

## Necessary and Sufficient Conditions Where One $\Gamma$ -Semigroup is a $\Gamma$ -Group

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**Abstract:** In this study, researchers have studied the  $\Gamma$ -algebraic structures and some characteristics of them. According to Sen and Saha, we defined algebraic structures:  $\Gamma$ -semigroup,  $\Gamma$ -regular semigroup,  $\Gamma$ -idempotent semigroup,  $\Gamma$ -invers semigroup and  $\Gamma$ -group. Theorem 2, 3 and 4 proves the existence of  $\Gamma$ -group and gives necessary and sufficient conditions where one  $\Gamma$ -semigroup is a  $\Gamma$ -group. Finally, theorem 5 shows necessary and sufficient conditions where one  $\Gamma$ -regular semigroup is a  $\Gamma$ -group. In addition, for every  $\Gamma$ - algebraic structure that we mentioned before we give an original example.

**Key words:**  $\Gamma$ -algebraic structures,  $\Gamma$ -semigroups,  $(\alpha, \beta)$ -inverse element,  $\alpha$ -idempotent element,  $\Gamma$ -semigroup idempotent,  $\Gamma$ -group

### INTRODUCTION

Idea for  $\Gamma$ -algebraic structures follows from  $\Gamma$ -ring which notion has been introduced by Nobusawa (1964). Theory of  $\Gamma$ -semigroup is expanded in a natural way while as foundation we take in the theory of semigroups.

**Definition 1:** Let:

$$S = \{a, b, c, \dots\}$$

and

$$\Gamma = \{\alpha, \beta, \gamma, \dots\}$$

two nonempty sets. A mapping  $f: S \times \Gamma \times S \rightarrow S$  or  $f: (a, \alpha, b) \rightarrow c$ ;  $a, b, c \in S$ ;  $\alpha \in \Gamma$  called ternary operation in  $S$  and  $\Gamma$ . This operation we denote by  $(.)_\Gamma$  or by  $(+)_\Gamma$ . The element  $(a, \alpha, b)$  we denote simple by  $a\alpha b$ . Operation  $f$  is commutative if  $\forall a, b \in S, \alpha \in \Gamma$ , satisfies condition:

$$a\alpha b = b\alpha a$$

Operation  $f$  is associative if it satisfies condition:

$$(a\alpha b)\beta c = a\alpha (b\beta c); \forall a, b, c \in S, \alpha, \beta \in \Gamma$$

**Definition 2:** Let:

$$S = \{a, b, c, \dots\}$$

and

$$\Gamma = \{\alpha, \beta, \gamma, \dots\}$$

two non-empty sets. Order pair  $(S, (.)_\Gamma)$  is called  $\Gamma$ -algebraic structure.

**Definition 3:** A  $\Gamma$ -algebraic structure  $(S, (.)_\Gamma)$  is called  $\Gamma$ -groupoid if it satisfies condition:

$$(i) \quad \forall a, b \in S, \alpha, \beta \in \Gamma \Rightarrow \alpha\alpha b \in S$$

**Example 1:** Let:

$$S = \{a = 4z + 3, z \in \mathbb{Z}\} = \{\dots -13, -9, -5, -1, 3, 7, 11, 15, \dots\}$$

$$\Gamma = \{\alpha = 4z + 1, z \in \mathbb{Z}\} = \{\dots -11, -7, -3, 1, 5, 9, 13, \dots\}$$

two sets. If  $a = 4z_1 + 3, b = 4z_2 + 3$  and  $\alpha = 4z_3 + 1$  where  $a, b \in S$  and  $\alpha \in \Gamma$ . Then:

$$\begin{aligned} a\alpha b &= 4z_1 + 3 + 4z_3 + 1 + 4z_2 + 3 \\ &= 4(z_1 + z_2 + z_3 + 1) + 3 = 4z + 3 \in S \end{aligned}$$

Therefore  $(S, (+)_\Gamma)$  is  $\Gamma$ -groupoid where operation  $(+)$ , or  $a\alpha b$  is addition of integers.

### MATERIALS AND METHODS

In definition 4, we defined  $\Gamma$ -semigroup by  $\Gamma$ -algebraic structure and examined few examples by using some of their characteristics. Next, briefly present definition of  $\Gamma$ -subsemigroup (Saha, 1987), also definition of ideal in  $\Gamma$ -semigroup (Saha, 1988).

**Definition 4:** A  $\Gamma$ -algebraic structure  $(S, (.)_\Gamma)$  is called  $\Gamma$ -semigroup if it satisfies condition:

$$(i) \quad \forall a, b \in S, \alpha \in \Gamma \Rightarrow a\alpha b \in S$$

(ii)  $\forall a, b, c \in S; \alpha, \beta \in \Gamma (\alpha\alpha b) \beta c = \alpha\alpha (b\beta c)$

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$\Gamma$ -semigroup we can definite also in this way: A  $\Gamma$ -groupoid  $(S, (\cdot)_{\Gamma})$  satisfying the associative law is a  $\Gamma$ -semigroup:

$$(\alpha\alpha b)\beta c = \alpha\alpha (b\beta c); (\forall a, b, c \in S; \alpha, \beta \in \Gamma)$$

**Example 2:** Let  $S$  be the set of all integers of the form  $6z + 1$ :

$$S = \{6z + 1 : z \in \mathbb{Z}\} = \{\dots, -11, -5, 1, 7, 13, \dots\}$$

and  $\Gamma$  be the set of all integers of the form  $6z + 5$ :

$$\Gamma = \{6z + 5 : z \in \mathbb{Z}\} = \{\dots, -13, -7, -1, 5, 11, 17, 23, \dots\}$$

Then order pair  $(S, (+)_{\Gamma})$  is a  $\Gamma$ -semigroup where  $\alpha\alpha b$  denotes the addition of integers.

**Solution**

**Closure property:** If

$$a = 6z_1 + 1, b = 6z_2 + 1, z_1, z_2 \in \mathbb{Z}$$

and

$$\alpha = 6z' + 5, z' \in \mathbb{Z}$$

then:

$$\begin{aligned} \alpha\alpha b &= 6z_1 + 1 + 6z' + 5 + 6z_2 + 1 \\ &= 6(z_1 + z_2 + z' + 1) + 1 = 6z + 1 \in S \end{aligned}$$

therefore, pair  $(S, (+)_{\Gamma})$  is a grupoid.

**Associative property:** If

$$a = 6z_1 + 1, b = 6z_2 + 1, z_1, z_2 \in \mathbb{Z}$$

and

$$\alpha = 6z' + 5, \beta = 6z'' + 5, z', z'' \in \mathbb{Z}$$

then:

$$\begin{aligned} (\alpha\alpha b)\beta c &= \{6(z_1 + z_2 + z' + 1) + 1\} + 6z'' + 5 + 6z_3 + 1 \\ &= 6(z_1 + z_2 + z_3 + z' + z'' + 2) + 1 = 6z + 1 \end{aligned}$$

From the other side:

$$\begin{aligned} \alpha\alpha(b\beta c) &= 6z_1 + 1 + 6z' + 5 + (6z_2 + 1 + 6z'' + 5 + 6z_3 + 1) \\ &= 6(z_1 + z_2 + z_3 + z' + z'' + 2) + 1 = 6z + 1 \end{aligned}$$

Consequently,

$$(\alpha\alpha b)\beta c = \alpha\alpha(b\beta c)$$

We now prove that the pair  $(S, (+)_{\Gamma})$  is a  $\Gamma$ -semigroup.

**Example 3:** Let  $S$  be the set of all matrices of type:

and  $\Gamma$  be the set of all matrices:

$$\Gamma = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

We can prove that the system  $(S, (\cdot)_{\Gamma})$  is  $\Gamma$ -semigroup, where operation  $(\cdot)_{\Gamma}$  is the product of the matrices.

**Solution**

**Closure property:** If

$$A_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & 0 \\ b_2 & 1 \end{pmatrix}, A_1, A_2 \in S$$

and

$$\alpha = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \alpha \in \Gamma$$

then:

$$\begin{aligned} A_1\alpha A_2 &= \begin{pmatrix} a_1 & 0 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ b_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1x & 0 \\ b_1x & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ b_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a_1xa_2 & 0 \\ b_1xa_2 + b_2 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \end{aligned}$$

where,

$$a_1xa_2 = a, b_1xa_2 + b_2 = b$$

and

$$A_1\alpha A_2 \in S$$

Associative property is true because the product of the matrices is associative and relation is evident:

$$(A_1\alpha A_2)\beta A_3 = A_1\alpha(A_2\beta A_3)$$

since the algebraic system  $(S, (\cdot)_{\Gamma})$  is  $\Gamma$ -semigroup.

Semigroup  $S$  can be considered always like  $\Gamma$ -semigroup if operation in  $S$  expands in  $S \cup \{1\}$  where  $11 = 1$  and  $1a = a1 = a, (1 \neq a), \forall a \in S$ . Hence,  $S \cup \{1\}$  is semigroup with identity element. If we define  $ab = a1b$  and  $\Gamma = \{1\}$  in this case  $S$  is  $\Gamma$ -semigroup. Therefore, principal results of the theory of semigroups can be expanded in the theory of a  $\Gamma$ -semigroup.

**Lemma 1:** If  $S$  is a semigroup,  $\Gamma = \{1\}$  and  $ab = a1b$  then  $S$  is  $\Gamma$ -semigroup.

**Proof:** From definition 4, it is evident.

**Definition 5:** Let  $S$  be a  $\Gamma$ -semigroup. Subset  $M$  of  $S$  is  $\Gamma$ -subsemigroup of  $\Gamma$ -semigroup  $S$  if  $M\Gamma M \subseteq M$ , where:

$$M\Gamma M = \{m\alpha n : m, n \in M; \alpha \in \Gamma\}$$

Other alternative way of defining  $\Gamma$ -subsemigroup  $M$  of semigroup  $S$  can be found by Saha (1987).

**Example 4:** Let  $S = [0, 1]$  and

$$\Gamma = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$$

Then  $S$  is  $\Gamma$ -semigroup. But subset:

$$M = \left[ 0, \frac{1}{2} \right]$$

is  $\Gamma$ -subsemigroup of  $\Gamma$ -semigroup  $S$ . We can see that:

$$M \subseteq S \text{ and } m\alpha n \in M, \forall m, n \in M; \forall \alpha \in \Gamma$$

Determination of ideal in  $\Gamma$ -semigroup  $S$ , although not according to the definition provided by Saha, the idea and some characteristics were taken from Saha (1988).

**Definition 6:** A left (right) ideal of a  $\Gamma$ -semigroup  $S$  is non-empty subset  $I$  of  $S$  ( $I \subseteq S, I \neq \emptyset$ ) such that  $S\Gamma I \subseteq I$  ( $I\Gamma S \subseteq I$ ). If  $I$  is both a left and right ideal, then we say that  $I$  is an ideal of  $S$ . Let  $Q$  non-empty set of  $S, Q$  is quasi-ideal of  $\Gamma$ -semigroup  $S$  if  $Q\Gamma S \cap S\Gamma Q \subseteq Q$ .

**Definition 7:** A  $\Gamma$ -semigroup  $S$  is left (right) simple if it has no proper left (right) ideal. A  $\Gamma$ -semigroup  $S$  is said to be simple if it has no proper ideal.

**Definition 8:** An element  $a \in S$  is said to be a regular in the  $\Gamma$ -semigroup  $S$  if  $a\alpha\Gamma S\Gamma a$ , where:

$$a\Gamma S\Gamma a = \{(a\alpha b)\beta a : b \in S; \alpha, \beta \in \Gamma\}$$

A  $\Gamma$ -semigroup  $S$  is said to be regular if every element of  $S$  is regular.

**Example 5:** Let  $A = \{1, 2, 3\}$  and  $B = \{4, 5\}$  be two non-empty sets. Let  $S = \{f, g, h\}$   $\Gamma = \{\alpha, \beta, \gamma, \delta, \theta, \phi\}$ , where  $f, g, h$  are mapping from the set  $A$  to the set  $B$  and  $\alpha, \beta, \gamma, \delta, \theta, \phi$  are mapping from the set  $B$  to the set  $A$ . They are defined by:

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 5 & 5 \end{pmatrix}, h = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 5 \end{pmatrix}$$

and:

$$\alpha = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 4 & 5 \\ 2 & 2 \end{pmatrix}, \gamma = \begin{pmatrix} 4 & 5 \\ 3 & 3 \end{pmatrix}, \\ \delta = \begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}, \theta = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}, \phi = \begin{pmatrix} 4 & 5 \\ 1 & 3 \end{pmatrix}$$

Where,  $1f = 4, 2f = 4, 3f = 4$  and similarly others. We can see that:

$$f\alpha f\delta f = f, g\alpha g\gamma g = g, h\theta h\phi h = h$$

**For example:** if  $1 \in A \wedge f \in S$ , we have  $1f = 4$ . From the other side:

$$1(f\alpha f\delta f) = 1f(\alpha f\delta f) = 4\alpha(f\delta f) \\ = 1f(\delta f) = 4\delta(f) = 2f = 4$$

We proof  $f\alpha f\delta f = f$ . Therefore,  $S$  is regular  $\Gamma$ -semigroup. We can give another definition about regular  $\Gamma$ -semigroups.

**Definition 9:** A  $\Gamma$ -semigroup  $S$  is called regular  $\Gamma$ -semigroup if for any  $a \in S$  exists  $b \in S, \alpha, \beta \in \Gamma$  such that  $a = (a\alpha b)\beta a$ .

**Example 6:** Let  $S$  be the set of all  $2 \times 3$  matrices over the field and  $\Gamma$  be the set of all  $3 \times 2$  matrices over the same field:

$$S = (a_{ij})_{2 \times 3}; \Gamma = (x_{ij})_{3 \times 2}$$

then  $S$  is regular  $\Gamma$ -semigroup where  $A\alpha B$  ( $A, B \in S, \alpha \in \Gamma$ ) denote the product of the matrices. Indeed for  $A \in S$ , we can chose  $\alpha \in \Gamma$  such that:

$$(A\alpha A)\alpha A = A\alpha A = A$$

**Definition 10:** An element  $e \in S$  is said to be an idempotent of  $\Gamma$ -semigroup  $S$ , if  $e\alpha e = e$  for some  $\alpha \in \Gamma$ . In this case, we call  $e$  an  $\alpha$ -idempotent.  $S$  is a idempotent  $\Gamma$ -semigroup if and only if every element of  $S$  is idempotent.

**Example 7:** Let

$$S = \left\{ E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \text{ and } \Gamma = \left\{ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Then  $S$  is a idempotent  $\Gamma$ -semigroup.

**Definition 11:** Let  $S$  be a  $\Gamma$ -semigroup and  $a \in S$ . If for  $b \in S$  exists  $\alpha, \beta \in \Gamma$  such that  $a = (a\alpha b)\beta a$  and  $b = (b\beta a)\alpha b$  then  $b$  is called an  $(\alpha, \beta)$  invers of  $a$ . In this case researcher write:

$$b \in V_{\alpha}^{\beta}(a)$$

**Proposition 1:** If S is a regular  $\Gamma$ -semigroup and  $a \in S$ , then  $V_{\alpha}^{\beta}(a) \neq \emptyset$  for some  $\alpha, \beta \in \Gamma$ .

**Proof:** If S is a regular  $\Gamma$ -semigroup, then for some  $a \in S$  exists  $b \in S, \alpha, \beta \in \Gamma$  such that  $a = (\alpha\beta)a$ . For element  $x = \beta\beta a \alpha \beta \in S$ , we can prove that  $x \in V_{\alpha}^{\beta}(a)$ . Indeed:

$$\begin{aligned} (\alpha\alpha x)\beta a &= (\alpha\alpha\beta\beta a\alpha\beta)\beta a = (\alpha\alpha\beta\beta a)\alpha\beta\beta a = \alpha\alpha\beta\beta a = a \\ \text{and} \\ x\beta a \alpha x &= \beta\beta(\alpha\alpha\beta\beta a)\alpha\beta\beta a \alpha\beta = \beta\beta(\alpha\alpha\beta\beta a)\alpha\beta = \beta\beta a \alpha\beta = x \end{aligned}$$

Then: 
$$x \in V_{\alpha}^{\beta}(a)$$

**Definition 12:** A  $\Gamma$ -semigroup S is called a inverse  $\Gamma$ -semigroup if every element a of S has a unique  $(\alpha, \beta)$ -inverse, whenever  $(\alpha, \beta)$ -inverse of a exists.

**Proposition 2:** A regular  $\Gamma$ -semigroup S is a inverse  $\Gamma$ -semigroup if  $|V_{\alpha}^{\beta}(a)|=1$  for all  $a \in S$  and for all  $\alpha, \beta \in \Gamma$ .

**Proof:** It is evident from definition 12.

**Theorem 1:** Let S be a inverse  $\Gamma$ -semigroup and  $a \in S, \alpha, \beta \in \Gamma$ . If  $\alpha^{-1} \in V_{\alpha}^{\beta}(\alpha)$ , then  $\alpha\alpha^{-1}$  is  $\beta$ -idempotent and  $a^{-1}\beta a$  is  $\alpha$ -idempotent of S.

**Proof:** If S is a inverse  $\Gamma$ -semigroup, then for  $a \in S$  and  $\alpha, \beta \in \Gamma$  exists unique inverse element  $\alpha^{-1} \in V_{\alpha}^{\beta}(\alpha)$  such that  $a = \alpha\alpha^{-1}\beta a$  and  $a^{-1} = a^{-1}\beta a\alpha^{-1}$ . Element  $(\alpha\alpha^{-1}) \in S$  is a  $\beta$ -idempotent:

$$(\alpha\alpha^{-1})\beta(\alpha\alpha^{-1}) = \alpha\alpha^{-1}\beta\alpha\alpha^{-1} = \alpha\alpha^{-1}$$

and element  $a^{-1}\beta a$  is a  $\alpha$ -idempotent i S:

$$(a^{-1}\beta a)\alpha(a^{-1}\beta a) = a^{-1}\beta(\alpha\alpha^{-1}\beta a) = a^{-1}\beta a$$

### RESULTS AND DISCUSSION

Initially we create semigroup  $S_{\alpha}$  (Chinram and Siammai, 2009). In Theorem 3 and 4, researchers proof necessary and sufficient condition where semigroup  $G_{\alpha}$  is a  $\Gamma$ -group. Furthermore, we have determined necessary and sufficient condition where regular and inverse semigroup  $G_{\alpha}$  is a  $\Gamma$ -group. Now we will create the semigroup  $S_{\alpha}$ .

Let S be a  $\Gamma$ -semigroup and  $\alpha$  be a fixed element of  $\Gamma$ . If  $a, b \in S$ , define operation  $\circ$  in S by,  $a \circ b = \alpha b, \forall a, b \in S$ . Then, S is a semigroup.

**Proof:**

- (i)  $\forall a, b \in S; \alpha \in \Gamma \Rightarrow a \circ b = \alpha b \in S, \alpha$ -fixed element of  $\Gamma$
- (ii)  $\forall a, b, c \in S, \alpha \in \Gamma$  we have  $a \circ (b \circ c) = a \circ (\alpha c) = \alpha(\alpha c)$   
On the other hand:

$$(a \circ b) \circ c = (\alpha b) \circ c = (\alpha b)\alpha c = \alpha(\alpha c)$$

Since:

$$(a \circ b) \circ c = a \circ (b \circ c) \Rightarrow (S, \circ) \text{ is semigroup}$$

We denote this semigroup by  $S_{\alpha}$ .

**Example 8:** If

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\} \text{ and } \alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is fixed element of  $\Gamma$ . Then  $S_{\alpha}$  is semigroup, where operation is the product of matrices. We took definition of  $\Gamma$ -group from Sen and Saha (1986) which in a free form appears as:

**Definition 13:** A  $\Gamma$ -semigroup G is called a  $\Gamma$ -group if  $G_{\alpha}$  is group for some  $\alpha \in \Gamma$  (Sen and Saha, 1986).

**Example 9:** If  $S = \{6z+1 : z \in \mathbb{Z}\}$  and  $\Gamma = \{5\}$  from example 2. We can see that  $S_5$  is semigroup:

(i)  $\forall a = 6z_1 + 1, b = 6z_2 + 1 \in S, \alpha = 5 \in \Gamma$

(ii)

$$a \circ b = \alpha b = (6z_1 + 1) + 5 + (6z_2 + 1) = 6(z_1 + z_2 + 1) + 1 = 6z + 1 \in S$$

$$\begin{aligned} a \circ (b \circ c) &= [(6z_1 + 1) \circ (6z_2 + 1 + 5 + 6z_3 + 1)] \\ &= 6(z_1 + z_2 + z_3 + 2) + 1 = 6z + 1 \\ &= [(6z_1 + 1) \circ (6z_2 + 1)] \circ (6z_3 + 1) \\ &= (6z_1 + 1 + 5 + 6z_2 + 1) \circ (6z_3 + 1) \\ &= 6(z_1 + z_2 + z_3 + 2) + 1 = 6z + 1 \end{aligned}$$

Consequently,  $S_5$  is  $\Gamma$ -semigroup and denoted by  $G_5$ . We can see that  $G_5$  is  $\Gamma$ -group. Identity element in  $G_5$  is  $-5 \in G_5$ :

$$-5 \circ (6z + 1) = -5 + 5 + 6z + 1 = 6z + 1 \in G_5$$

$$\forall a = 6z + 1 \in G_5, \exists ! b = -6(z + 2) + 1 \in G_5$$

$$a \circ b = 6z + 1 + 5 + [-6(z + 2) + 1] = 6z + 6 - 6z - 12 + 1 = -5$$

Consequently,  $G_5$  is  $\Gamma$ -group. Similarly, we can see that  $G_{\alpha}$  is  $\Gamma$ -group for all  $\alpha \in \Gamma$ .

**Theorem 2:**  $G_{\alpha}$  is a  $\Gamma$ -group if and only if G is simple  $\Gamma$ -semigroup.

**Proof:** Suppose  $G$  is simple, then  $G$  is right simple and left simple. Let  $a \in G$ , consider the set  $a\alpha G$ . We can show that  $a\alpha G$  is a right ideal in  $G$ . Since  $G$  is right simple we have  $a\alpha G = G$ . Similarly, we can show that  $G\alpha a = G$ . Hence,  $a \circ G = G$  and  $G \circ a = G$  for any  $a \in G$ . Then from the known (Ljapin, 1960) result it follows that  $G_\alpha$  is a  $\Gamma$ -group.

Conversely assume that  $G_\alpha$  is a group and  $e_\alpha$  be the identity element in  $G_\alpha$ . Let  $I$  be a left ideal in the  $\Gamma$ -semigroup  $G$  and  $a \in I$ . Then there exists  $b \in G$  such that  $b \circ a = e_\alpha$ . Hence,  $e_\alpha = b \circ a = b\alpha a = \epsilon I$ . Let  $c \in G$ . Then  $c = c \circ e_\alpha = c\alpha e_\alpha \in I$ . This shows that  $G = I$ . Similarly, we can show that  $G$  has no proper right ideal. Hence  $G$  both left simple and right simple  $\Gamma$ -semigroup. Then  $G$  is simple  $\Gamma$ -semigroup.

**Corollary 1:**  $G_\alpha$  is a  $\Gamma$ -group if and only if  $G$  is a  $\Gamma$ -semigroup and if it has not proper ideal.

**Proof :** From Theorem 2 and definition 6 and 7 is evident.

**Theorem 3:** Let  $G$  a  $\Gamma$ -semigroup and  $\alpha \in \Gamma$ .  $G_\alpha$  is  $\Gamma$ -group if and only if  $G$  does not have proper quasi-ideal.

**Proof:** Suppose that  $G$  does not have proper quasi-ideal. Let  $a \in G$ , consider the set  $a\alpha G$ . Then:

$$(a\alpha G)\Gamma G \cap G\Gamma(a\alpha G) \subseteq (a\alpha G)\Gamma G = a\alpha(G\Gamma G) \subseteq a\alpha G$$

Therefore,  $a\alpha G$  is quasi-ideal. But  $G$  does not have proper quasi-ideal, then  $a\alpha G = G$ . Similarly we can show that  $G\alpha a = G$ . Hence, for every  $a \in G$  we have  $a \circ G = G \circ a = G$ . It shows that  $G_\alpha$  is a  $\Gamma$ -group.

Conversely, suppose that  $G_\alpha$  is a group and  $Q$  is quasi-ideal of  $G$ . Let  $a \in Q$ , then:

$$G = G\alpha a = a\alpha G = a\alpha G \cap G\alpha a \subseteq Q\Gamma G \cap G\Gamma Q \subseteq Q$$

Consequently,  $G = Q$ . This shows that  $G$  does not have proper quasi-ideal.

**Theorem 4:** Let  $G$  be a  $\Gamma$ -semigroup if  $G_\alpha$  is  $\Gamma$ -group for some  $\alpha \in \Gamma$  then  $G_\alpha$  is  $\Gamma$ -group for all  $\alpha \in \Gamma$ .

**Proof:** Let  $G_\alpha$  be a group. Consider the sets  $a\beta G$  and  $G\beta a$ ;  $a \in G$ ,  $\beta \in \Gamma$ . Now  $(a\beta G)\alpha G = a\beta(G\alpha G) \subseteq a\beta G$  and  $G\alpha(G\beta a) = (G\alpha G)\beta a \subseteq G\beta a$ . Hence,  $a\beta G$  is a right ideal and  $G\beta a$  is a left ideal in  $G_\alpha$ . Since  $G_\alpha$  is a group we have  $a\beta G = G$  and  $G\beta a = G$ . Then  $a \circ G_\beta = G$  and  $G_\beta \circ a = G$  for all  $a \in G$ . Hence, from known result it follows that  $G_\beta$  is a group.

**Theorem 5:** A regular  $\Gamma$ -semigroup  $G$  will be a  $\Gamma$ -group if and only if  $e\alpha f = f\alpha e = f$  and  $e\beta f = f\beta e = e$  for any two idempotents  $e = e\alpha e$  and  $f = f\beta f$  of  $G$ .

**Proof:** Suppose  $G$  is a  $\Gamma$ -group. Let  $e = e\alpha e$  and  $f = f\beta f$  of  $G$  be two idempotents in  $G$ . Then  $e$  is the identity element of the group  $G_\alpha$  and  $f$  is the identity element of the group  $G_\beta$ . Now  $f \in G_\alpha$ . Hence  $e \circ f = f \circ e$  ( $e$ -identity of  $G_\alpha$ ) and  $e \circ f = e\alpha f$ ,  $f \circ e = f\alpha e$ . This shows that  $e\alpha f = f\alpha e = f$ . Similarly  $e\beta f = f\beta e = e$ . Conversely, suppose that the given condition holds in a regular  $\Gamma$ -semigroup  $G$ . Let  $a \in G$ . Then there exist  $\alpha, \beta \in \Gamma$  and  $b \in G$  such that  $a = a\alpha(b\beta a) = (a\alpha b)\beta a$ . Let  $e = b\beta a$  and  $f = a\alpha b$ . Hence  $e\alpha e = e$  and  $f\beta f = f$  are two idempotents in  $G$ . We shall show that  $G_\alpha$  is a group. Let  $c = (c\gamma d)\delta c$  be an element in  $G$  where  $\gamma, \delta \in \Gamma$  and  $d \in G$ . Then:

$$(c\gamma d)\delta(c\gamma d) = [(c\gamma d)\delta c]\gamma d = c\gamma d \text{ and } (d\delta c)\gamma(d\delta c) = d\delta c$$

are idempotents. Now:

$$e\alpha c = e\alpha[(c\gamma d)\delta c] = [e\alpha(c\gamma d)]\delta c = (c\gamma d)\delta c = c$$

and

$$c\alpha e = [c\gamma(d\delta c)]\alpha e = c\gamma[(d\delta c)\alpha e] = c\gamma(d\delta c) = c$$

Hence,  $e \circ c = e\epsilon c = c \circ e = c\alpha e = c$  for all. Again  $f\beta e = e$ . Hence,  $(a\alpha b)\beta e = e$ . Then  $a\alpha(b\beta e) = e$ . Also

$$(b\beta e)\alpha a = b\beta[e\alpha(a\alpha b)\beta a] = b\beta[e\alpha(f\beta a)] = b\beta[(e\alpha f)\beta a] = b\beta(f\beta a) = b\beta a = e$$

Hence, for a there exists  $b\beta e$  in  $G_\alpha$  such that:

$$a \circ (b\beta e) = (b\beta e) \circ a = e$$

Hence,  $G_\alpha$  is a group.

**Example 10:** If

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}, \Gamma = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{Z} \right\}$$

then  $S$  is  $\Gamma$ -semigroup. For fixed element:

$$\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \alpha \in \Gamma$$

We can see that  $G_\alpha$  is  $\Gamma$ -group. Identity element is:

$$E = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} E \circ A = E + \alpha + A = A$$

For:

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \exists! A^{-1} = \begin{pmatrix} -4+a & 0 \\ -b & -4+c \end{pmatrix}; A, A^{-1} \in G_\alpha : A \circ A^{-1} = A$$

**Theorem 6:** An invers  $\Gamma$ -semigroup  $G$  will be a  $\Gamma$ -group if and only if  $e\alpha f = f\alpha e = f$  and  $e\beta f = f\beta e = e$  for any two idempotents  $e = e\alpha e$  and  $f = f\beta f$  of  $G$ .

**Proof:** Inverse  $\Gamma$ -semigroup  $G$  always is a regular  $\Gamma$ -semigroup. Hence, the remaining part of proof of Theorem 6 follows from Theorem 5.

### CONCLUSION

A  $\Gamma$ -semigroup  $G$  is a  $\Gamma$ -group, if  $G$  is simple or  $G$  does not have proper quasi-ideal or ideal. A regular or invers  $\Gamma$ - semigroup  $G$  will be a  $\Gamma$ -group if  $G$  satisfies condition in theorem 5 or 6.

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