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Linearized Oscillations in Nonlinear Neutral Delay Impulsive Differential Equations

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Abstract: Our aim in this study, is to develop a linearized oscillation theory for nonlinear neutral delay impulsive differential equations. Precisely, we prove that a certain nonlinear neutral delay impulsive differential equation has the same oscillatory character as its associated linear impulsive equation.

Key words: Linearized oscillations, neutral delay impulsive differential equations

INTRODUCTION

Recently, linearized oscillation theory for first order differential equations with and without impulses have been discussed by Ladde *et al*. (1987), Bainov and Hristova (1987), Bainov and Simeonov (1998), Zhang *et al*. (2004), Xia *et al*. (2007) and Agarwal *et al*. (2000). However, there appear to be little or no results in linearized oscillations for first order nonlinear neutral delay impulsive differential equations. In this study, we develop some results on linearized oscillation theory which parallels the so-called linearized stability theory of differential and difference equations. Roughly speaking, we prove that certain nonlinear neutral delay impulsive differential equations have the same oscillatory character as the associated linear neutral delay impulsive differential equations.

Before the formulation of the problem considered in this study, we present some basic definitions and concepts that will be useful in our discussions throughout.

Let, S: = $\{t_k\}_{k\in E}$ denote the set of time points of impulses, where E represents a subscript set which can be the set of natural numbers N or the set of integers Z and satisfy the properties:

C1.1: If $\{t_k\}_{k\in E}$ is defined with E = N, then $0 < t_1 < t_2 < \ldots$ and

$$\lim_{k\to+\infty} t_k = +\infty$$

C1.2: If $\{t_k\}_{k\in E}$ is defined with E = Z, then $t_0 \le 0 < t_1, t_k < t_{k+1}$ for $k \in Z$, $k \neq 0$ and

$$\lim_{k\to\pm\infty} t_k = \pm\infty$$

Let, f: R × R → R and f_k: R → R, k ∈ Z be continuous functions and let x: R → R, then

$$\begin{cases} x' = f(t,x), \; t^{1} \neq t_k \\ \Delta x(t)\big|_{t=t_k} = f_k(x), \; t = t_k \end{cases} \quad (1.1)$$

Where,

$$\Delta x = x(t_k + 0) - x(t_k - 0)$$

Definition 1.1: The function x = φ(t) is a solution of (1.1) in the interval J: = (α, β) if

- φ(t) is differentiable in J, $t \neq t_k$, k ∈ N and satisfies the condition
  $\varphi'(t) = f(t, \varphi(t))$ for all t ∈ J, $t \neq t_k$ and k ∈ N
- φ(t) satisfies the relation
  $\varphi(t_k + 0) - \varphi(t_k - 0) = f_k(\varphi(t_k - 0))$, $t_k \in J$ and k ∈ Z

Definition 1.2: A solution x is said to be finally positive, if there exists T≥0 such that x(t) is defined for t≥T and x(t) > 0 for all t≥T:

- Finally, negative, if there exists T≥0 such that x(t) is defined for t≥T and x(t) < 0 for all t≥T
- Non-oscillatory, if it is either finally positive or finally negative
- Oscillatory, if it is neither finally positive nor finally negative (Lakshmikantham *et al*., 1989)

Usually, the solution x(t) for t ∈ J, t∉S of a given impulsive differential equation or its first derivative x'(t) is a piece-wise continuous function with points of discontinuity t_k, $t_k \in J\cap S$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r, which are defined as follows:

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Let, $r \in N$, $D \subset R$ and the sequence S be fixed.

Definition 1.3: PC (D, R) is the set of those functions which are continuous for all $t \in D$, $t \notin S$, $\forall k \in N$ and have discontinuity of the first kind for $t \in S$ and $k \in N$.

Definition 1.4: PC^r (D, R) is the set of those functions which are r-times continuously differentiable for all $t \in D$, $t \notin S$, $\forall k \in N$ and have discontinuity of the first kind for $t \in S$ and $k \in N$ (Bainov and Simeonov, 1998; Lakshmikantham *et al*., 1989).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols PC^r (D, R; S) and PC^r (D, R; S), $r \in N$.

Now let us consider the nonlinear neutral delay impulsive differential equation.

$$\begin{cases} \left[x(t)-p(t)g(x(t-\tau))\right]' + q(t)h(x(t-\sigma)) = 0, \ t \notin S \\ \Delta\left[x(t_k)-p(t_k)g(x(t_k-\tau))\right] + q_k h(x(t_k-\sigma)) = 0, \ t_k \in S \end{cases} \tag{1.2}$$

where, $\tau > 0$ and σ, $q_k \geq 0$ and associate with it, the following hypotheses:

$$\begin{cases} p \in PC^1(I_0, R_+), \ q \in PC(I_0, R_+), \ g \in C^1(R, R) \\ h \in C(R, R), \ \tau > 0, \ \sigma, q_k \geq 0 \end{cases} \tag{1.3}$$

where, $I_0 = [t_0, \infty)$ and

$$\lim_{t\to+\infty} p(t) \equiv p_0 \in [0,1), \quad \lim_{t\to+\infty} q(t) \equiv q_0 \in (0,+\infty) \tag{1.4}$$

$ug(u) > 0$ for $u \neq 0$, $g(u) \leq u$ for $u \geq 0$ and $g(u) \geq u$ for $u \leq 0$,

$$\lim_{u\to 0} \frac{g(u)}{u} = 1 \tag{1.5}$$

$uh(u) > 0$ for $u \neq 0$ and

$$\lim_{u\to 0} \frac{h(u)}{u} = 1 \tag{1.6}$$

Always when at least one of the conditions (1.5) or (1.6) holds, we will refer to Eq (1.7):

$$\begin{cases} \left[y(t)-p_0 y(t-\tau)\right]' + q_0 y(t-\sigma) = 0, t \notin S \\ \Delta\left[y(t_k)-p_0 y(t_k-t)\right] + q_{0k} y(t_k-\sigma) = 0, \ t_k \in S \end{cases} \tag{1.7}$$

as linearized in respect of Eq. (1.1).

The following lemma and theorem extracted from Bainov and Simeonov (1998), are needed in establishing the oscillatory conditions of the problem in question. They may also have further applications in analysis.

Lemma 1.1: Let, $p \in [0, 1)$, $\tau \in (0, \infty)$, $t_0 \in R$, $x \in C([t_0 - \tau, \infty), R_+)$ and assume that for every $\varepsilon > 0$ there exists a $t_\varepsilon \geq t_0$ such that

$$x(t) \leq (p+\varepsilon)x(t-\tau) + \varepsilon, \ t \geq t_\varepsilon \tag{1.8}$$

Then,

$$\lim_{t\to\infty} x(t) = 0$$

Consider the linear impulsive differential equation with delay

$$\begin{cases} x'(t) + p(t)x(t-\tau) = 0, \ t \notin S \\ \Delta x(t_k) + p_k x(t_k-\tau) = 0, \ t_k \in S \end{cases} \tag{1.9}$$

together with the corresponding inequalities

$$\begin{cases} x'(t) + p(t)x(t-\tau) \leq 0, \ t \notin S \\ \Delta x(t_k) + p_k x(t_k-\tau) \leq 0, \ t_k \in S \end{cases} \tag{1.10}$$

And

$$\begin{cases} x'(t) + p(t)x(t-\tau) \geq 0, \ t \notin S \\ \Delta x(t_k) + p_k x(t_k-\tau) \geq 0, \ t_k \in S \end{cases} \tag{1.11}$$

We assume that the following condition is fulfilled:

C1.3: $p \in PC(R_+, R)$ and $\tau \geq 0$.

Our aim, here is to establish the following results.

Theorem 1.1: Let condition C1.3 be fulfilled and let there exist a sequence of disjoint intervals $J_n = [\zeta_n, \eta_n]$ with $\eta_n - \zeta_n = 2\tau$, such that:

- For each $n \in N$, $t \in J_n$ and $t_k \in J_n$

$$p(t) \geq 0, \ p_k \geq 0 \tag{1.12}$$

- There exists $v_1 \in N$ such that for $n \geq v_1$

$$\int_{\eta_n-\tau}^{\eta_n} p(s)ds + \sum_{\eta_n-\tau \leq t_k < \eta_n} p_k \geq 1 \tag{1.13}$$

Then,

- The inequality (1.10) has no finally positive solution
- The inequality (1.11) has no finally negative solution
- Each regular solution of Eq. (1.9) is oscillatory

Corollary 1.1: Consider the Eq. (1.14):

$$\begin{cases} \left[x(t) - \sum_{j=1}^{M} p_j x(t-\tau_j)\right]' + \sum_{i=1}^{N} q_i x(t-\sigma_i) = 0, t \notin S \\ \Delta\left[x(t_k) - \sum_{j=1}^{M} p_j x(t-\tau_j)\right] + \sum_{i=1}^{N} q_{i0} x(t_k - \sigma_i) = 0, \forall t_k \in S \end{cases} \tag{1.14}$$

Let, p_j, $q_i \in (0, \infty)$ and $q_{i0} \in [0, \infty)$, τ_j, $\sigma_i \geq 0$ for $1 \leq j \leq M$ and $1 \leq i \leq N$ and assume that

$$\sum_{j=1}^{M} p_j \leq 1 \tag{1.15}$$

Then the following statements are equivalent:

- The Eq. (1.14) has a finally positive solution
- The characteristic system

$$\begin{cases} -\lambda + \lambda \sum_{j=1}^{M} p_j e^{\lambda\tau_j} (1-\mu)^{-v_j\xi} + \sum_{i=1}^{N} q_i e^{\lambda\sigma_i} (1-\mu)^{-v_i\xi} = 0, \\ -\mu + \mu \sum_{j=1}^{M} p_j e^{\lambda\tau_j} (1-\mu)^{-v_j\xi} + \sum_{i=1}^{N} q_{i0} e^{\lambda\sigma_i} (1-\mu)^{-v_i\xi} = 0 \end{cases} \tag{1.16}$$

where, λ and $\mu<1$ are constants, has a solution $(\lambda, \mu) \in R \times (-\infty, 0]$.

- The inequality

$$\begin{cases} \left[x(t) - \sum_{j=1}^{M} p_j x(t-\tau_j)\right]' + \sum_{i=1}^{N} q_i x(t-\sigma_i) \leq 0, t \notin S \\ \Delta\left[x(t_k) - \sum_{j=1}^{M} p_j x(t-\tau_j)\right] + \sum_{i=1}^{N} q_{i0} x(t_k - \sigma_i) \leq 0, \forall t_k \in S \end{cases} \tag{1.17}$$

has a non-increasing finally positive solution

- There exists an $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \in [0, \varepsilon_0]$, the inequality

$$\begin{cases} \left[x(t) - \sum_{j=1}^{M} (1-\varepsilon) p_j x(t-\tau_j)\right]' \\ + \sum_{i=1}^{N} (1-\varepsilon) q_i x(t-\sigma_i) \leq 0, t \notin S \\ \Delta\left[x(t_k) - \sum_{j=1}^{M} (1-\varepsilon) p_j x(t-\tau_j)\right] \\ + \sum_{i=1}^{N} (1-\varepsilon) q_{i0} x(t_k - \sigma_i) \leq 0, \forall t_k \in S \end{cases}$$

has a non-increasing finally positive solution.

RESULTS

Our aim in this study, is to establish conditions for the oscillation of all solutions of Eq. (1.2) in terms of the oscillation of all the solutions of Eq. (1.7) and vice versa.

We recall (Bainov and Simeonov, 1998) that every solution of Eq. (1.7) oscillates if and only if the characteristic equation:

$$H(\lambda) \equiv -\lambda - \lambda p_0 e^{\lambda\tau} (1-\mu)^{n_1} + q_0 e^{\lambda\sigma} (1-\mu)^{n_2} = 0 \tag{2.1}$$

where,

$$n_1 = \begin{cases} -m_1 = -i[t-\tau, t), & \text{if } \tau \geq 0, \\ m_1 = i[t, t-\tau), & \text{if } \tau < 0 \end{cases}$$

and

$$n_2 = \begin{cases} -m_2 = -i[t-\sigma, t), & \text{if } \sigma \geq 0, \\ m_2 = i[t, t-\sigma), & \text{if } \sigma < 0 \end{cases}$$

has no real roots. As $p_0 \in [0, 1)$, Eq. (2.1) has no roots in $[0, \infty)$ and so every solution of Eq. (1.7) oscillates if and only if Eq. (2.1) has no negative roots.

Notice that Eq. (2.1) is built from the characteristic system

$$\begin{cases} -\lambda - \lambda p_0 e^{\lambda\tau} (1-\mu)^{n_1} + q_0 e^{\lambda\sigma} (1-\mu)^{n_2} = 0 \\ -\mu - \mu p_0 e^{\lambda\tau} (1-\mu)^{n_1} + q_{0k} e^{\lambda\sigma} (1-\mu)^{n_2} = 0 \end{cases}$$

with the solution (λ, μ) satisfying the relation

$$\mu = \frac{q_{0k}}{q_0} \lambda$$

Theorem 2.1: Let conditions (1.3-1.6) be satisfied and that Eq. (2.1) has no real roots. Then every regular solution of Eq. (1.2) oscillates.

Proof: Let us assume on the contrary, that Eq. (1.2) has a non-oscillatory solution x(t). We further assume that x(t) is finally positive. The case where x(t) is finally negative is similar and is omitted. Set

$$z(t) = x(t) - p(t) g(x(t-\tau))$$

Then finally,

$$\begin{cases} z'(t) = -q(t) h(x(t-\sigma)) \leq 0 \\ \Delta z(t_k) = -q_k h(x(t_k - \sigma)) \leq 0 \end{cases} \tag{2.2}$$

and so z (t) is a decreasing function. We claim that x (t) is bounded. Otherwise, there exists a sequence of points $\{t_n\}_{n=1}^{\infty}$ for which

$$\lim_{n\to+\infty} t_n = +\infty, \quad \lim_{t_n\to+\infty} x(t_n) = +\infty$$

and

$$x(t_n) = \max_{s\le t_n} x(s)$$

Then from conditions (1.4) and (1.5)

$$\begin{aligned} z(t_n) &= x(t_n) - p(t_n)g(x(t_n - \tau)) \ge x(t_n) - p(t_n)x(t_n - \tau) \\ &\ge x(t_n)[1 - p(t_n)] \to +\infty \text{ as } n \to +\infty \end{aligned}$$

which contradicts the fact that z (t) is decreasing. Thus, x (t) is bounded and so

$$\lim_{t\to+\infty} z(t) \equiv L \in R \tag{2.3}$$

But then, the condition L < 0 cannot occur since

$$\int_{T_0}^{\infty} z(s)ds = -\infty \tag{2.4}$$

follows from L<∞.

On the other hand,

$$\int_{T_0}^{\infty} \left(x(s) - p(s)g(x(s-\tau))\right)ds =$$

$$\lim_{T\to\infty}\left(\int_{T_0}^{T} x(s)ds - \int_{T_0}^{T} p(s)g(x(s-\tau))ds\right)$$

$$= \lim_{T\to\infty}\left(\int_{T_0}^{T} x(s)ds - \int_{T_0-\tau}^{T-\tau} p(s+\tau)g(x(s))ds\right) = \lim_{T\to\infty}\left(\int_{T_0}^{T} x(s)ds - \right.$$

$$\left. - \int_{T_0-\tau}^{T_0} p(s+\tau)g(x(s))ds - \int_{T_0}^{T-\tau} p(s+\tau)g(x(s))ds - \int_{T-\tau}^{T} x(s)ds\right)$$

$$= \lim_{T\to\infty}\left(\int_{T_0}^{T-\tau} \left(x(s) - p(s+\tau)g(x(s))\right)ds + \int_{T-\tau}^{T} x(s)ds\right) -$$

$$- \int_{T_0-\tau}^{T_0} p(s+\tau)g(x(s))ds$$

The

$$\lim_{T\to\infty}\left(\int_{T_0}^{T-\tau} \left(x(s) - p(s+\tau)g(x(s))\right)ds + \int_{T-\tau}^{T} x(s)ds\right) \ge 0$$

and

$$\int_{T_0-\tau}^{T_0} p(s+\tau)g(x(s))ds \le 0$$

This contradicts condition (2.4), hence, L ≥ 0.
If L>0, then

$$\int_{T_0}^{\infty} x(s)ds = \infty$$

Since, x is bounded, $0 < x(s) \le M$, hence,

$$\inf_{0<u\le M} \frac{h(u)}{u} = m > 0$$

by Eq. (1.6) and what is more, h (u) > 0, u >0, Therefore, h (u)≥mu and 0≤u≤M, which implies

$$\begin{aligned} &-\int_{T_0}^{\infty} q(s)h(x(s-\sigma))ds \le \\ &-\int_{T_0}^{\infty} (q_0 - \varepsilon)m \cdot x(s-\sigma)ds = -\infty \end{aligned} \tag{2.5}$$

This contradicts the statement

$$\lim_{t\to+\infty} z(t) = L > 0$$

hence, L = 0. Thus,

$$L = \lim_{t\to+\infty}\left[x(t) + p(t)g(x(t-\tau))\right] = 0 \tag{2.6}$$

From Eq. (2.6) follows

$$x(t) + p(t)g(x(t-\tau)) < \varepsilon$$

if $t > \tilde{T}$, hence,

$$\begin{aligned} 0 < x(t) &< -p(t)g(x(t-\tau)) \\ &+ \varepsilon \le (\varepsilon - p_0)x(t-\tau) + \varepsilon \end{aligned}$$

where, $\varepsilon \in (0; 1 + p_0)$ is given. Then by Lemma 1.1,

$$\lim_{t\to+\infty} x(t) = 0 \tag{2.7}$$

Set

$$P(t) = p(t)\frac{g(x(t-\tau))}{x(t-\tau)} \text{ and } Q(t) = q(t)\frac{h(x(t-\sigma))}{x(t-\sigma)}$$

From the hypotheses and Eq. (2.7), it follows that

$$\lim_{t\to+\infty} P(t) = p_0 \in [0,1), \quad \lim_{t\to+\infty} Q(t) = q_0 \in (0,\infty)$$

and

$$\begin{cases} [x(t) - P(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0, \ t \notin S \\ \Delta[x(t_k) - P(t_k)x(t_k-\tau)] + Q_k x(t_k-\sigma) = 0, \ t_k \in S \end{cases} \quad (2.8)$$

where, $Q_k = q_k$. We integrate both sides of Eq. (2.8) from t to $+\infty$, with sufficiently large $t \in [T_1, \infty)$ and from Eq. (2.6) obtain

$$x(t) - P(t)x(t-\tau) + \int_t^{+\infty} Q(s)x(s-\sigma) \, ds + \sum_{t\le t_k} Q_k x(t_k-\sigma) = 0 \quad (2.9)$$

Again, set

$$w(t) = \int_t^{+\infty} Q(s)x(s-\sigma)ds + \sum_{t\le t_k} Q_k x(t_k-\sigma) \quad (2.10)$$

Then finally, w (t) > 0 and

$$\begin{cases} w'(t) = -Q(t)x(t-\sigma) < 0, t \notin S \\ \Delta w(t_k) = -Q_k x(t_k-\sigma) < 0, t_k \in S \end{cases}$$

Hence,

$$x(t) = -\frac{w'(t+\sigma)}{Q(t+\sigma)}, \ x(t_k) = -\frac{\Delta w(t_k+\sigma)}{Q(t_k+\sigma)} \quad (2.11)$$

Substituting Eq. (2.10) and (2.11) into Eq. (2.9), we obtain, for t sufficiently large,

$$\begin{cases} w'(t) - P(t-\sigma)\dfrac{Q(t)}{Q(t-\tau)}w'(t-\tau) + Q(t)w(t-\sigma) = 0, t \notin S \\ \Delta w(t_k) - P(t_k-\sigma)\dfrac{Q_k}{Q(t_k-\tau)}\Delta w(t_k-\tau) + Q_k w(t_k-\sigma) = 0, \\ t_k \in S \end{cases} \quad (2.12)$$

Clearly,

$$\lim_{t\to+\infty}\left[P(t-\sigma)\frac{Q(t)}{Q(t-\tau)}\right] = p_0$$

Let us first assume that $p_0>0$. Then, for any $\varepsilon \in (0, 1)$. Eq. (2.12) yields

$$\begin{cases} w'(t) - (1-\varepsilon)p_0 w'(t-\tau) + (1-\varepsilon)q_0 w(t-\sigma) \le 0, \ t \notin S \\ \Delta w(t_k) - (1-\varepsilon)p_0 \Delta w(t_k-\tau) + (1-\varepsilon)q_{0k} w(t_k-\sigma) \le 0, \\ t_k \in S \end{cases} \quad (2.13)$$

By virtue of Corollary 1.1, Eq. (2.1) has a real root. This is a contradiction and the proof is complete when $p_0>0$.

Next, we assume that $p_0 = 0$. Then, the inequality (2.13) reduces to

$$\begin{cases} w'(t) + (1-\varepsilon)\, q_0 w(t-\sigma) \le 0, \ t \notin S \\ \Delta w(t_k) + (1-\varepsilon)\, q_{0k} w(t_k-\sigma) \le 0, \ t_k \in S \end{cases} \quad (2.14)$$

with its characteristic system as

$$\begin{cases} -\lambda + (1-\varepsilon)q_0 e^{\lambda\sigma}(1-\mu)^{n_2} = 0 \\ -\mu + (1-\varepsilon)q_{0k} e^{\lambda\sigma}(1-\mu)^{n_2} = 0 \end{cases}$$

Clearly, the solution (λ, μ) of the system satisfies the relation

$$\mu = \frac{q_{0k}}{q_0}\lambda$$

This enables us to build the characteristic equation

$$H(\lambda) \equiv -\lambda + (1-\varepsilon)\, q_0 e^{\lambda\sigma}\left(1 - \frac{q_{0k}}{q_0}\lambda\right)^{n_2} = 0$$

of the corresponding inequality (2.14). As it stands, the equation has no negative roots. Hence, by theorem 1.1 (i), the inequality (2.14) cannot have a finally positive solution.

This contradicts the fact that w(t) > 0 and the proof of theorem 2.1 is complete.

The following result is a partial converse to theorem 2.1.

Theorem 2.2: Consider the neutral impulsive differential equation

$$\begin{cases} [x(t) - p_0 x(t-\tau)]' + q(t)h(x(t-\sigma)) = 0, t \notin S \\ \Delta[x(t_k) - p_0 x(t_k-\tau)] + q_k\, h(x(t_k-\sigma)) = 0, t_k \in S \end{cases} \quad (2.15)$$

where,

$$p_0 \in (0,1),\ \tau \in (0,\infty),\ q \in PC([t_0,\infty), R_+),\ \sigma,\ q_k \geq 0,\ h \in C(R,R) \quad (2.16)$$

Assume that there exist positive numbers q_0 and δ such that

$$q(t) \leq q_0 \text{ for } t \geq t_0 \quad (2.17)$$

and

$$\left.\begin{array}{l}\text{either } 0 \leq h(u) \leq u, \text{ for } 0 \leq u \leq \delta \\ \text{or } 0 \geq h(u) \geq u, \text{ for } -\delta \leq u \leq 0\end{array}\right\} \quad (2.18)$$

Suppose, that Eq. (2.1) has a real root. Then, Eq. (2.15) has a finally positive solution on $[t_0 - r, \infty]$, where

$$r = \max\{\tau, \sigma\}$$

Proof: Let us assume that condition (2.18) holds with $0 \leq h(u) \leq u$ for $0 \leq u \leq \delta$. The case where, $0 \geq h(u) \geq u$ for $-\delta \leq u \leq 0$ is similar and is omitted. Choose d, $d_0 \in (0, \delta)$ and consider the solution x (t) of Eq. (2.15) with the initial conditions,

$$x(t) = d \text{ for } t_0 - r \leq t \leq t_0$$

and

$$x(t_k) = d_0 \text{ For } t_0 - r \leq t_k \leq t_0$$

Consequently, x (t) is left continuous on $[t_0 - r, \infty)$ and satisfies the equation

$$\begin{cases} x'(t) - p_0 x'(t-\tau) + q(t) h(x(t-\sigma)) = 0,\ t \notin S \\ \Delta x(t_k) - p_0 \Delta x(t_k - \tau) + q_k h(x(t_k - \sigma)) = 0,\ t_k \in S \end{cases} \quad (2.19)$$

almost everywhere on $[t_0, \infty)$.

Now, we prove that x $(t)>0$ for $t \geq t_0 - r$. First, we claim that as long as x $(t)>0$, it remains strictly less then δ. Otherwise, there exists a T_1 such that

$$0 < x(t) < \delta \text{ for } t_0 - r \leq t < T_1 \text{ and } x(T_1) = \delta$$

Set

$$z(t) = x(t) - p_0 x(t - \tau)$$

Then, for $t_0 \leq t \leq T_1$

$$\begin{cases} z'(t) = -q(t) h(x(t-\sigma)) \leq 0,\ t \notin S \\ \Delta z(t_k) = -q_k h(x(t_k - \sigma)) \leq 0,\ t_k \in S \end{cases}$$

and so

$$z(T_1) \leq z(t_0)$$

Hence,

$$\begin{aligned} \delta = x(T_1) &\leq p_0 x(T_1 - \tau) + x(t_0) - p_0 x(t_0 - \tau) \\ &< p_0 \delta + \delta - p_0 \delta = \delta \end{aligned}$$

which is a contradiction.

Now assume conversely, that there exists $T > t_0$ such that

$$0 < x(t) < \delta \text{ for } t_0 - r < T \text{ and } x(T) = 0 \quad (2.20)$$

From Eq. (2.19) and conditions (2.17) (2.18) and (2.20), we have

$$\begin{cases} x'(t) - p_0 x'(t-\tau) + q_0 x(t-\sigma) \geq 0 \\ \Delta x(t_k) - p_0 \Delta x(t_k - \tau) + q_{0k} x(t_k - \sigma) \geq 0 \end{cases}$$

almost everywhere, on $[t_0, \infty)$. By our assumption, the characteristic equation

$$H(\lambda) \equiv -\lambda - \lambda p_0 e^{\lambda\tau} \left(1 - \frac{q_{0k}}{q_0}\lambda\right)^{n_1} + q_0 e^{\lambda\sigma} \left(1 - \frac{q_{0k}}{q_0}\lambda\right)^{n_2} = 0$$

is assumed to have a real root, say λ_0. As $p_0 \in (0, 1)$ and $q_0 q_{0k} > 0$, it is seen that $\lambda_0 < 0$. Therefore,

$$x(t) = e^{-\lambda_0 t} \left(1 - \frac{q_{0k}}{q_0}\lambda_0\right)^{i[t_0 - \tau,\, t)}$$

is a positive, left continuous and non-increasing solution of Eq. (1.7). From Corollary 1.1, it follows that $x(t)>0$ for all $t \geq t_0$ and the proof of Theorem 2.2 is complete.

A combination of Theorem 2.1 and 2.2 yields the following linearized oscillation result for neutral impulsive differential equations.

Corollary 2.1: Assume that conditions (1.6, 2.16-2.18) are satisfied and suppose

$$\lim_{t \to +\infty} q(t) = q_0 \in (0, \infty).$$

Then every solution of Eq. (2.15) oscillates if and only if every solution of Eq. (1.2) oscillates if and only if Eq. (2.1) has no negative real roots.

CONCLUSION

Certain nonlinear neutral impulsive differential equations with deviating arguments have the same oscillatory character as the associated linear neutral impulsive differential equations with deviating arguments.

Precisely, we have been able to establish the necessary conditions for the oscillation of all solutions of the nonlinear neutral delay impulsive differential equations in terms of the oscillatory conditions of the solutions of the corresponding linear neutral impulsive differential equations with deviating arguments and vice versa.

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