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Probabilistic Periodic Review (Q_m, N) Backorders and Lost Sales Inventory Models under Constraint and Varying Holding Cost and Normally Distributed Protection Interval Demand

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Abstract: This study derives the probabilistic periodic review Backorders and Lost sales inventory models when the holding cost is a function of the inventory cycle. The expected total cost is composed of four components; the expected purchase cost, the expected ordering cost, the expected reviewing cost, the expected holding cost and the expected shortage cost. The objective is to minimize the expected annual total cost under a restriction on the expected annual reviewing cost when the protection interval demand follows the normal distribution. The Lagrangian multipliers are used to solve this constrained model in a closed form. Finally, some special cases are deduced and an illustrative numerical example is added with some graphs.

Key words: Probabilistic periodic review inventory system, protection interval demand, varying holding cost, review cost, Backorders inventory model, Lost sales inventory model, Lagrangian multiplier technique, Egypt

INTRODUCTION

Most of the literature dealing with probabilistic inventory models assumes that the demand is probabilistic since, the probability distribution of the future demand rather than the exact value of demand rate itself is known. Most of the probabilistic inventory models assume that the purchase cost of units is constant independent of the quantity ordered. Many researchers have studied unconstrained probabilistic inventory models assuming the holding cost to be constant independent of the number of periods. Hadley and Whitin (1963) and Taha (1997) have examined unconstrained probabilistic inventory models. Ben-Daya and Raouf (1994) examined unconstrained inventory model with constant units of cost, demand follows a normal distribution and the lead-time is one of the decision variables. Recently, Fergany and El-Saadani (2005) studied constrained probabilistic inventory model with continuous distributions and varying holding cost. Also, Abuo-El-Ata *et al*. (2003) has examined the constrained probabilistic multi-item inventory model.

Recently, El-Sodany (2011) has examined the periodic review probabilistic multi-item inventory system with zero lead time under constraint and varying holding cost. Liang *et al*. (2008) studied the periodic review inventory model with backorders and lost sales. Also, Lin (2008, 2010) studied the periodic review inventory models with backorders and variable lead time. The inventory model analyzed in this study assumes a periodic review system where demand is defined as a continuous random variable such that the inventory levels are reviewed at equal time intervals and orders are placed at such intervals, the quantity ordered each time depends on the available inventory level at the time of review.

One operating doctrine is that an order be placed at each review time if there have been any demands at all in the past period of time. A sufficient quantity is ordered to bring the inventory level up to level Q_m where, Q_m is the maximum inventory level after the arrival of the ordered quantity. The quantity ordered can vary from one review period to the next.

This operating doctrine will be called an order up to Q_m policy. Inventory models which use an order up to Q_m policy are referred to as (Q_m, N) inventory models where, N is the time between reviews. In this study, we investigate the constrained periodic review (Q_m, N) backorders and lost sales inventory models with varying holding cost.

Notations and assumptions: The following notations are adopted for developing the models:

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D	=	The expected annual demand rate
Q_m	=	The maximum inventory level
N	=	The inventory cycle
1/N	=	The average number of cycles per year
x	=	The continuous random variable represents the protection interval, L+ N, demand
f (x; L+N)	=	The probability density function of the protection interval demand x in the protection interval L+N
μ	=	The expected value protection interval demand
σ	=	The standard deviation of the protection interval demand per year
L	=	The lead time between the placement of an order and its receipt
c_r	=	The cost of making a review
c_o	=	The order cost per cycle
c_p	=	The purchase cost of the item
c_l	=	The cost of a lost sale per cycle
c_b	=	The cost of a backorder per cycle
c_h	=	The holding unit cost per year
$C_h(N) = c_h N^{\beta}$	=	The varying holding unit cost per year
$\bar{H}$	=	The average on hand inventory; min on hand+max on hand/2
β	=	A constant real number selected to provide the benefit of estimated cost function
K_r	=	The limitation on the expected annual review cost
E (PC)	=	The expected annual purchase cost
E (OC)	=	The expected annual ordering cost
E (HC)	=	The expected annual holding cost
E (BC)	=	The expected backorder cost
E (LC)	=	The expected lost sales cost
E (TC)	=	The expected annual total cost
Min E (TC)	=	The minimum expected annual total cost

The system is periodic review which means that the inventory levels are reviewed at equal time intervals and orders are placed at such intervals, the quantity ordered each time depends on the available inventory level at the time of review. An order is placed at each review time if there have been any demands at all in the past period of time, a sufficient quantity is ordered to bring the inventory level up to level Q_m. The problem is to determine the optimal values of Q_m and N which minimize the expected annual total cost. The following assumptions are made in the simple treatments for developing the mathematical model:

- The inventory cycle N is defined as the time between the placement of two successive orders
- The average number of cycles per year can be written as 1/N
- The lead time L is constant
- The purchase cost c_p of the item is constant independent of the quantity ordered
- There is never >1 order outstanding
- The cost c_r of making a review is independent of the variables Q_m and N
- The holding cost per unit is a varying function of the review time N. The varying holding unit cost per year takes the form:

$$C_h(N) = c_h N^{\beta}, \qquad 0.01 \le \beta \le 0.1$$

- The cost c_b of a backorder is independent of the time at which the backorder exist

Model I: Probabilistic periodic review (Q_m, N) backorders inventory models under constrained reviewing cost and varying holding cost: In this model, the demands occurring when the system is out of stock are backordered until a replenishment quantity arrives. Backorders imply that when an order arrives, it is always sufficient to meet any out standing backorders. The relevant annual expected total cost is the sum of the expected purchase cost, expected holding cost, expected backorder cost, expected review cost and the expected ordering cost that is:

$$E(TC) = E(PC)+E(RC)+E(OC)+E(HC)+E(BC)$$

Since, the purchase cost of the unit is constant independent of the quantity ordered; then the relevant annual expected total cost can be given by:

$$E(TC) = E(RC)+E(OC)+E(HC)+E(BC) \qquad (1)$$

The expected annual reviewing cost is given by:

$$E(RC) = c_r/N \qquad (2)$$

and the expected ordering cost is given by:

$$E(OC) = c_o/N \qquad (3)$$

The expected annual holding cost will be found by computing the expected holding cost per period and then multiplying by the number of orders per year. The expected holding cost per period $C_h(N)\,\bar{H}$ is where, $\bar{H}$ is the average on hand inventory per period given by:

$$\bar{H} = N\left(Q_m - DL - \frac{DN}{2}\right)$$

The expected holding cost per period:

$$C_h(N)\bar{H} = c_h N^{\beta+1}\left(Q_m - DL - \frac{DN}{2}\right)$$

Then, the expected annual holding cost is given by:

$$E(HC) = c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right) \quad (4)$$

The expected number of backorders incurred per year is the expected number of backorders incurred per period multiplied by the number of orders per year. Since, lead time L is constant then an order placed at time t will arrive in the system at time t+L and the next order quantity will arrive in the system at time t+L+N. After an order placed at time t, the inventory position of the system will be Q_m, hence to compute the expected number of backorders occurring between t+L and t+L+N, a backorder will occur in this period of time if and only if the protection interval demand in protection interval L+N exceeds Q_m. Then the expected number of backorders incurred per period is given by:

$$\int_{Q_m}^{\infty}(x - Q_m)f(x; L+N)dx \quad (5)$$

The expected number of backorders incurred per year is given by:

$$E(Q_m, N) = \frac{1}{N}\int_{Q_m}^{\infty}(x - Q_m)f(x; L+N)dx \quad (6)$$

Then, the expected annual cost of backorders is given by:

$$\begin{aligned} E(BC) &= c_b E(Q_m, N) \\ &= \frac{c_b}{N}\int_{Q_m}^{\infty}(x - Q_m)h(x; N)dx \end{aligned} \quad (7)$$

Then the expected annual total cost is given by:

$$\begin{aligned} E(TC) = {} & \frac{c_r}{N} + \frac{c_o}{N} + c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right) \\ & + \frac{c_b}{N}\int_{Q_m}^{\infty}(x - Q_m)f(x; L+N)dx \end{aligned} \quad (8)$$

The aim is to determine the optimal values of Q_m and N that minimize the expected annual total cost under the following constraint:

$$E(RC) \le K_r$$

In the study, we will consider Eq. 8 is a convex function. Let us write it in the following form:

$$\begin{aligned} E(TC) = {} & \frac{c_r}{N} + \frac{c_o}{N} + c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right) \\ & + \frac{c_b}{N}\int_{Q_m}^{\infty}(x - Q_m)f(x; L+N)dx \end{aligned} \quad (9)$$

Subject to:

$$\frac{c_r}{N} \le K_r \quad (10)$$

To find the optimal values Q^*_m and N* which minimize Eq. 9 under Eq. 10, we use the Lagrangian multiplier technique with the Kuhn-Tucker conditions as follows: The Lagrangian function is:

$$\begin{aligned} L_{(Q_m, N)} = E(TC) = {} & \frac{c_r}{N} + \frac{c_o}{N} + c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right) \\ & + \frac{c_b}{N}\int_{Q_m}^{\infty}(x - Q_m)f(x; L+N)dx + \lambda\left[\frac{c_r}{N} - K_r\right] \end{aligned} \quad (11)$$

where, λ is the Lagrangian multiplier. The optimal values Q^*_m and N^* are found by setting the corresponding 1st partial derivatives of $L_{(Qm, N)}$ equal to zero as follows:

$$\frac{\partial L_{(Q_m, N)}}{\partial Q_m}\Big|Q_m = Q^*_m, N = N^* = 0$$

Hence, the optimal maximum inventory level is the solution of the following equation:

$$\int_{Q^*_m}^{\infty} h(x; N^*)dx = \frac{c_h N^{*\beta+1}}{c_b} \quad (12)$$

also:

$$\frac{\partial L_{(Q_m, N)}}{\partial N}\Big|Q_m = Q^*_m, N = N^* = 0$$

Hence:

$$\begin{aligned} & c_b\int_{Q^*_m}^{\infty}(x - Q^*_m)f(x; L+N^*)dx + c_o + (1+\lambda)c_r \\ & = c_h\beta N^{*\beta+1}\left(Q^*_m - DL - \frac{DN^*}{2}\right) - \frac{c_h D N^{*\beta+2}}{2} \end{aligned} \quad (13)$$

Substituting from Eq. 12 into Eq. 13 then the optimal inventory cycle is the solution of the following equation:

$$c_b\left(\mu-\int_{-\infty}^{Q_m^*} xf\,(x;L+N^*)dx\right)-Q_m^* c_h N^{*\beta+1}+ \\ c_o+(1+\lambda)c_r= \\ c_h\beta N^{*\beta+1}\left(Q_m^*-DL-\frac{DN^*}{2}\right)-\frac{c_h DN^{*\beta+2}}{2} \tag{14}$$

However, $Q_m^*>0$ and $N^*>0$ minimize the expected annual total cost (Eq. 8) since;

$$\frac{\partial^2 E(TC)}{\partial Q_m^2}=\frac{c_b}{N}f\,(Q_m;L+N)>0 \tag{15}$$

$$\frac{\partial^2 E(TC)}{\partial N^2}=\frac{1}{N^3}\Big[2c_r+2c_o-c_h\beta N^{\beta+2}D+ \\ c_h\beta(\beta-1)N^{\beta+1}\left(Q_m-DL-\frac{DN}{2}\right)+ \\ 2c_b\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx\Big]>0 \tag{16}$$

$$\frac{\partial^2 E(TC)}{\partial N\partial Q_m}=\frac{\partial^2 E(TC)}{\partial Q_m\partial N}=c_h\beta N^{\beta-1}+\frac{c_b}{N^2} \\ \int_{Q_m}^{\infty} f(x;L+N)dx>0 \tag{17}$$

Therefore from the Hesian matrix, we have:

$$\Delta=\begin{vmatrix}\frac{\partial^2 E(TC)}{\partial Q_m^2} & \frac{\partial^2 E(TC)}{\partial Q_m\partial N}\\ \frac{\partial^2 E(TC)}{\partial N\partial Q_m} & \frac{\partial^2 E(TC)}{\partial N^2}\end{vmatrix} \\ =\frac{\partial^2 E(TC)}{\partial Q_m^2}\frac{\partial^2 E(TC)}{\partial N^2}-\left(\frac{\partial^2 E(TC)}{\partial Q_m\partial N}\right)^2>0 \tag{18}$$

Clearly, there is no closed form solution of Eq. 12 and 14. But by the following algorithm due to Hadley and Whitin (1963), we can obtain a closed approximate solution of these equations in a finite number of iterations as follows:

Step 1: Assume that the initial value of the inventory cycle N is any constant number then from Eq. 12, we have the initial maximum inventory level Q_{m1}.

Step 2: Substituting by Q_{m1} into Eq. 14 to get N_1.

Step 3: Substituting by N_1 into Eq. 12 to get Q_{m2}. The procedure is to vary λ in steps 2-3 until the smallest value of $\lambda>0$ is found such that the constraint holds for the different values of β.

Step 4: Repeat steps 2 and 3 until successive values of Q_m and N are sufficiently close which are the optimal values that gives the minimum annual expected total cost numerically.

Model I with normally distributed protection interval demand: Assume that the protection interval demand follows the Normal distribution with mean μ = D (L+N) and standard deviation $\sigma\sqrt{L+N}$. So, we can minimize the expected annual total cost mathematically as follows consider:

$$f\,(x;L+N)=\frac{1}{\sigma\sqrt{L+N}\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},\ 0<x<\infty \tag{19}$$

Then from Eq. 12, the optimal maximum inventory level is the solution of the following equation:

$$1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)=\frac{c_h N^{*\beta+1}}{c_b} \tag{20}$$

and also from Eq. 14, the optimal inventory cycle is the solution of the following equation:

$$c_b\left[\sigma\sqrt{L+N}\phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)+ (\mu-Q_m^*)\left(1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)\right)\right]+ \\ c_o+(1+\lambda)c_r=c_h\beta N^{*\beta+1} \\ \left(Q_m^*-DL-\frac{DN^*}{2}\right)-\frac{c_h D}{2}N^{*\beta+2} \tag{21}$$

and the minimum expected annual total cost is given by:

$$\min E(TC)=\frac{c_r}{N^*}+\frac{c_o}{N^*}+c_h N^{*\beta}\left(Q_m^*-DL-\frac{DN^*}{2}\right)+ \\ \frac{c_b}{N^*}\left[\sigma\sqrt{L+N}\phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)+ (\mu+Q_m^*)\left(1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)\right)\right] \tag{22}$$

Clearly, there is no closed form solution of Eq. 20 and 21 and we minimize the expected annual total cost numerically using the iteration method as the previous algorithm.

Model II: Probabilistic periodic review (Q_m, N) lost sales inventory models under constrained reviewing cost and varying holding cost: In the backorders case, it was assumed that all demands incurred when the system was out of stock were backordered but in the lost sales case a demand which occurs when the system is out of stock is lost forever.

We shall assume that the cost of a lost sale has the form c_l that the lost sale cost will be constant independent of time since a demand which occurs when the system is out of stock in the lost sales case is lost forever.

The relevant annual expected total cost is the sum of the expected purchase cost, expected review cost, expected ordering cost, expected holding cost and the expected lost sale cost that is:

$$E(TC) = E(PC)+E(RC)+E(OC)+E(HC)+E(LC)$$

Since, the purchase cost of the unit is constant independent of the quantity ordered then the relevant annual expected total cost can be given by:

$$E(TC) = E(RC)+E(OC)+E(HC)+E(LC) \quad (23)$$

The expected annual costs in the lost sales case are the same as the expected annual costs of the backorders case except the holding cost. For the lost sales case, the holding cost will be affected by the extra stock, since the demand which occurs when the system is out of stock is lost forever that there will be an increase in the holding cost.

The expected annual holding cost will be found by computing the expected holding cost per period and then multiplying by the number of orders per year. The expected holding cost per period is $c_h(N)\bar{H}$. Where, $\bar{H}$ is the average on hand inventory per period given by:

$$\bar{H}=N\left(Q_m - DL - \frac{DN}{2} + \int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx\right)$$

Then, the expected holding cost per period can be given by:

$$c_h N^{\beta+1}\left(Q_m - DL - \frac{DN}{2} + \int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx\right)$$

The expected annual holding cost is given by:

$$E(HC)=c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right) + c_h N^{\beta}\left(\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx\right) \quad (24)$$

The expected annual lost sales cost is given by:

$$E(LC)=\frac{c_l}{N}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx \quad (25)$$

Then the expected annual total cost is given by:

$$E(TC)=\frac{c_r}{N}+\frac{c_o}{N}+c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right)+ c_h N^{\beta}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx + \frac{c_l}{N}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx \quad (26)$$

The aim is to determine the optimal values of Q_m and N that minimize the expected annual total cost under the following constraint:

$$E(RC)\le k_r$$

In the study, we will consider Eq. 26 is a convex function. Let us write it in the following form:

$$E(TC)=\frac{c_r}{N}+\frac{c_o}{N}+c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right)+ c_h N^{\beta}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx + \frac{c_l}{N}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx \quad (27)$$

Subject to: $$\frac{c_r}{N}\le K_r \quad (28)$$

To find the optimal values Q^*_m and N^* which minimize Eq. 27 under Eq. 28, we use the Lagrangian multiplier technique with the Kuhn-Tucker conditions as follows: The Lagrangian function is:

$$L_{(Q_m,N)}=\frac{c_r}{N}+\frac{c_o}{N}+c_h N^{\beta}\left(Q_m - DL - \frac{DN}{2}\right)+ c_h N^{\beta}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx + \frac{c_l}{N}\int_{Q_m}^{\infty}(x-Q_m)f(x;L+N)dx + \lambda\left[\frac{c_r}{N}-K_r\right] \quad (29)$$

The optimal values: Q^*_m and N^* are found by setting the corresponding 1st partial derivatives of $L_{(Qm,\ N)}$ equal to zero as follows:

$$\frac{\partial L_{(Q_m,N)}}{\partial Q_m}\Big| Q_m = Q_m^*, N = N^* = 0$$

Hence, the optimal maximum inventory level is the solution of the following equation:

$$\int_{Q_m^*}^{\infty} f(x; L + N^*)dx = \frac{c_h N^{*\beta+1}}{c_I + c_h N^{*\beta+1}} \qquad (30)$$

also:

$$\frac{\partial L_{(Q_m,\ N)}}{\partial N}\Big| Q_m = Q_m^*,\ N = N^* = 0$$

Hence, the optimal inventory cycle is the solution of the following equation:

$$\begin{aligned}&(c_I - c_h\beta N^{*\beta+1}) \int_{Q_m^*}^{\infty} (x - Q_m^*) f(x; L + N^*)\\ &dx + c_o + (1+\lambda)c_r\\ &= c_h\beta N^{*\beta+1}\left(Q_m^* - DL - \frac{DN^*}{2}\right) - \frac{c_h D}{2} N^{*\beta+2}\end{aligned} \qquad (31)$$

However, $Q^*_m > 0$ and $N^* > 0$ minimize the expected annual total cost (Eq. 27) since:

$$\frac{\partial^2 E(TC)}{\partial Q_m^2} = (c_h N^\beta + \frac{c_I}{N}) h(Q_m; N) > 0 \qquad (32)$$

$$\begin{aligned}\frac{\partial^2 E(TC)}{\partial N^2} = \frac{1}{N^3}\Bigg[&2\left(c_r + c_o + c_I\int_{Q_m}^{\infty}(x - Q_m)h(x;N)dx\right) +\\ &c_h\beta(\beta-1)N^{\beta+1}\left(\begin{matrix}Q_m - \mu - \frac{DN}{2} + \\ \int_{Q_m}^{\infty}(x - Q_m)h(x;N)dx\end{matrix}\right) -\\ &\frac{c_h\beta D N^{\beta+2}}{2}\Bigg] > 0\end{aligned} \qquad (33)$$

$$\begin{aligned}\frac{\partial^2 E(TC)}{\partial N \partial Q_m} &= \frac{\partial^2 E(TC)}{\partial Q_m \partial N} = \left(c_h\beta N^{\beta-1} - \frac{c_I}{N^2}\right)\\ &\int_{-\infty}^{Q_m} h(x;\ N)dx + \frac{c_I}{N^2} > 0\end{aligned} \qquad (34)$$

Therefore from the Hesian matrix, we have:

$$\Delta = \begin{vmatrix}\frac{\partial^2 E(TC)}{\partial Q_m^2} & \frac{\partial^2 E(TC)}{\partial Q_m \partial N}\\ \frac{\partial^2 E(TC)}{\partial N \partial Q_m} & \frac{\partial^2 E(TC)}{\partial N^2}\end{vmatrix}$$

$$= \frac{\partial^2 E(TC)}{\partial Q_m^2}\frac{\partial^2 E(TC)}{\partial N^2} - \left(\frac{\partial^2 E(TC)}{\partial Q_m \partial N}\right)^2 > 0 \qquad (35)$$

Clearly, there is no closed form solution of Eq. 30 and 31 and we minimize the expected annual total cost numerically using the iteration method as the algorithm used in model I.

Model II with normally distributed protection interval demand: Assume that the protection interval demand follows the Normal distribution with mean $\mu = D(L+N)$ and standard deviation $\sigma\sqrt{L+N}$ with the probability density function given by Eq.19. Then from Eq. 30, the optimal maximum inventory level is the solution of the following equation:

$$1 - \Phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right) = \frac{c_h N^{*\beta+1}}{c_I + c_h N^{*\beta+1}} \qquad (36)$$

and from Eq. 31, the optimal inventory cycle is the solution of the following equation:

$$\begin{aligned}&(c_I - c_h\beta N^{*\beta+1})\left[\begin{matrix}\sigma\sqrt{L+N}\phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right) + \\ (\mu - Q_m^*)\left(1 - \Phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right)\right)\end{matrix}\right] +\\ &c_o + (1+\lambda)c_r = c_h\beta N^{*\beta+1}\left(Q_m^* - \mu - \frac{DN^*}{2}\right) - \frac{c_h D}{2} N^{*\beta+2}\end{aligned} \qquad (37)$$

and the minimum annual expected total cost is given by:

$$\begin{aligned}\min E(TC) = &\frac{c_r}{N^*} + \frac{c_o}{N^*} +\\ &c_h N^{*\beta}\left(Q_m^* - \mu - \frac{DN^*}{2}\right) +\\ &\left(c_h N^{*\beta} + \frac{c_I}{N^*}\right)\\ &\left[\begin{matrix}\sigma\sqrt{L+N}\phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right) + \\ (\mu - Q_m^*)\left(1 - \Phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right)\right)\end{matrix}\right]\end{aligned} \qquad (38)$$

Clearly, there is no closed form solution of Eq. 36 and 37 and we minimize the expected annual total cost numerically using the iteration method as the algorithm used in model I.

Special cases: We deduce two special cases of the models as follows:

Case 1: For model I, assume that:

$$\beta = 0 \text{ and } K_r \to \infty \Rightarrow C_h(N) = c_h \text{ and } \lambda = 0$$

Thus from Eq. 20, the following expression for the optimal maximum inventory level can be obtained:

$$1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)=\frac{c_h N^*}{c_b}$$

Also from Eq. 21, the following expression for the optimal inventory cycle can be obtained:

$$c_b\left[\sigma\sqrt{L+N}\phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)+\left(\mu-Q_m^*\right)\left(1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)\right)\right]+$$
$$c_o+c_r=\frac{c_h D}{2}N^{*2}$$

This is unconstrained probabilistic periodic review (Q_m, N) backorders inventory model with protection interval demand follows the normal distribution and the holding unit cost is constant.

Case 2: For model II, assume that:

$$\beta = 0 \text{ and } K_r \to \infty \Rightarrow C_h(N) = c_h \text{ and } \lambda = 0$$

Thus from Eq. 36, the following expression for the optimal maximum inventory level can be obtained:

$$1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)=\frac{c_h N^*}{c_l + c_h N^*}$$

Also from Eq. 37, the following expression for the optimal inventory cycle can be obtained:

$$c_l\left[\begin{array}{l}\sigma\sqrt{L+N}\phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)+\\ (\mu-Q_m^*)\left(1-\Phi\left(\frac{Q_m^*-\mu}{\sigma\sqrt{L+N}}\right)\right)\end{array}\right]+$$
$$c_0+c_r=-\frac{c_h D}{2}N^{*2}$$

This is unconstrained probabilistic periodic review (Q_m, N) lost sales inventory model with protection interval demand follows the normal distribution and the holding unit cost is constant.

An illustrative example: A large California warehouse follows a policy of reviewing all items over fixed periods of time. It uses an order up to Q_m policy. Consider one item, it carries say a particular type of tractor tire. The mean demand rate has been constant over time at the value of 600 year^{-1}.

The warehouse orders tires directly from the manufacturer and the lead time L is nearly constant and has the value 6 months.

The demand in the protection interval L+N can be represented quite well by a normal distribution with mean 450 unit and standard deviation 25.981 unit. The warehouse uses an inventory holding cost is \$3. The ordering cost is \$13 and the reviewing cost is \$12. The researchers will determine the optimal maximum inventory level and the optimal inventory cycle that minimize the expected annual total cost in the following cases:

- All demands occurring when the system is out of stock are backordered and the cost of a backorder is \$25, under the expected review cost limitation K_r = \$44.5
- All demands occurring when the system is out of stock are lost for ever and the cost of a lost sale is \$25, under the expected review cost limitation K_r = \$44.3

Solution: Using the results of the models, the optimal values and the minimum annual expected total cost for the periodic review (Q_m, N) inventory models with varying holding cost can be shown in Table 1 and 2. And the

Table 1: The optimal solutions and the min E (TC) for the backorders periodic review (Q_m, N) inventory model when the protection interval demand follows the normal distribution

β	Q_m	N	Min E (TC)
0.01	498.097	0.271283	467.619
0.02	498.218	0.271653	462.833
0.03	498.358	0.271686	458.523
0.04	498.511	0.271563	454.313
0.05	498.722	0.270638	450.911
0.06	498.925	0.269719	447.579
0.07	499.006	0.271200	441.606
0.08	499.206	0.269878	439.064
0.09	499.303	0.271969	428.543
0.10	499.403	0.271969	428.543

Table 2: The optimal solutions and the min E (TC) for the lost sales periodic review (Q_m, N) inventory model when the protection interval demand follows the normal distribution

β	Q_m	N	Min E (TC)
0.01	494.414	0.271164	469.550
0.02	498.583	0.271155	464.566
0.03	498.755	0.271147	459.830
0.04	498.917	0.270540	455.575
0.05	499.052	0.271166	451.025
0.06	499.189	0.271112	446.862
0.07	499.327	0.271085	442.739
0.08	499.487	0.270938	438.608
0.09	499.630	0.271165	434.087
0.10	499.767	0.271023	430.295

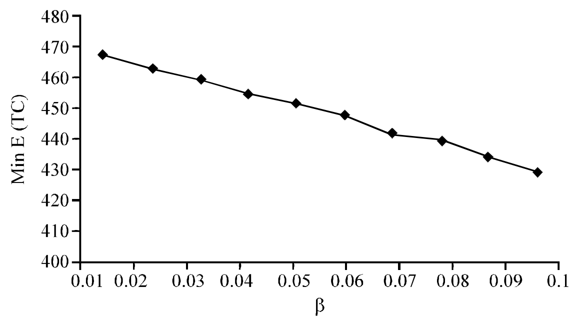

Fig. 1: The values of min E (TC) for the backorders periodic review (Q_m, N) inventory model when the protection interval demand follows the normal distribution at each value of β

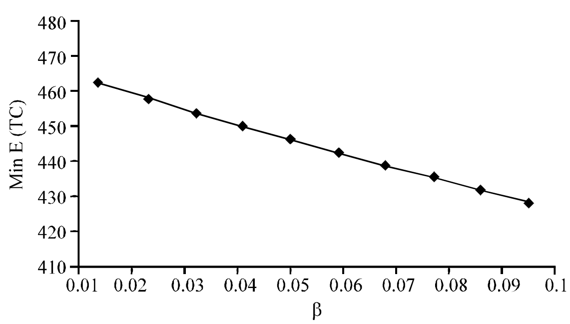

Fig. 2: The values of min E (TC) for the lost sales periodic review (Q_m, N) inventory model when the protection interval demand follows the normal distribution at each value of β

solution of the problem may be determined more readily by plotting min E (TC) against β for the two inventory shows as in Fig. 1 and 2.

MATERIALS AND METHODS

The aim of this study is to investigate probabilistic periodic review (Q_m, N) backorders inventory models and probabilistic periodic review (Q_m, N) lost sales inventory models under the expected reviewing cost constraint when the holding cost is a varying function of the inventory cycle.

The lagrangian multiplier and probabilistic periodic review (Q_m, N) lost sales inventory models under the expected reviewing cost constraint when the holding cost is a varying function of the inventory cycle. The lagrangian multiplier approach is used to determine the optimal inventory cycle and the optimal maximum inventory level which minimize the expected annual total cost under the expected review cost constraint.

RESULTS AND DISCUSSION

The basic results of this study are: The minimum expected annual total cost of the probabilistic periodic review (Q_m, N) backorders inventory model under constrained reviewing cost and varying holding cost when the protection interval demand follows the normal distribution is given by:

$$\min E(TC) = \frac{c_r}{N^*} + \frac{c_o}{N^*} + c_h N^{*\beta}\left(Q_m^* - DL - \frac{DN^*}{2}\right) + \frac{c_b}{N^*}\left[\sigma\sqrt{L+N}\,\phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right) + \left(\mu - Q_m^*\right)\left(1 - \Phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right)\right)\right]$$

The minimum expected annual total cost of the probabilistic periodic review (Q_m, N) lost sales inventory model under constrained reviewing cost and varying holding cost when the protection interval demand follows the normal distribution is given by:

$$\min E(TC) = \frac{c_r}{N^*} + \frac{c_o}{N^*} + c_h N^{*\beta}\left(Q_m^* - \mu - \frac{DN^*}{2}\right) + \left(c_h N^{*\beta} + \frac{c_l}{N^*}\right)\left[\sigma\sqrt{L+N}\,\phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right) + \left(\mu - Q_m^*\right)\left(1 - \Phi\left(\frac{Q_m^* - \mu}{\sigma\sqrt{L+N}}\right)\right)\right]$$

At the end of this study, special cases of previously research are added. Also, a numerical illustrative example is added with some graphs by using Mathematica program.

CONCLUSION

We have evaluated the optimal inventory cycle and the optimal maximum inventory level; then we deduced the minimum annual expected total cost min E (TC) of the considered backorders periodic review inventory model and lost sales periodic review inventory model. We draw the curves min E (TC) against β for both models which indicate the value of β that gives the minimum value of the expected annual total cost of our numerical example.

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