coefficient of

$$c_1^{y_1} c_2^{y_2} \ldots\ldots\ldots\ldots c_n^{y_n}$$

required on the right side of (3.5). This argument holds for fixed ($y_1, y_2, \ldots\ldots y_n$).

DISCUSSION

$$\begin{aligned}
&\textbf{When } n = 1, \\
&x_1 + 2x_2 + 3x_3 + \ldots\ldots + n\, x_n = n \\
&\text{and } x_1 + x_2 + \ldots\ldots + x_n = q \text{ gives} \\
&x_1 = 1 \quad q = 1 \\
&|a_2| \le \frac{\tau\,(\alpha e^{-i\lambda}, 0) c_1\, x^1}{x_1!} \\
&= \frac{2e^{-i\lambda}(1-\beta)\cos\lambda}{(1 + \alpha e^{-i\lambda})}
\end{aligned}$$

As we discussedin the introduction, When $\lambda = \beta = 0$ the aboveine quality becomes α-convex function.

$$|a_2| \le \frac{2}{1+\alpha} \text{ - } \alpha - \text{convex functions}$$

When $\alpha = 0$

$$|a_2| \leq 2e^{-i\lambda}(1-\beta)\cos\lambda$$

λ-spiral function of order β
Similarly,
When n = 2,

$$x_1 + 2x_2 = 2,\ x_1 + x_2 = q$$

when

$$x_1 = 2 \quad x_2 = 0$$

when

$$x_1 = 0 \quad x_2 = 1$$

$$|a_3| \leq \frac{(2-2\beta)\cos\lambda}{2!} \left| \frac{1+(2-2\beta)\cos\lambda\ e^{-i\lambda} + 3\alpha\,(2-2\beta)\cos\lambda e^{-2i\lambda} + 2\alpha e^{-i\lambda} + \alpha^2 e^{-2i\lambda}}{(1+\alpha e^{-i\lambda})^2(1+2\alpha\ e^{-i\lambda})} \right|$$

when $\alpha = 0$

$$|a_3| \leq \frac{(2-2\beta)\cos\lambda}{2!} \left| \frac{1+(2-2\beta)\cos\lambda\ e^{-i\lambda}}{1} \right| - \lambda - \text{spiral functions}$$

when $\lambda = \beta$ 0

$$|a_3| \leq \frac{3+8a+a^2}{(1+a)^2(1+2\alpha)}$$

α-convex function ans so on.

REFERENCES

Eenigenburg, P., S. Miller, P. Mocanu and M. Reade, 1972. On a subclass of Bazilevic functions, Notices. Am. Math. Soc., 19: 706.

Libra, R., 1967. Univalent α-spiral functions. Can. J. Math., 19: 449-456.

Silvia, E.M., 1974. On a subclass of spiral-like functions. Proc. Am. Math. Soc., 44: 411- 420.

Singh, R., 1969. A note on spiral-like functions. J. Indian Math. Soc., 33: 49-55.