

## Inventory Production Control Model with Back-Order when Shortages are Allowed

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**Abstract:** It is prohibited to have shortage of inventory since inventory cost is induced from the amount of product stored. This study presents inventory control theory in production inventory problem when shortages are allowed and backorder takes place. Three assumptions are considered here on shortage and backorders and this leads to three models. The first: when demand is fixed and known, production is infinite and shortages are allowed although, the cost of shortage is finite. Second when time (t) interval is fixed, replenishment is allowed and production rate is infinite. Third, when production rate is finite. It makes economic sense from the applications that for any production where shortages are allowed, backorder must follow to avoid lost in sales.

**Key words:** Inventory control, backorder, production, shortage, demand, infinite

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### INTRODUCTION

Due to the quest for efficiency accelerated by the so called financial crisis, inventory control is a vital function in almost all kinds of productions. Inventory models majorly focused on minimizing the total inventory cost and to balance the economics of large orders or large production runs against the cost of holding inventory and the cost of going short. The method has been efficiently and successfully applied by some researchers in many areas of operation (Davis and Elzinga, 1971; Kaplan, 1969; Bounkhel and Tadj, 2006; Derzko and Sethi, 1981; Gupta and Hira, 2005; Rapp, 1974; Goyal and Giri, 2001).

Production and inventory planning and control procedures for a target firm depends on whether production is make to stock or make to order (which in turn depends on the relation between customer promise time and production lead time) and whether demand is for known production or anticipated production.

**Literature review:** This study introduce some typical research involve in the topics with different subcategories, lost sales, backorders, shortages and deterioration as well as periodic review and continuous review. One critical factor playing major roles on the inventory theory is backorders. Much of the literatures on inventory models ignore backorders. Backorders means delay in meeting demands or inability to meet it at all. Most inventory models discuss two extreme situations when items are stock out.

They are all demand within shortage period is backorder. All demand within shortage period is lost

sales. In real inventory systems, demands during the period of stock out can be partially captive. If demand is fully captive, the next replenishment will fulfill unsatisfied demands during the period of backorders. On the contrary, unsatisfied demands will be completely lost if demand cannot be fully captive yet demand rate during the period of stock out is not a fix constant if to take backorders into consideration.

The recent survey of Kaplan (1969) and Katircioglu and Atkins (1996) and many other researchers have developed inventory models on related field and initiated the concept of demands which will be changed through time cycle into model, also included backorder status, study on (Sethi, 1978) optimal control of production inventory system with deteriorating items and dynamic cost and a study on optimal control of production inventory system with deterioration items using Weibull distribution (Al-Khedhairi and Tadj, 2007; Benhadid *et al.*, 2008).

Also there was an inventory model of replenishing the stock after a period of backorder (Goyal, 1992) which is that deplete cycle always started from the period of backorder. A modification of the complete backorder assumptions and proposed the concept of partial backorders (Wee, 1995) which assumed the backorder ratio is a constant between 0 and 1. The assumption is that usually the time scale of backorder will become consumers' main pondering factor to accept backorder. This study looked into assumptions and models for production inventory of a single item when shortages are allowed and there is an order to meet exogenous demand at a minimum cost.

**MATERIALS AND METHODS**

Inventory control models assumed that demand from customer are known for planning period both at present and past period. It is prohibited to have shortage of inventory since inventory cost is induced from the amount of product storage. Three assumptions were considered (Fig. 1).

**Notations and assumptions:** To develop the proposed models, the following notations and assumptions are used in this study:

- $I(t)$  = Inventory level at time  $t$
- $R_t$  = Demand rate or the number of items required per unit time
- $C_1$  = Holding cost per unit time
- $C_2$  = Shortage cost per unit item per time
- $C_3$  = Production set up cost per run
- $t$  = Interval between runs
- $q$  = Number of items produced per production run if a production is made at time interval  $t$ , a quantity
- $q = Rt$  must be produced in each run

Since the stock in small time  $dt$  is  $Rtdt$ , the stock in time period  $t$  is:

$$\int_0^t Rtdt = \frac{1}{2} Rt^2 = \frac{1}{2} qt$$

**Assumptions 1:** In this model, we assume that demand is fixed and known, production is infinite and shortages are allowed, although the cost of shortage is finite i.e.,

- The inventory system involves only one item
- Replenishment occurs instantaneously on ordering i.e., lead-time is zero
- Demand rate  $R(t)$  is deterministic and given by  $R(t) = ; 0 < t < T$
- Shortages are allowed and completely backlogged
- The planning period is of infinite length. The planning horizon is divided into sub-intervals of length  $T$  units. Orders are placed at time points  $t_1$  and  $t_2$ , the order quantity at each re-order point being just sufficient to bring the stock height to a certain maximum level  $S$ . If:

$$t = t_1 + t_2$$

then;

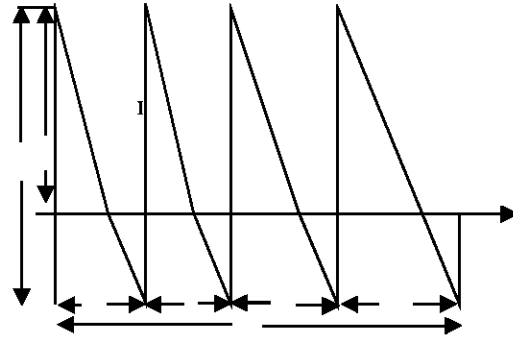


Fig. 1: Variation of inventory with time

$$\frac{t_1}{t} = \frac{I_t}{q}, \quad t_1 \frac{I_t}{q}$$

Also:

$$\frac{t_2}{t} = \frac{q - I_t}{q}, \quad t_2 \frac{q - I_t}{q}$$

Total inventory during time:

$$t = \text{Area of } \Delta AOB = \frac{1}{2} I_t t_1$$

Inventory holding cost during time:

$$t = \frac{1}{2} C_1 I_t t_1$$

Similarly, total shortage during time:

$$t = \text{Area of } \Delta BCD = \frac{q - I_t}{2} t_2$$

Shortage cost during time:

$$t = C_2 \frac{(q - I_t)}{2} t_2$$

Total cost during time:

$$t = \frac{1}{2} C_1 I_t t_1 + C_2 \frac{(q - I_t)}{2} t_2 + C_3$$

Average total cost during time:

$$t = \frac{1}{t} \left[ \frac{1}{2} C_1 I_t t_1 + C_2 \frac{(q - I_t)}{2} t_2 + C_3 \right]$$

$$C(I_t, q) = \frac{C_1 I_t t}{2q} + \frac{C_2 (q - I_t)^2}{2q} + \frac{C_3 R}{q} \tag{1}$$

Differentiate Eq. 1 partially w.r.t.  $I_t$ ,  $q$  and equate to zero to obtain optimal inventory level  $I_t$  and optimum lot size ( $q$ ):

$$\frac{\partial C(I_t, q)}{\partial I_t} = 0$$

$$I_t = \frac{C_2 q}{C_1 + C_2}$$

which is positive for the second derivative, it shows that the optimal value of inventory level is:

$$I_{t0} = \frac{C_2}{C_1 + C_2} q \tag{2}$$

$$\frac{\partial C(I_t, q)}{\partial q} = 0$$

Similarly:

$$q = \sqrt{\frac{C_1 + C_2}{C_1 C_2}} \cdot \sqrt{2C_3 R} \tag{3}$$

The optimal value of lot size  $q$  is:

$$q_0 = \sqrt{\frac{C_1 + C_2}{C_1 C_2}} \cdot \sqrt{2C_3 R} = \sqrt{\frac{C_1 + C_2}{C_2}} \cdot \sqrt{\frac{2C_3 R}{C_1}} \tag{4}$$

Hence, Eq. 2 can be written as:

$$I_{t0} = \sqrt{\frac{C_2}{C_1(C_1 + C_2)}} \cdot \sqrt{2C_3 R} \tag{5}$$

Substituting the values of  $I_{t0}$ ,  $q_0$  in Eq. 1, we obtain the minimum average cost per unit time i.e.,

$$C_0(I_{t0}, q_0) = \sqrt{\frac{C_2}{C_1 + C_2}} \cdot \sqrt{2C_1 C_3 R} \tag{6}$$

Optimum time interval between runs is given by:

$$t_0 = \frac{q_0}{R} = \sqrt{\frac{C_1 + C_2}{C_1 C_2}} \cdot \sqrt{\frac{2C_3}{R}} \tag{7}$$

**Assumption 2:** Fixed time interval  $t$  when it is fixed, it means inventory is to be replenished after every fixed time  $t$ . All other assumptions in 1 above hold. Total inventory holding cost during time:

$$t = \frac{1}{2} C_1 I_t t_1$$

Total shortage cost during time:

$$t = \frac{1}{2} C_2 (q - I_t) t_2$$

Set up cost  $C_3$  and time interval  $t$  are both constant therefore, average set up cost per unit time  $C_3/t$  is also constant. It needs not to be considered. Total average cost per unit:

$$C(I_t) = \frac{1}{t} \left[ \frac{1}{2} C_1 I_t t_1 + \frac{1}{2} C_2 (q - I_t) t_2 \right]$$

or;

$$= \frac{C_1}{2q} \cdot I_t^2 + \frac{C_2}{2q(q - I_t)^2} \tag{8}$$

$$\frac{\partial}{\partial I_t} (C I_t) = 0$$

$$I_t = \frac{C_2 q}{C_1 + C_2} \tag{9}$$

Hence, the minimum inventory level or order quantity given is:

$$I_{t0} = \frac{C_2 q}{C_1 + C_2} \text{ or } \frac{C_2 R t}{C_1 + C_2} \tag{10}$$

The minimum average cost per unit time from Eq. 8 is:

$$C_0(I_t) = \frac{C_1}{2q} \left( \frac{C_2 q}{C_1 + C_2} \right)^2 q^2 + \frac{C_2}{2q} \left( q - \frac{C_2}{C_1 + C_2} \cdot q \right)^2 \tag{11}$$

$$\frac{1}{2} \cdot \frac{C_1 C_2}{C_1 + C_2} \cdot q \text{ or } \frac{1}{2} \cdot \frac{C_1 C_2}{C_1 + C_2} \cdot R t$$

**Assumption 3**

**Finite production/planning rate:** The model here follows the assumptions in 1 except that production rate is finite. With this assumption, we found that inventory is zero at the beginning. It increases at a constant rate  $(K-R)$  for  $t_1$  time until it reaches a level  $I_t$ . No replenishment during time  $t_2$ , inventory decreases at the rate  $R$  until it reaches zero. Shortage start piling up at constant rate  $R$  during  $t_3$  until this backlog reaches a levels. Lastly, production

start and backlog is filled at a constant rate  $K-R$  during  $t_4$  till backlog become zero. This completes cycle. The total time taken is:

$$t = t_1 + t_2 + t_3 + t_4$$

$$\text{Holding cost} = \frac{1}{2} C_1 I_1 (t_1 + t_2)$$

Shortage cost during time interval:

$$t = \frac{1}{2} \cdot C_2 s (t_3 + t_4)$$

Set up cost =  $C_3$ . Hence, total average cost per unit time  $t$ :

$$C = \frac{\frac{1}{2} C_1 I_1 (t_1 + t_2) + \frac{1}{2} C_2 s (t_3 + t_4) + C_3}{t_1 + t_2 + t_3 + t_4} \quad (12)$$

Equation 12 is a function of six variables i.e.,  $I_1, s, t_1 + t_2 + t_3 + t_4$ . Inventory level at time  $t_1$ :

$$I_1 = (K - R)t_1 \quad (13)$$

Also at time  $t_2$  is:

$$I_1 = Rt_2 \quad (14)$$

$$\therefore (K - R)t_1 = Rt_2 \quad (15)$$

Also:

$$S = Rt_3 \quad (16)$$

And:

$$s = (K - R)t_4 \quad (17)$$

$$\therefore (K - R)t_4 = Rt_3 \quad (18)$$

Adding Eq. 15 and 18:

$$(K - R)[(t_1 + t_4)] = R(t_2 + t_3)$$

Manufacturer's rate multiply by manufacturer's time gives manufactured quantity produced:

$$q = Kt_1 + Kt_4 = [(t_1 + t_4)K] \quad (19)$$

Adding Eq. 14 and 16:

$$I_1 + s = R(t_2 + t_3)$$

$$I_1 = R(t_2 + t_3) - s$$

$$I_1 = (K - R)[(t_1 + t_4)] - s$$

$$I_1 = \frac{q}{k}(K - R) - s$$

$$I_1 = \left(\frac{q}{k}\right)(K - R) - s$$

$$I_1 = q \left(1 - \frac{R}{k}\right) - s \quad (20)$$

From Eq. 13 and 14:

$$\begin{aligned} t_1 + t_2 &= \frac{I_1}{K - R} + \frac{I_1}{R}(t_2 + t_3) \\ &= \frac{s}{K - R} + \frac{s}{R} \end{aligned} \quad (21)$$

Hence,  $t = t_1 + t_2 + t_3 + t_4$

$$= \left(\frac{1}{K - R} + \frac{1}{R}\right) \left(q \cdot \frac{K - R}{K}\right) = \frac{q}{R} \quad (22)$$

Hence, Eq. 12 becomes:

$$\begin{aligned} C(q, s) &= \frac{1}{2q} \cdot \frac{K}{K - R} \\ \left[ C_1 \left\{ q \cdot \frac{K - R}{K} - s \right\}^2 + C_2 s^2 \right] + \frac{R}{q} C_3 \frac{\partial C(q, s)}{\partial q} &= 0 \end{aligned} \quad (23)$$

Minimum lot size is:

$$q_0 = \sqrt{\frac{2C_3(C_1 + C_2)}{C_1 C_2}} \cdot \sqrt{\frac{KR}{K - R}} \quad (24)$$

and:

$$\frac{\partial C(q, s)}{\partial s} = 0$$

implies;

$$s_0 = q \cdot \frac{K - R}{K} \cdot \frac{C_1}{(C_1 + C_2)}$$

$$s_0 = \sqrt{2C_3 \cdot \frac{C_1}{(C_1 + C_2)C_2}} \cdot \sqrt{\frac{R(K - R)}{K}} \quad (25)$$

Substituting  $q_0$  and  $s_0$  into Eq. 17, we have the optimum shortage cost:

$$C_0(q,s) = \sqrt{\frac{2C_1 C_2 C_3}{(C_1+C_2)}} \cdot \sqrt{\frac{R(K-R)}{K}} = \sqrt{\frac{2C_1 C_2 C_3 R(K-R)}{K(C_1+C_2)}} \quad (26)$$

Optimum time interval  $t_0$  is:

$$t_0 = \frac{q_0}{R} = \sqrt{\frac{[2KC_3(C_1+C_2)]}{C_1 C_2 R(K-R)}} \quad (27)$$

Optimum inventory level:

$$I_{10} = q \left(1 - \frac{R}{K}\right) - s_0 = \sqrt{\frac{C_2}{C_1+C_2}} \cdot \sqrt{\frac{(K-R)}{K}} \cdot \sqrt{\frac{2C_3 R}{C_1}} \quad (28)$$

$$= \sqrt{\frac{2C_2 C_3 R(K-R)}{K C_1 (C_1+C_2)}}$$

**Numerical applications**

**Example 1:** If a particular soap items has demand of 9000 units year<sup>-1</sup>. The cost of one procurement is £100 and holding cost per unit is £2.40 year<sup>-1</sup>. The replacement is instantaneous and the cost of shortage is also £5/unit/year. We are required to determining the following:

- Economic lot size/Optimum lot size
- The number of orders per year
- The time between the orders
- The total cost per year if the cost of one unit is £1

**Solution: Step I:**

Demand rate, R = £9000 units year<sup>-1</sup>  
 Holding cost, C<sub>1</sub> = £2.40/unit/year  
 Shortage cost, C<sub>2</sub> = £5/unit/year  
 Production set up = £100/procurement  
 cost per run, C<sub>3</sub>

From Eq. 4,  $q_0 = 1,053$  units/run i.e., the optimum lot size/run is 1,053 units. The number of order per year = 8.55 units year<sup>-1</sup> (Eq. 7). Hence, the number of order per year is 8.55 or ≈ 9 number of times ordered per year. Time period between the order is as follows, from Eq. 7 is 0.117 year i.e., there is approximately 1 month and 13 days period between the order. From Eq. 6, the total cost per year if the cost of one unit is £1 = £10710 year<sup>-1</sup>. Hence, the total cost per year if the cost of one unit is £1 is £10,710.

**Example 2:** Consider an inventory system with the following data in usual notations:

R = 20 engines day<sup>-1</sup>  
 C<sub>2</sub> = £10/engine/day

C<sub>1</sub> = £12/month or 12/30 = £0.4 day<sup>-1</sup>  
 t = 1 month = 30 days i.e., lead time ≠ 0 (fixed)

We now want to check for the inventory level at the beginning of each month and the optimum cost per unit. Recall From Eq .10;

$$I_{10} = \frac{10}{0.4+10} \times 20 \times 30 = \frac{577 \text{ engines}}{\text{month}}$$

Hence, the optimum inventory level at the beginning of each month is 577 engines. Also from Eq .11.

$$C_0(I_1) = \frac{1}{2} \cdot \frac{C_1 C_2}{C_1 + C_2} R t = \frac{1}{2} \cdot \frac{0.4 \times 10}{0.4 + 10} \times 20 \times 30$$

While £115.38/unit of engines i.e., the minimum cost of producing an engine is £115.38.

**Example 3:** A company has a demand of 12,000 units year<sup>-1</sup> from an item and it can produce 2,000 such items per month. The cost of one set up is £400 and the holding cost/unit/month is £0.15, the shortage cost of one unit is £20 year<sup>-1</sup>. Find the optimum lot size and the total cost per year, assuming the cost of one unit if £4. We can also find the maximum inventory manufacturing time and total time. Given the following:

R = 12,000  
 K = 2000 × 12 = 24,000/units/year  
 C<sub>1</sub> = 0.15 × 12 = 1.8/unit/year  
 C<sub>2</sub> = £20 year<sup>-1</sup>  
 C<sub>3</sub> = £400/set-up

Using Eq. 19:

$$q_0 = \sqrt{\frac{2 \times 400 \times (1.8 + 20)}{1.8 \times 20}} \cdot \sqrt{\frac{24000 \times 1200}{24000 - 1200}}$$

$$= 3,410 \text{ units}$$

The optimum lot size is 3,410 units. The total cost per year is considered by using Eq. 21:

$$C_0(q,s) = 12000 \times 4 + \sqrt{\frac{2C_1 C_2 C_3 R(K-R)}{K(C_1+C_2)}}$$

$$C_0(q,s) = 1200 \times 4 + \sqrt{\frac{2 \times 1.8 \times 20 \times 400 \times 12000(24000 - 12000)}{24000(20 + 1.8)}}$$

While £50,185 year<sup>-1</sup>. The total cost per year is

£50,185 when the cost of one item is £4. Using the Eq. 23, optimum inventory level at time t is:

$$I_{t_0} = \sqrt{\frac{2 \times 20 \times 400 \times 12000 \times 10}{2 \times 1.8 \times 10.9}}$$

While 1,564 unit/production run, manufacturing time interval  $t_1 + t_4$ . From Eq. 15:

$$\begin{aligned} [(t)_1 + t_4] &= \frac{q}{K} \\ &= \frac{3410}{24000} \end{aligned}$$

Hence, the optimum inventory level at time t is 0.1421 year. Which is approximately 52 days or 1 month and 3 weeks. Optimum time interval is given by:

$$\frac{q}{R} = \frac{3410}{1200}, 0.2842 \text{ years}$$

This means that the minimum time interval required is 103 days i.e., 3 months and 8 days.

### CONCLUSION

It can be deduced that when replenishment cost and demand rate per unit time R increase, order quantity q and relevant total cost C will increase. An increment of inventory holding cost per unit (h), backorder cost and penalty cost will lead to the phenomenon of increasing before diminishing.

This idea can induce cost items in inventory depletion period having a trade off relationship with cost items in backorders status. Also the decision about when an order should be placed will also be based on how low the inventory should be allowed to be depleted before the order arrives. The idea is to place an order early enough so that the expected number of units demanded during the replenishment lead time will not result in stock out every often. This research contribute to knowledge in many areas of production or daily life activities where failure to meet up with demand/supply (activities) induced a nebulous cost and pay-price or replenishment has to be done. Many industries can benefit from this through proper implementations/applications.

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