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Quadrature Methods for One-Dimensional Volterra's Integral Equations of the First Kind

I.O. Isaac and A.O. Unanam
Department of Mathematics, Statistics and Computer Science, University of Calabar,
P.M.B. 1115, Calabar, Cross River State, Nigeria

Abstract: In this study, we construct numerical algorithms for solving Volterra's integral equations of the first kind with kernels of rather general nature. The trapezoidal quadrature method is used and a concrete example is provided for the case of nonlinear difference kernel.

Key words: Volterra's integral equations of the first kind, trapezoidal method, quadratue method, statistics, integral equation, Nigeria

INTRODUCTION

The exchange of integrals with quadrature formula which underlines the quadrature methods is one of the most direct practical and effective techniques (of all the known numerical approaches) of preparing Volterra's integral equations of the first kind for evaluation in computer. Recall that Isaac (2006) has examined the general overview and the description of relevant components associated with quadrature methods of solving Volterra's integral equations of the second kind. In the study, the general approach for constructing computational formulas under the mechanical quadrature method among others was worked out.

In this study, we are adopting similar strategy to construct an effective algorithm for the numerical calculation of Volterra's integral equations of the first kind. A one-dimensional linear Volterra's integral equation of the first kind is an equation of the form:

$$\int_a^x K(x, s)y(s)ds = f(x), x \in [a, b] \quad (1)$$

The most important and most widely used in practical purposes of Eq. 1 are the convolution type equations among which are the equations (Diogo *et al*., 2006):

$$\int_{-\infty}^{x} K(x-s)y(s)ds = f(x), -\infty < x < \infty \quad (2)$$

And:

$$\int_0^x K(x-s)y(s)ds = f(x), x \in [0, b] \quad (3)$$

The Volterra's equation of the first kind with Hammerstein operator (Volterra-Hammerstein equation of the first kind) is an equation of the form:

$$\int_a^x K(x, s)F[s, y(s)]ds = f(x), x \in [a, b] \quad (4)$$

While the first kind Volterra-Hammerstein equation of convolution type has the form:

$$\int_{-\infty}^{x} K(x-s)F[s, y(s)]ds = f(x), -\infty < x < \infty \quad (5)$$

or

$$\int_0^x K(x-s)F[s, y(s)]ds = f(x), x \in [0, b] \quad (6)$$

We conclude this X-ray of Volterra's integral equations of the first kind by mentioning the nonlinear Volterra's equation of the first kind:

$$\int_a^x K[x, s, y(s)]ds = f(x), x \in [a, b] \quad (7)$$

This is usually referred to as the Volterra-Uryson's equation. Unlike Volterra's integral equations of the second kind, first kind Volterra's equations possess a major pitfall especially in connection with the preparation of the said computational algorithms.

In general terms, this occurs when trying to apply the methods of exchanging integral sums with systems of algebraic equations. If the segment [a, b] is subdivided by

Corresponding Author: I.O. Isaac, Department of Mathematics, Statistics and Computer Science, University of Calabar, P.M.B. 1115, Calabar, Cross River State, Nigeria

the network of points $x = x_i$ (I = 1, 2,..., n) where $x_1 = a$ and $x_n = b$ into n-1 parts then the linear equation:

$$Ay \equiv \int_a^x K(x, s)y(s)ds = f, x \in [a, b] \quad (8)$$

may be transformed into the form:

$$\int_a^{x_i} K(x_i, s)y(s)ds = f(x_i) \quad (9)$$

from where with the aid of the quadrature statement:

$$\int_a^b \varphi(x)dx = \sum_{i=1}^n A_i\varphi(x_i) + R(\varphi) \quad (10)$$

we obtain the system of equations:

$$\sum_{j=1}^i A_j K_{ij} y_j = f_i \ (i = 1, 2, \cdots, n) \quad (11)$$

Here, A_j are the weights obtained from the expression for the quadrature expansion of the kernel:

$$K_{ij} = K(x_i, x_j), \ j = 1, 2, \cdots, i$$

and $f_i = f(x_i)$ and $y_i = y(x_i)$ are the approximate values of the free term and the unknown function at the nodes (x_i), respectively. The complication associated with the use of Eq. 11 is the inability to determine the initial value y_i which as a matter of necessity, remains indispensable in the recurrent generation of the other values $y_2, y_3, ...y_n$. Indeed when $x = x_1 = a$ the integral in Eq. 8 is equal to zero and:

$$f(a) = f_1 = 0$$

We can circumvent this unpleasant development by differentiating Eq. 8 with respect to x and then work rather with the expression:

$$\int_a^x \frac{\partial K(x, s)}{\partial x} y(s)ds + K(x, y)y(x) = f'(x) \quad (12)$$

Hence, when x = a:

$$y_1 = y(a) = \frac{f'(a)}{K(a, a)} = \frac{f'(a)}{K_{11}} \quad (13)$$

As it stands, we can now successively compute the values of the other approximate solutions as follows:

$$\begin{cases} y_2 = \dfrac{1}{A_2K_{22}}(f_2 - A_1K_{21}y_1) \\ y_3 = \dfrac{1}{A_3K_{33}}(f_3 - A_1K_{31}y_1 - A_2K_{32}y_2) \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ y_n = \dfrac{1}{A_nK_{nn}}\left(f_n - \sum_{j=1}^{n-1} A_jK_{nj}y_j\right) \end{cases} \quad (14)$$

Here, the understanding is that $A_i K_{ii} \neq 0$ where i = 1, 2,..., n. The above computation is facilitated due to the participation of the equations in system (Eq. 11) whose matrix of coefficients is already known to be triangular.

STATEMENT OF THE PROBLEM

The choice of a specific quadrature formula among the different quadrature methods depends on the nature of initial conditions attached to the given problem itself. Although, the application of some quadrature formulas allows for an easy compromise between the exact solution and the stability of the numerical results, it is required that when choosing any quadrature technique, we should be mindful of such computational pitfalls like the type mentioned above. The most preferable choices in this perspective are those methods whose algorithms are based on midpoint rectangular rules. The general outlay of the algorithms associated with such formulas may be written in the form:

$$h\sum_{j=0}^{i-1} K\left(x_i, x_{j+\frac{1}{2}}\right) y\left(x_{j+\frac{1}{2}}\right) = f(x_i), i = 1, 2, \cdots \quad (15)$$

Where its simple numerical structure immediately permits us to determine the values of the unknown function at the nodal points:

$$x_{i+\frac{1}{2}} = \left(i + \frac{1}{2}\right)h, \ h = \text{const}$$

In general however, one most effective technique in overcoming the computational fallout associated with the determination of the initial value of the unknown function at x = a is the transformation from the equation of the first kind to that of the second kind. If such an operation exists and is successfully undertaken then it will certainly guarantee a transformation to a well-posed problem. It is noteworthy that if in the transformation to the second

kind integral equations, the differentiation technique (which implies the differentiation of the kernel and the free term) is preferred then we must ensure that the associated computational error is minimal. This is always possible if we work rather with those approximate formulas that attempt to achieve equality with the exact functions under consideration.

An important feature common to nearly all quadrature methods of solving Volterra's integral equations of the first kind with arbitrary kernel is the cumulative dependence of computations in every step on the total number of steps involved.

By this, we mean that the totality of the calculations in each step is repeated again in the next step and with an addition of one unit to the existing number of its operations as it progresses. If in Eq. 15, the kernel is separable that is has the form:

$$K(x, s)=\sum_{i=1}^{m}\alpha_i(x)\beta_\ell(s),\ \ell=1, 2, \cdots, m \tag{16}$$

or it is possible to approximate it to a form that will guarantee the attainment of a separable kernel then we can construct an algorithm whose number of operations is certainly independent on the number of the discrete points. By virtue of condition (Eq. 16), Eq. 15 is written in the form:

$$\sum_{i=1}^{m}\alpha_i(x)\int_0^x\beta_\ell(s)y(s)ds=f(x) \tag{17}$$

Now applying a quadrature formula to the above statement, we obtain a recurrent expression for the determination of the approximate values of the unknown function. Thus:

$$\begin{cases} y(0)=\dfrac{f'(0)}{\sum_{\ell=1}^{m}\alpha_\ell(0)\beta_\ell(0)}, \\ y(x_i)=\dfrac{1}{\sum_{\ell=1}^{m}A_\ell\alpha_\ell(x_i)\beta_\ell(x_i)}\left[f(x_i)-\sum_{\ell=1}^{m}\alpha_\ell(x_i)*\right. \\ \left.*\sum_{j=1}^{i-1}A_j\beta_\ell(x_i)y(x_i)\right] \end{cases} \tag{18}$$

The realization of the above algorithm in computer requires only a little memory space. In addition, the number of arithmetic operations involved is minimal therefore, it takes a little computer time to execute and in addition, leads to a high precision of the computational error.

RESULTS

We employ the trapezoidal formulas to test the validity of our concept. Also, we utilize the results of the accompanying worked example to justify our earlier comments about the computational error. The problem of solving Volterra's integral equations of the first kind in a sense, appears to be situated mid-way between the problem of solving Volterra's integral equations of the second kind and that of Fredholm's integral equations of the first kind. Precisely, let us assume that we have a Volterra's integral equation of second kind that is well-posed and can effectively be solved by any classical means (quadrature, iterative, etc.).

Further, let a given Fredholm's integral equation of the first kind be ill-posed in a given preconceived functional space and solvable only by special methods (Tikhonov regularization, quasi-solution, etc.). Then a given Volterra's integral equation of the first kind may be either well-posed or ill-conditioned depending on the choice of the solution space and on the nature of the technique used (Apersin, 1979, 1981; Apersin and Bakushinskii, 1972; Tien, 1975). The conditionality for well posed problems include:

- The existence of a solution
- The uniqueness of such a solution
- The stability of the solution

As it stands, we now present in what follows the question of well posed or ill posed problem as it concerns the Volterra's integral equations of the first kind. Let in Eq. 8:

$$\begin{aligned} &y(s)\in C[a, b],\ f(x)\in C^1[a, b] \\ &K(x, s)\in C^1([a, b]x[a, b]) \end{aligned} \tag{19}$$

where, X defines a Cartesian product what is more:

$$\|f(x)\|_{C^1}\le k_1 \tag{20}$$

$$\|K(x, s)\|_{C^1}\le k_2 \tag{21}$$

$$\min_{x\in[a, b]}|K(x, x)|\le k_3\neq 0 \tag{22}$$

and:

$$f(a)=0 \tag{23}$$

Then the problem of solving Volterra's integral equations of the first kind using quadrature trapezoidal formulas is well-posed. Under this circumstance, the width

h of the interval between the nodal points is a function depending on the errors δ and ξ of approximating the right-hand side and the kernel, respectively that is:

$$h = h(\delta, \xi) \tag{24}$$

However, if the values of f(x) are defined in a tabular form at nodes with an unequal step size then Eq. 8 can still be solved though by ignoring the step size in Eq. 24. Under this circumstance, it is advisable to first approximate f(x) using any interpolation procedure or represent it in any convenient approximate form at the tabular nodes. Finally, the integral equation is then solved using the nodes of f(x) of course with the understanding that the computational error may be high. However, this could greatly be improved if the chosen values of δ and ξ are significantly small. The associated numerical algorithms for the approximate solutions under this circumstance are worked out as follows. Let the interval [a, b] be divided by the points $x_1 = a < x_2 < ... < x_n = b$ which in general may have unequal step size. Also, let the values of the function f (x) be specified at these nodes that is $f_i = f(x_i)$, i = 1, 2,..., n. Then if we:

- Apply trapezoidal formulas with unequal step size to replace integral in Eq. 8
- Denote by $y_j = y(s_j)$, j = 1, 2,..., n. The values of the unknown function y(s) or the framework for the values of the approximate solutions of Eq. 8 at the points $s_1 = x_1 = a$, $s_2 = x_2$,..., $s_n = x_n = b$

We obtain the following system of linear algebraic equations with a triangular matrix of the coefficients in respect of y_j j, j = 1, 2,..., n:

$$\begin{cases} \dfrac{h}{2}(K_{21}y_1 + K_{22}y_2) = f_2 \\ \dfrac{h}{2}K_{i1}y_1 + \displaystyle\sum_{j=2}^{i-1}\frac{(x_{j+1} - x_{j-1})}{2}K_{ij}y_j + \frac{h_i}{2}K_{ii}y_i = f_i, \\ i = 3, 4, \cdots, n \end{cases} \tag{25}$$

where, $h_i = x_{i-1}$, $K_{ij} = K(x_i, s_j)$. The above system of algebraic equations is under defined that is the number of the unknowns, y_j, y_2,..., y_n is by one unit more that the umber of the equations which is equal to n-1.

We can adopt the following technique to determine the initial value y_j. Place f (x) close to x = a by Lagrange interpolation polynomial of degree 2:

$$f(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}f_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}f_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}f_3 \tag{26}$$

Differentiating Eq. 26 with respect to x and setting $x = x_1 = a$, we obtain:

$$f'(a) = -\frac{x_2 - x_1 + x_3 - x_1}{(x_2 - x_1)(x_3 - x_1)}f_1 + \frac{x_3 - x_1}{(x_2 - x_1)(x_3 - x_2)}f_2 - \frac{x_2 - x_1}{(x_3 - x_1)(x_3 - x_2)}f_3 \tag{27}$$

Now differentiating Eq. 8 with respect to x and setting x = a, we obtain:

$$K_{11}y_1 = f'(a) \tag{28}$$

Next, we use the properties of the triangular matrix of the system of algebraic Eq. 25 and condition Eq. 22 to obtain, taking statements Eq. 14 and 28 into account, the algorithm for the solution of Eq. 8 by quadrature trapezoidal method at the nodal points x_i, i = 1, 2,..., n as follows:

$$\begin{cases} y_1 = \dfrac{f'(a)}{K_{11}}, \\ y_2 = \dfrac{f_2 - \dfrac{h_2}{2}K_{21}y_1}{\dfrac{h_2}{2}K_{22}}, \\ y_i = \dfrac{f_i - \dfrac{h_2}{2}K_{i1}y_1 - \displaystyle\sum_{j=2}^{i-1}\frac{(x_{j+1} - x_{j-1})}{2}K_{ij}y_j}{\dfrac{h_i}{2}K_{ii}}, \\ i = 3, 4, \cdots, n, \end{cases} \tag{29}$$

Where f'(a) is defined by Eq. 27. Note that in Eq. 29 above, it is required that:

$$\frac{h_i K_{ii}}{2} \neq 0 \tag{30}$$

The expression for the error estimate by the quadrature trapezoidal method is stated as follows. We subtract Eq. 11 from the more accurate system:

$$\sum_{j=1}^{i} A_j K_{ij} y_j = f_i - R_i,\ i = 1,\ 2,\ \cdots,\ n$$

Where, R_i is the quadrature error and we obtain:

$$\sum_{j=1}^{i} A_j K_{ij} \Delta y_j = -R_i,\ i = 1,\ 2,\ \cdots,\ n$$

where, Δy_j is the computational error. Consequently, we have:

$$\begin{cases} \Delta y_1 \approx \Delta y_2 \approx 0, \\ \Delta y_i \approx \dfrac{-R_i + \sum\limits_{j=2}^{i-1} \dfrac{(x_{j+1} - x_{j-1})}{2} K_{ij} \Delta y_j}{\dfrac{h_i}{2} K_{ii}}, \\ i = 3,\ 4,\ \cdots,\ n \end{cases} \tag{31}$$

and R_i is computed by the formula:

$$R_i \approx \begin{cases} 2\left[\dfrac{K_{ij-1} y_{j-1}}{h_j (h_j + h_{j-1})} - \dfrac{K_{ij} y_j}{h_j h_{j+1}} + \dfrac{K_{ij+1} y_{j+1}}{(h_j + h_{j+1}) h_{j+1}} \right], \\ j = 2,\ \cdots,\ i-1, \\ 2\left[\dfrac{K_{ii-2} y_{i-2}}{h_{i-1} (h_{i-1} + h_i)} - \dfrac{K_{ii-1} y_{i-1}}{h_{i-1} h_i} + \dfrac{K_{ii} y_i}{(h_{i-1} + h_i) h_i} \right], \\ j = i \end{cases} \tag{32}$$

When deducing system Eq. 31, the right side error δ_i was assumed to be zero while the quadrature error alone remained the only active parameter.

However if $\delta_i \neq 0$, then we obtain yet another formula close to that of Eq. 32 as follows (Emelyanov and Ilyin, 1967):

$$\begin{cases} |\Delta y_1| = \dfrac{\left| \dfrac{2x_1 - x_2 - x_3}{h_2 (h_2 + h_3)} \right| - \left| \dfrac{x_1 - x_3}{h_2 h_3} \right| + \left| \dfrac{x_1 - x_2}{(h_2 + h_3) h_3} \right|}{|K_{11}|} \delta_i, \\ |\Delta y_2| = \dfrac{\delta_i + \left| \dfrac{h_2}{2} K_{21} \right| \cdot |\Delta y_1|}{\left| \dfrac{h_2}{2} K_{22} \right|}, \\ |\Delta y_i| = \dfrac{|R_i| + \delta_i + \left| \dfrac{h_2}{2} K_{i1} \right| \cdot |\Delta y_1| + \sum\limits_{j=2}^{i-1} \left(\dfrac{x_{j+1} - x_{j-1}}{2} \right) K_{ij} |\Delta y_j|}{\left| \dfrac{h_i}{2} K_{ii} \right|}, \\ i = 3,\ 4,\ \cdots,\ n \end{cases}$$

For an equal step size of width $h_i = h = \text{const}$, we have:

$$\begin{cases} y_1 = \dfrac{f'(a)}{K_{11}}, \\ y_i = \dfrac{2}{K_{ii}} \left(\dfrac{f_i}{h} - \sum\limits_{j=2}^{i-1} a_j K_{ij} y_j \right),\ i = 2,\ 3,\ \cdots,\ n, \\ x_i = a + (i-1)h, \\ a = \begin{cases} 0.5, \text{ if } j = 1 \\ 1.0, \text{ if } j > 1, \end{cases} \end{cases} \tag{33}$$

where:

$$f'(a) = \frac{-3f_1 + 4f_2 - f_3}{2h}$$

ILLUSTRATION

We consider a one-dimensional Volterra's integral equation of the first kind:

$$\int_a^x \left(2 + x^2 - s^2\right) s e^{-\frac{s^2}{2}} ds = x^2,\ x \in [a,\ b]$$

where, A = 0, B = 1.00 and the exact solution is defined by the function:

$$y(x) = x e^{-\frac{x^2}{2}}$$

A careful application of the above scheme leads to a system of linear algebraic equations:

$$\begin{cases} y_1 = \dfrac{f'(a)}{K_{11}} \\ y_2 + 0.5K_{21} y_1 = \dfrac{2f_2}{hK_{22}} \\ y_3 + 0.5K_{31} y_1 + K_{32} y_2 = \dfrac{2f_3}{hK_{33}} \\ y_4 + 0.5K_{41} y_1 + K_{42} y_2 + K_{43} y_3 = \dfrac{2f_4}{hK_{44}} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ y_n + 0.5K_{n1} y_1 + K_{n2} y_2 + \cdots + K_{n,n-1} y_{n-1} = \dfrac{2f_n}{hK_{nn}} \end{cases} \tag{34}$$

or;

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0.5K_{21} & 1 & 0 & 0 & \cdots & 0 \\ 0.5K_{31} & K_{32} & 1 & 0 & \cdots & 0 \\ 0.5K_{41} & K_{42} & K_{43} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.5K_{n1} & K_{n2} & K_{n3} & K_{n4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} f'(a)/K_{11} \\ f_2/hK_{22} \\ f_3/hK_{33} \\ f_4/hK_{44} \\ \cdots \\ f_n/hK_{nn} \end{bmatrix} \tag{35}$$

Table 1: Values of approximate and exact solution at 21 nodal points

x_i	$\tilde{y}(x_i)$	y_i	$\tilde{y}(x_i) - y_i$
0.00	0.00000000	0.00000000	0.00000000
0.05	0.05000000	0.04993754	0.00006246
0.10	0.09962499	0.09950125	0.00012374
0.15	0.14850500	0.14832196	0.00018304
0.20	0.19627750	0.19603973	0.00023777
0.25	0.24259910	0.24230831	0.00029079
0.30	0.28713170	0.28679924	0.00033246
0.35	0.32958180	0.32920582	0.00037598
0.40	0.36965440	0.36924654	0.00040786
0.45	0.40710130	0.40666819	0.00043311
0.50	0.44170640	0.44124845	0.00045795
0.55	0.47324870	0.47279802	0.00045068
0.60	0.50164010	0.50116213	0.00047797
0.65	0.52666670	0.52622157	0.00044513
0.70	0.54835490	0.54789318	0.00046172
0.75	0.56654970	0.56612970	0.00042000
0.80	0.58133040	0.58091923	0.00041117
0.85	0.59264800	0.59228406	0.00036394
0.90	0.60062560	0.60027913	0.00034647
0.95	0.60527690	0.60499003	0.00028687
1.00	0.60679690	0.60653066	0.00026624

for the determination of the approximate solutions $y_i \equiv \tilde{y}(x_i), i = 1, 2, \cdots, n$. In Table 1, we compute the values of the approximate and the exact solutions at 21 nodal points. We also, include the difference column to enable us assess the effectiveness of the algorithm. The approximate expansions of the kernel K and the free term f contain no rounding and finally, the step size h is defined by the formula h = const = 0.05.

CONCLUSION

Clearly, the picture shown in the findings implies that the application of the quadrature methods for solving Volterra's integral equations of the first kind is effective, especially with regards to the type of the kernel used in the example. It is equally believed that similar accuracy may be achieved if the method is applied to equations with other forms of kernels.

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