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A New Method for Solutions of Differential Equation by Fast Fourier Transform

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Abstract: A new numerical method has been carried out to solve the differential equation using Fast Fourier Transform (FFT). The new algorithm has been accompanied by a numerical example. Firstly, we solve a Cauchy problem for an elastic vibrating system using the finite difference method. Then with the values of the approximate solution obtained in the equidistant points from the interval (0, 1), we shall find an interpolation polynomial using FFT. Also we study, the approximation of the numerical solution and stability of the difference scheme which correspond of a second-order differential equation.

Key words: Numerical method, fourier transform, cauchy problem, equidistant points, polynomial, Iran

INTRODUCTION

Let on the interval $[0,\ 2\pi]$, 2N equidistant points, $x_0 = 0, x_1, ..., x_{2N-1}, x_{2N} = 2\pi$. A periodical function $f: R \to R$ with the period $T = 2\pi$, attain the given values, $f_j = f(x_j)$ at the points x_j, $j\in\{0, 1, ..., 2N-1\}$. The interpolation trigonometric polynomial of the function f is defined by the equality:

$$P_N = \frac{A_0}{2} + \sum_{k=1}^{N-1}(A_k\cos(kx)+B_k\sin(kx)) + \frac{A_N}{2}\cos(Nx) \quad (1)$$

Where:

$$A_k = \frac{1}{N}\sum_{j=0}^{2N-1} f_j \cos kx_j, k\in\{0,1,\ ...,\ N\}$$
$$B_k = \frac{1}{N}\sum_{j=0}^{2N-1} f_j \sin kx_j, k\in\{1,2,\ ...,\ N-1\} \quad (2)$$

To halve the computations number for calculation coefficients A_k, B_k on separates the components of f_j with even index from those with odd index and on defines (Look *et al*., 1998):

$$yk = f_{2k} + if_{2k+1},\ k\in\{0,1,....,N-1\} \quad (3)$$

With Cooley and Tukey algorithm, we calculate the sums of the form:

$$y_j = \sum_{k=0}^{N-1} C_k\omega^{kj},\ \omega = \exp\left(\frac{2\pi i}{N}\right), j\in\{0, 1, ...,N-1\} \quad (4)$$

Where:

$$C_j = \frac{1}{N}\sum_{k=0}^{N-1} y_k\,\omega^{-kj},\ j\in\{0,1,\ .....,N-1\} \quad (5)$$

Interpolation of reticular function: Consider a steel beam which is simply supported at the ends of length l = 25 cm and diameter d = 8 mm. At his half is placed a weight G = 3 daN which is acted by o force varying harmonically with time:

$$F(t) = F_0.\sin pt$$

Where:
F_0 = 5 daN
p = 100 rad sec^{-1}

The elastic constant:

$$k = \frac{48EI_z}{l^3} = \frac{48\times2.1\times10^6\times0.02}{25^3} = 129\ daN\,cm^{-1}$$

The equation of motion then becomes (Brennan 1988):

$$\frac{P}{g}\cdot\ddot{u}+ku = F_0\,\sin pt \quad (6)$$

Where, g is the acceleration of gravity (Chen *et al*., 1998). The circular frequency is defined by:

$$\omega = \sqrt{\frac{k\cdot g}{P}} = \sqrt{\frac{129.981}{3}} = 205.4\ rad\ sec^{-1} \quad (7)$$

And the period is:

$$T_1 = \frac{2p}{\omega} = \frac{2p}{205.4} = 0.03 \text{ sec} \quad (8)$$

Multiplying Eq. 6 with ratio P/g we obtain:

$$\ddot{u}+\omega^2 u = q\sin pt,\ q=\frac{F_0 g}{P} \quad (9)$$

and introducing the numerical values we get Cauchy problem:

$$\ddot{u}+205.4^2\ u = 1635 \sin 100t$$

$$U(0) = 0 \quad (10)$$

The complete solution of the problem Eq. 9 is composed of 2 functions (Chattopadhyay and Queisser, 1981). The first of these represents a natural vibratory motion:

$$u_1(t) = -\frac{p}{\omega}\cdot\frac{q}{\omega^2-p^2}\cdot \sin\omega t = A_1 \sin\omega t \quad (11)$$

The second vibratory motion is due to exciting force F(t):

$$u_2(t) = \frac{q}{\omega^2-p^2}\cdot \sin pt = A_2 \sin pt \quad (12)$$

With the period:

$$T_2 = \frac{2\pi}{p} = \frac{2\pi}{100} = 0.06 \text{ sec} \quad (13)$$

The interpolation polynomial will be the sum:

$$P(t) = P_1(t) + P_2(t) \quad (14)$$

Where, P_1, P_2 are the polynomials which correspond to u_1, u_2, respectively. For the natural vibratory motion with $T_1 = 0.03$, we change the interval $[0, T_1]$ in the interval [0, 1] by the relation: $x = t/T_1$. Let on the interval [0, 1], $N = 2^3$ equidistant points with the step of division h = 1/8. We solve the problem:

$$\begin{aligned} &\ddot{u}_1+\omega^2 u_1 = 0 \\ &u_1(0) = 0, \quad \dot{u}_1(0) = 0 \end{aligned} \quad (15)$$

by finite difference method and the results of calculations are entered in the appropriate rows of Table 1. In accordance with the difference scheme for the Eq. 14 and 15 we have (Makino *et al*., 2001):

$$u''(t) \approx \frac{u(t+h)-2u(t)+u(t-h)}{h^2}$$

Table 1: Finitr difference method and the results of calculations

I	0	1	2	3	4	5	6	7
x_i	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8
u_i	0	-0.017	-0.024	-0.017	0	0.017	0.024	0.017

and from Taylor's formula:

$$u(t+h) = u(t) + hu'(t) + \frac{h^2}{2}u''(t) + \frac{h^3}{3!}u'''(t) + \frac{h^4}{4!}u^{(4)}(\xi_1)$$

$$u(t-h) = u(t) + hu'(t) + \frac{h^2}{2}u''(t) + \frac{h^3}{3!}u'''(t) + \frac{h^4}{4!}u^{(4)}(\xi_2)$$

Hence:

$$\frac{u(t+h)-2u(t)+u(t-h)}{h^2} = u''(t) + \frac{h^2}{4!}\left(u^{(4)}(\xi_1)+u^{(4)}(\xi_2)\right),\ u\in C^4\left((0,T)\right)$$

where, $\xi_1, \xi_2 \in (t-h, t+h)$ and C^4 (0, T) is the set of the function with the derivatives to four order continuous. From Darboux property of function u there is $\xi\in[\xi_1, \xi_2]$ for which (Di and Brennan, 1991):

$$u^{(4)}(\xi) = \frac{u^{(4)}(\xi_1)+u^{(4)}(\xi_2)}{2}$$

Hence:

$$\frac{u(t+h)-2u(t)+u(t-h)}{h^2} = u''(t) + \frac{h^2}{12}u^{(4)}(\xi)$$

In order to demonstrate this property for the difference problem which corresponding to Eq. 14 we consider in the interval [0, 1], $N = 2^3$ equidistant points, t_n, n = 0, 1, 2, ..., N, $t_0 = 0$, $t_N = 1$. Let Δ be this partition of [0, 1] with the step h and we shall denote: $u_n = u(t_n)$, $B = \omega^2$ and $\varphi_n = q\sin pt_n$. Where:

$$Lu = \begin{cases} \dfrac{u_{n+1}-2u_n+u_{n-1}}{h^2} + B.u_n, n\in\{1,2,...N-1\} \\ u(0) \\ \dfrac{u_1-u_0}{h} \end{cases} \quad (16)$$

$$f = \begin{cases} \varphi_n \\ 0,\ 0\le t\le 1 \\ 0 \end{cases}$$

In accordance with the scheme is stable if for any f there is a unique solution and:

$$\|u\| < C\|f\| \tag{17}$$

An equivalent schema for Eq. 16 is:

$$\begin{cases} u_{n+1} = (2 - Bh^2)u_n - u_{n-1} + h^2\varphi_n \\ u_0 = 0 \\ u_1 - u_0 = 0 \end{cases} \tag{18}$$

Now we define:

$$z_n = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \begin{bmatrix} 2 - Bh^2 & -1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} + h \begin{bmatrix} h\varphi_n \\ 0 \end{bmatrix} \tag{19}$$
$$= R_h z_{n-1} + h.\rho_{n-1}$$

$$z_0 = \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let, Y be the bi-dimensional space with u_n, $\rho_n \in Y$ and we define the norm:

$$\left\| \begin{matrix} a \\ b \end{matrix} \right\|_Y = \max(|a|, |b|) \tag{20}$$

Since $\|R_h\| > 1$ $\|R_h\|^n \to \infty$ we shall define a norm which depends on h-step of partition of the interval [0, 1]:

$$\left\| \begin{matrix} a \\ b \end{matrix} \right\|_{Y_h} = \max\left(|a|, \frac{|b - a|}{h} \right) \tag{21}$$

Relationship between these norms is:

$$\left\| \begin{matrix} a \\ b \end{matrix} \right\|_{Y_h} \left\| \begin{bmatrix} 1 & 0 \\ 1/h & -1/h \end{bmatrix} . \begin{bmatrix} a \\ b \end{bmatrix} \right\|_Y = \left\| S. \begin{bmatrix} a \\ b \end{bmatrix} \right\|_Y \tag{22}$$

Where:

$$S = \begin{bmatrix} 1 & 0 \\ 1/h & -1/h \end{bmatrix}$$

We shall prove that:

$$\|T\|_{Y_h} = \|STS^{-1}\|_Y \tag{23}$$

for any linear operator T. Indeed:

$$\left\| T \begin{bmatrix} a \\ b \end{bmatrix} \right\|_{Y_h} = \left\| ST \begin{bmatrix} a \\ b \end{bmatrix} \right\|_Y = \left\| STS^{-1} \left(S \begin{bmatrix} a \\ b \end{bmatrix} \right) \right\|_Y$$

Then in accordance with Gonze *et al.* (2002), we obtain from the definition of the norm in Y_h and Eq. 23:

$$\|T\|_{Yh} = \max \frac{\|Tx\|_{Y_h}}{\|x\|_{Y_h}} = \max_{x \in Y} \frac{\|STS^{-1}(Sx)\|_Y}{\|Sx\|_Y}$$
$$= \max_{v \in Y} \frac{\|STS^{-1}v\|_Y}{\|v\|_Y} = \|STS^{-1}\|_Y$$

and thus, Eq. 23 is true. Also we have:

$$\|R_h\|_{Y_h} = \left\| SR_h S^{-1} \right\|_Y \tag{24}$$

Recall that for any matrix T;

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

We have:

$$\|T\|_Y = \max\{|t_{11}| + |t_{12}|, |t_{21}| + |t_{22}|\} \tag{25}$$

Hence if:

$$SR_h S^{-1} = \begin{bmatrix} 1 & 0 \\ 1/h & -1/h \end{bmatrix} . \begin{bmatrix} 2 - Bh^2 & -1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 1 & -h \end{bmatrix}$$

Or:

$$SR_h S^{-1} = \begin{bmatrix} 1 - Bh^2 & h \\ -Bh & 1 \end{bmatrix}$$

$$\|SR_h.S^{-1}\|_Y = \max\{|1 - Bh^2| + h, Bh + 1\} = Bh + 1$$

and from Eq. 25;

$$\|R_h^n\|_{Y_h} = \|SR_h^n S^{-1}\| = \left\|(S.R_h S^{-1})^n\right\| \le \|SR_h.S^{-1}\|^n \le$$
$$(1+Bh)^N = e^{\ln(1+Bh)^{\frac{1}{Bh} NhB}} \le e^B$$

because Nh = 1. From the Eq. 25:

$$z_n = R_h(R_h z_{n-2} + \rho_{n-2}h) + \rho_{n-1}h$$
$$= R_h^2 z_{n-2} + h(R_h.\rho_{n-2} + \rho_{n-1})$$
$$= R_h^n z_0 + (R_h^{n-1}\rho_0 + \ldots + \rho_{n-1})h$$

We define:

$$\|u\|_{Y_h} = \max_{0\le n\le N}|u_n| = \max_{0\le n\le N}|z_n|$$

And:

$$\|f\|_{Y_h} = \max\left\{u_0, \max_n \rho_n\right\}$$

Then:

$$\begin{aligned}
&\|u\|_{Y_h} = \max_n |z_n| \\
&= \max_n \left|R_h^n z_0 + h\left(R_h^{n-1}\rho_0 + R_h^{n-2}\rho_1 + \ldots + \rho_{n-1}\right)\right| \\
&\le \max_{0\le n\le N}\left|R_h^n\right|.\left(|z_0| + Nh\max_n|\rho|_n\right) \Rightarrow \qquad (26)\\
&\|u\|_{Y_h} \le \left\|R_h^n\right\|\left(\|z_0\| + \|\rho_n\|\right) \\
&\le \left\|R_h^n\right\|.2\|f_h\| \le 2e^B\|f_h\|
\end{aligned}$$

Because:

$$\|\rho_n\| \le \|f\|, \|z_0\| \le \|f\|$$

Therefore, there is a constant $C = 2.e^B$ such that is fulfilled and the difference scheme for Cauchy problem is stable.

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