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On the Construction of Balanced Incomplete Block Designs Using Lotto Designs

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Abstract: Researchers present two algorithms for constructing Balanced Incomplete Block Designs (BIBD); the first, for determining the BIBDs that qualify to be Lotto Designs (LD) and the second for generating BIBDs from the LD parameters (n, k, p, t). The algorithms are tested using (υ = 6, b = 20, r = 10, k = 3, λ = 4) and (υ = 13, b = 130, r = 30, k = 3, λ = 4) BIBDs. One of the results, the (υ = 4, b = 4, r = 3, k = 3, λ = 2) BIBD which is pair wise balanced is shown to be D-optimal. Also, the (13, 130, 30, 3, 5) BIBD yielded (13, 56, 21, 3, 6), (13, 84, 28, 3, 7), (13, 120, 36, 3, 8) and (13, 165, 45, 3, 9) BIBDs; the first three being less cumbersome and more economical for experimental purposes. In general, a BIBD that qualifies as a LD can be used to generate other BIBDs.

Key words: Lotto designs, design matrix, information matrix, universal optimality, D-optimal, pair wise BIBD

INTRODUCTION

Some experiments require fixed number of block sizes regardless of the number of treatments available. For instance, some block sizes are determined based on biological grounds in sociological and psychological experiments where pairs of identical twins have variously been used. Some incomplete block designs are available for these situations (Hinkelmann and Kempthorne, 2005). Since all the treatments are usually of equal importance, the balanced incomplete block design has been evolved and this design ensures that the treatment combinations for each block are selected in a balanced way (Eno *et al*., 2009). Balanced Incomplete Block Designs (BIBDs) are used when the block size k is less than the number of treatments υ available (John, 1971). In this study, the researchers present the construction of some BIBDs from their parent BIBDs using lotto designs and they are able to show that one of the results, the (4, 4, 3, 3, 2) BIBD is D-optimal.

Lotto designs: Lotto is a gambling game in which a player randomly picks a pre-set quantity of numbers k from a pool of large numbers n. K constitutes the ticket that a player buys so as to be able to participate in a lottery game. The organisers of the game stop the sale of tickets at a certain point and p winning numbers are selected by the organisers randomly from the n numbers. If any of the tickets sold match t or more of the winning numbers, a prize is given to the holder of the matching ticket. The larger the value of t, the larger the prize; usually t must be three or more to receive a prize (Li, 1999). Mathematically, an (n, k, p, t) lotto design is an n set, V of elements and a set B of k element subsets (blocks) of V, so that for any p subset P of V, there is a block B∈B, for which $|P \cap B| \geq t$ (Li and van Rees, 2007). Li (1999) discussed many techniques for constructing LDs one of which is the use of BIBDs. He proved the following theorem as a general condition that must be satisfied by a BIBD for it to qualify as a LD.

Theorem: If B is the set of blocks of a (υ, b, r, k, λ) BIBD and p, t are positive integers, where:

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1)}{2} < \binom{p}{2} \lambda$$

then B is the set of blocks of a (υ, k, p, t) Lotto design.

Balanced incomplete block designs: A Balanced Incomplete Block Design (BIBD) is a pair (V, B) where V is a v-set and B is a collection of bk-subsets of V (blocks) such that each element of V is contained in exactly r blocks and any 2 subset of V is contained in exactly λ blocks (Mathon and Rosa, 2007). Necessary conditions for the existence of a BIBD (υ, b, r, k, λ) are:

$$\upsilon r = bk \qquad \text{(i)}$$

$$r(k-1) = \lambda(\upsilon-1) \qquad \text{(ii)}$$

A randomized incomplete block design is said to be pair-wise balanced if the treatment combinations are

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selected in each block such that every pair of treatments occur together in the same number of blocks (Eno *et al*., 2009). There are many construction methods for BIBDs (Colbourn and Dinitz, 1996). A design is said to be universally optimal if its information matrix, $X^T X$ where X is the design matrix is completely symmetric with maximal trace (Kiefer, 1974). A design is said to be D-optimal if the determinant of its information matrix is greater than or equal to the information matrix of any alternative design (Eno *et al*., 2009). A computer algorithm was used to construct D-optimal designs for $\upsilon + b \le 31$ and $\upsilon r \le 44$ (Mitchell, 1973). Results of systematic search for optimal incomplete block designs for $\upsilon \le 12$, $r \le 10$ and $\upsilon \le b$ have been given (John and Mitchell, 1977).

MATERIALS AND METHODS

An algorithm that enables us to determine easily the BIBDs that qualify as LD is as follows:

- Define parameters p, t, r, λ
  - p is fixed to run from 3-11
  - t is fixed to run from 3-8
  - r is selected from a list of BIBDs
  - λ is selected from a list of BIBDs

- Compute formula:

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \left(\frac{pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1)}{2} \right) < \binom{p}{2} \lambda$$

to identify items less than:

$$\binom{p}{2} \lambda$$

- Select elements using the criterion before
- Selected sets qualify as lotto designs
- Refine parameters in step 1 to select others
- End

Two BIBDs are selected randomly from a list of those that satisfied Li's theorem. They are (6, 20, 10, 3, 4) and (13, 130, 30, 3, 5) BIBDs. If the researchers let υ and k in the (υ, b, r, k, λ) BIBD correspond to n and k, respectively in the LD (n, k, p, t), then the (6, 20, 10, 3, 4) BIBD qualify as LD (6, 3, 4, 3), LD (6, 3, 5, 3) and LD (6, 3, 6, 3). Also, the (13, 130, 30, 3, 5) BIBD qualify as LD (13, 3, 8, 3), LD (13, 3, 9, 3), LD (13, 3, 10, 3) and LD (13, 3, 11, 3).

The second algorithm, adapted from Rosen (1991), uses lexicographic ordering to construct specific BIBDs from the Lotto design parameters. The algorithm which generates all possible k element combinations from n objects is as follows:

An algorithm to generate combinations: Starting from the first combination of k objects from a set of n objects, the algorithm generates the next combination in lexicographic order:

```
a0<--1, a1<--2,...,ak-1<--k %first %combination is: 1, 2, 3, ..., k loop
i<-- k-1 % start at last item
while ai = (n-k+i+1) %find next item %to increment decr i
end while,
exit when (i<0)  % all done
incr ai          % increment
for j<-- (i+1)...(k-1)do %reset %elements to the %right
aj<-- ai+j-i
end for
end loop
```

RESULTS AND DISCUSSION

Table 1 shows some BIBDs constructed from particular LDs. The BIBDs constructed from LD (6, 3, 4, 3) and LD (6, 3, 5, 3) cannot utilize all the six treatments but the BIBD got from LD (6, 3, 6, 3) produced the same set of parameters the parent BIBD possessed. The design matrix of the (4, 4, 3, 3, 2) BIBD derived from LD (6, 3, 4, 3) is:

$$X_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

This matrix is the same regardless of the comparison value used. For instance, comparison values (1, 2, 3, 4) are shown:

- Comparison value (1 2 3 4 t = 3):

1 2 3
1 2 4
1 3 4
2 3 4

Also, comparison values (1, 2, 4, 6) are shown:

- Comparison value (1 2 4 6 t = 3):

1 2 4
1 2 6
1 4 6
2 4 6

Table 1: Some BIBDs constructed from LDS

Lotto designs (n, k, p, t)	BIBD produced
Parent BIBD (υ, b, r, k, λ): (6, 20, 10, 3, 4) BIBD	
(6, 3, 4, 3)	(4, 4, 3, 3, 2)
(6, 3, 5, 3)	(5, 10, 6, 3, 3)
(6, 3, 6, 3)	(6, 20, 10, 3, 4)
Parent BIBD (υ, b, r, k, λ): (13, 130, 30, 3, 5) BIBD	
(13, 3, 8, 3)	(13, 56, 21, 3, 6)
(13, 3, 9, 3)	(13, 84, 28, 3, 7)
(13, 3, 10, 3)	(13, 120, 36, 3, 8)
(13, 3, 11, 3)	(13, 165, 45, 3, 9)

The design matrix X_1 is an alternative design to the design matrix X of the (4, 4, 3, 3, 2) BIBD constructed by Eno *et al.* (2009) using a non-linear, non-pre-emptive, binary integer goal programming model. The determinant of its information matrix $X_1^T X_1$ is thus equal to that of $X^T X$. Therefore, the (4, 4, 3, 3, 2) BIBD constructed through LD is D-optimal. Also, the (13, 130, 30, 3, 5) BIBD yielded (13, 56, 21, 3, 6), (13, 84, 28, 3, 7), (13, 120, 36, 3, 8) and (13, 165, 45, 3, 9) BIBDs. The first three of these designs have smaller blocks compared to their parent BIBD, thus they are more economical and less cumbersome to use in practice. In general, BIBDs that qualify as LDs can be used to generate other BIBDs.

CONCLUSION

In this study, researchers developed two algorithms; the first for determining the BIBDs that qualified as LDs using Li's theorem and the second one for constructing BIBDs from particular lotto parameters. The (6, 20, 10, 3, 4) BIBD produced the (4, 4, 3, 3, 2), (5, 10, 6, 3, 3) and (6, 20, 10, 3, 4) BIBDs. The (4, 4, 3, 3, 2) BIBD constructed through LD (6, 3, 4, 3) is D-optimal. Also, the (13, 130, 30, 3, 5) BIBD yielded (13, 56, 21, 3, 6), (13, 84, 28, 3, 7), (13, 120, 36, 3, 8) and (13, 165, 45, 3, 9) BIBDs; the first three being less cumbersome and more economical. In general, a BIBD that qualifies as a LD can be used to generate other BIBDs.

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