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On Mcshane Equiintegrable Sequences

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Abstract: In this study, we define the equiintegrability sequence for the Banach-valued McShane integrable functions and prove the theorem limit for the McShane integrable functions.

Key words: Equiintegrable sequences, McShane integrale, R-integrable, Riemanns technique, banach space, Kosovo

INTRODUCTION

Traditional Riemann integration while powerful, leaves us with much to be desired. The class of functions that can be evaluated using Riemann's technique, for example is very small. Another problem is that a convergent sequence of Riemann integrable functions (we will denote this class of functions as) does not necessarily converge to an function. Gordon (1990) generalized the definition of the McShane integrale for real-valued functions to functions taking values in Banach spaces and investigated some of its properties. Many researchers have studied McShane integral (Gordon, 1990; Bartle, 2001).

In this study, we define the equiintegrability sequence for the Banach-valued McShane integrable functions and prove the theorem limit for the McShane integrable functions. The typical form of a limit theorem for an $\aleph$ integral through measure μ is: If sequence (f_n) is the $\aleph$ integrable functions sequence for which the following statements are true:

- $\lim\limits_{n\to\infty} f_n(t) = f(t)$ Almost, everywhere regarding μ
- Statement Q, then the function f is $\aleph$ integrable, moreover we have that:

$$\lim_{n\to\infty}(\aleph)\int_I f_n(t)d\mu = (\aleph)\int_I f(t)d\mu$$

The Limit theorems for particular type of integrals are important because we can use them in more powerful mathematical techniques for integral studies. The Lebesgue integral (Dunford) is a type of integral with most powerful theorems of convergence (Limit); statement Q is in fact about limitation of the sequence (f_n) with a function g which is Lebesgue integrable and exactly the statement Q can be replaced with $(\forall n \in N)[|f_n(t)| \le g(t)$ almost everywhere, regarding μ. We consider functions f: I→X where $I \subset R$ is a compact interval and X is a banach space with the norm $\|.\|_X$. Based on the fact that respective definitions on banach space are well known if we consider Schwabik and Guoju (2005) and Temaj and Tato (2008). By μ let the Lebesgue measure in R be denoted. A system (finite collection) of point-interval pairs $\{(I_i, t_i): I = 1, 2, ..., r\}$ is called an M-system in I if I_i are non-overlapping (int $I_i \cap$ int $I_j = \phi$ for $i \neq j$, int I_i is the interior of I_j), t_i are arbitrary points in I. An M-system in I is called an M-partition of I if:

$$\bigcup_{i=1}^r I_i = I$$

Given f: I→ X and partition $P = \{(I_i, t_i): i = 1, ..., r\}$ in I, we set:

$$\sigma(f, P) = \sum_{i=1}^r f(t_i)\mu(I_i)$$

and call this number the Riemann sum of f associated with p. Given δ: I→ (0, +∞), called a gauge, an M-system $\{(I_i, t_{ij}): I = 1, 2,...., r\}$ in I is called δ-fine if:

$$I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i)) \;\; i = 1, 2, ..., r$$

The following Lemma is crucial and leads us to the McShane integral definition.

Lemma 1: Given a gauge δ : I→ (0, + ∞) there exists a δ-fine M-partition of I (Schwabik and Guoju, 2005) (Cousin).

Definition 1: f : I→X is McShane intergable and $\omega \in X$ is its McShane integral over I if for every $\epsilon>0$ there exists a gauge δ: I→ (0, +∞) such that for every δ-fine M-partition $P = \{(I_i, t_i): i = 1,..., r\}$ of the inequality:

$$\|\sigma(f, P) - \omega\|_X < \varepsilon$$

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holds. Denote: $\omega = (M)\int_I f(t)d\mu_L$ and M denotes the set of all McShane integrable functions. Given a set $E \subset I$ we denote by χ_E its characteristic function:

$$(\chi_E(t) = 1 \text{ for } t \in E,\ \chi_E(t) = 0 \text{ otherwise})$$

A function f: I→X is called McShane integral over the set $E \subset I$ if the function $f \cdot \chi_E$: I→X is McShane intergable. In the case we write:

$$\int_I f \cdot \chi_E = \int_I f$$

Lemma 2: Assume that f: I→X is McShane intergable. Given ε>0 assume that the gauge δ in I is such that (Schwabik and Guoju, 2005) (Saks-Henstock):

$$| \sigma(f, P) - (M)\int_I f(t)d\mu_L \|_X < \varepsilon$$

for every δ-fine M-partition P = $\{(I_i, t_i)$: i = 1, ..., r$\}$ of I. Then if P = $\{(r_i, K_j)/j = 1, 2, \ldots, m\}$ is δ-fine M-system we have:

$$\| \sum_{j=1}^{m} \left(f(r_j)\mu_L(K_j) - (M)\int_{K_j} f(t)d\mu_L \right) \|_X \le \varepsilon$$

MATERIALS AND METHODS

A limit theorem for McShane integral: Let us consider the problem for a limit theorem regarding M-integral. Is it possible to assume that for this type of integral only the verity of statement (a) is sufficient? The answer is negative and the following example will prove this.

Example 1: For every k ∈ N we define the function:

$$f_k : [0, 1] \to R,\ f_k(x) = k.x_{(0,\frac{1}{k})}(x),\ x \in [0,1]$$

We can see that the function sequence (f_K) converges in [0, 1] to the function f = 0 and at the other hand we have:

$$(M)\int_{[0,1]} f_k(s)d\mu_L = 1$$

for k ∈ N and (M):

$$(M)\int_{[0,1]} f(s)d\mu_L = 0$$

So, we have:

$$\lim_{k\to\infty}(M)\int_{[0,1]} f_k(s)d\mu_L \ne (M)\int_{[0,1]} f(s)d\mu_L$$

One limit theorem for M-integral in R is given by (Kurzweil and Schwabik, 2004) in which instead of the statement Q is another statement that has to do with the uniformity of the M-integration of the function sequence or equiintegrability of a function sequence. In this study, we will prove the limit theorem for M-integral in banach space. Let us start with the following:

Definition 2: A collection B of functions f: I→X is called equiintegrable if every f ∈ B is McShane integrable and for every ε>0 there is a gauge δ such that for any f ∈ B the inequality:

$$\| \sigma(f, P) - (M)\int_I f(t)d\mu_L \|_X < \varepsilon$$

holds provided $P = \{(I_i, t_i) : i = 1, \ldots, r\}$ is δ-fine M-partition of I. Now, let us prove the limit theorem for McShane integrale.

RESULTS AND DISCUSSION

Theorem 1: Assume that B = $\{f_n$: I → X/n ∈ N$\}$ is an equiintegrable sequence such that:

$$\lim_{n\to\infty} f_n(t) = f(t),\ t \in I \tag{1}$$

Then the function f:→ X is McShane integrable and holds:

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = (M)\int_I f(t)d\mu_L \tag{2}$$

Proof: Let ε>0 be given. Then according to definition 2, there can be found a gauge δ: I → (0, +∞) such that for every δ-fine M-partition $P_0 = \{(I_i, t_i): i = 1, \ldots, k\}$ of Ithe inequality:

$$\| \sigma(f_n, P_0) - (M)\int_I f_n(x_i)d\mu_L \| < \varepsilon \tag{3}$$

is true for any n∈N. If the partition $P_0 = \{(I_i, t_i): I = 1, \ldots, k\}$ is fixed then the pointwise convergence (Schwabik and Guoju, 2005) yields:

$$\lim_{n\to\infty} \sigma(f_n, P_0) = \sigma(f, P_0) \tag{4}$$

So we have:

$$\| \sigma(f_n, P_0) - \sigma(f_m, P_0) \| = \| \sum_{i=1}^{k} f_n(t_i)\mu(I_i) - \sum_{i=1}^{k} f_m(t_i)\mu(I_i) \| \le \sum_{i=1}^{k} \| f_n(t_i) - f_m(t_i) \| \mu(I_i)$$

For each t_i, there exists a positive integer K_i (t_i) such that:

$$\| \sigma(f_n, t_i) - \sigma(f_m, t_i) \| \le \frac{\varepsilon}{k} \text{ for all } m, n \ge K_i$$

$$N = \max\{K_i / 1 \le i \le k\}$$

That:

$$\| \sigma(f_n, P_0) - \sigma(f_m, P_0) \| < \varepsilon \text{ for all m, n} \geq N$$

There exists a positive integer N such that:

$$\| \sigma(f_n, P_0) - \sigma(f_m, P_0) \| < \varepsilon$$

For all m, n≥N. Then:

$$\| (M)\int_I f_n(t)d\mu - (M)\int_I f_m(t)d\mu_L \| \leq$$
$$\leq \| (M)\int_I f_n(t)d\mu - \sigma(f_n, P_0) \| + \| \sigma(f_n, P_0)$$
$$-\sigma(f_m, P_0) \| + \| \sigma(f_m, P_0) - (M)\int_I f_m(t)d\mu \| < 3\varepsilon$$

For all m, n≥N, it follows that:

$$\{(M)\int_s f_n(t)d\mu\}$$

is a Cauchy sequence in Banach space X. Let:

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = \omega \in X$$

We need to show that:

$$(M)\int_I f(t)d\mu_L = \omega$$

Hence, it is sufficient to show that:

$$(M)\int_I f(t)d\mu_L = \omega$$

Let ε>0 by hypothesis, there exists a gauge **δ**: I→ (0, + ∞) on I such that:

$$\| \sigma(f_n, P) - (M)\int_I f_n(t_i)d\mu_L \| < \varepsilon$$

For all n whenever $P = \{(I_i, t_i): i = 1, ..., r\}$ is **δ**-fine M-partition of I. Since, $\{f_n\}$ convergese pointwise to f, exists k≥N such that:

$$\| \sigma(f, P) - \sigma(f_k, P) \|_X < \varepsilon$$

Hence:

$$\| \sigma(f, P) - \omega \|_X \leq \| \sigma(f, P) - \sigma(f_k, P) \|_X + \|$$
$$\sigma(f_k, P) - (M)\int_I f_k(t)d\mu_L \|_X + \|$$
$$(M)\int_I f_k(t)d\mu_L - \omega \|_X < 3\varepsilon$$

It follows that f is McShane integrable on I and:

$$\lim_{n\to\infty}(M)\int_I f_n(t)d\mu_L = (M)\int_I f(t)d\mu_L$$

The following proof of proposition is a slight modification of the proof the of Theorem 2.

Proposition 1: Let functions f_n: I→X, n = 1, 2, 3, ..., be given where the integrals $\int_I f_n(t)d\mu$ exists for n = 1, 2, 3, ..., assume that:

$$\lim_{n\to\infty} f_n(t) = 0 \tag{5}$$

for t∈I and that the sequence of functions: f_n: I→X is equiintegrable. Then for every ε>0 and for every M-system $\{(I_j, t_j): j = 1, 2, ..., I\}$ in I there is an $N \in \mathbb{N}$ such that for every n>N the inequality:

$$\| \sum_{j=1}^{I} \int_{I_j} f_n(t)d\mu \|_X < \varepsilon$$

Proof: Let ε>0 be given. By the definition 2, there exists a gauge **δ**: I → (0, + ∞) such that for every **δ**-fine M-partition $P_0 = \{(I_i, t_i): i = 1, ..., k\}$ of I we have:

$$\left\| \sigma(f_n, P_0) - \int_I f_n(t)d\mu \right\|_X < \frac{\varepsilon}{3} \tag{6}$$

for n = 1, 2, 3, ..., assume that an arbitrary M-system $\{(I_j, t_j): j = 1, 2, ..., I\}$ in I is given. Let:

$$P_0^j = \{(I_i^j, t_i^j) : i = 1,...,k\}$$

be a **δ**-fine M-partition of $\{(I_j, t_j) ; j = 1, 2,..., I\}$. Using Eq. 6 and lemma 2 (Saks-Henstok) we obtain:

$$\left\| \sum_{j=1}^{I} [\sigma(f_n, P_0^j) - \int_{I_j} f_n(t)d\mu] \right\|_X = \tag{7}$$
$$\left\| \sum_{j=1}^{I} \sum_{i=1}^{k_j} f_n(t_i^j)\mu(I_i^j) - \sum_{j=1}^{I} \int_{I_j} f_n(t)d\mu \right\|_X \leq \frac{\varepsilon}{3} < \frac{\varepsilon}{2}$$

By Eq. 5 for every fixed system M-partition of:

$$P_0^j,\ j = 1, 2, \cdots, 1$$

with the properties given above, there exists a positive integer N ϵ N such that for n>N the inequality:

$$\left\| \sum_{j=1}^{1} \sigma(f_n, P_0^j) \right\|_X = \left\| \sum_{j=1}^{1} \sum_{i=1}^{k_j} f_n(t_i^j)\mu(I_i^j) \right\|_X < \frac{\varepsilon}{2}$$

holds. Hence using Eq. 7, we get for any n>N.

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