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On the Convoluted Beta-Exponential Distribution

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Abstract: Many useful properties of statistical distribution are revealed by transformation of random variables, however not many of the logic of beta distribution have been investigated by convolution techniques. This study investigates the statistical properties of the beta-exponential distribution defined by Nadarajah and Kotz. Specifically, it studies the distribution of the sum of two random variables from beta-exponential distribution using the convolution method. The probability density function (pdf) and the cumulative distribution (cdf) of the convoluted distribution were obtained. Also, derived are various statistical properties of the distribution which include moment, moment and characteristic generating function, skewness and kurtosis, hazard function and the entropy. The parameters of the distribution were estimated using the maximum likelihood method. The convoluted random variable was found to be unimodal and leptokurtic which makes it a more powerful distribution for analysis of financial data. The hazard function behaves in much the same way as that of Convoluted Beta-Weibull Distribution (CBWD).

Key words: Convolution, beta distribution, exponential distribution, beta-exponential, Nigeria

INTRODUCTION

Modelling of interrelationship among naturally occurring phenomena is made possible by the use of distribution function and their properties. Thus a number of statistical distributions function and their generalization have been proposed and defined in the literature, most of which are found to be useful in studying naturally occurring phenomena particularly where the variables assure values between 0 and 1.

Various methods exist in defining statistical distribution. Some arise from modelling naturally occurring events while some arise as functions of one or more distributions. Sun (2011) give example of a random variable T which is said to have a t-distribution if $T = Z/\sqrt{W/N}$ where t has the standard normal distribution and W has the Chi-squared (χ^2) distribution with n degrees of freedom. Many useful properties of such distribution are revealed by transformation of random variables while another approach is the convolution technique which is rarely used. Among the useful distributions applicable to real life data and the logit of beta distribution are the beta-pareto (Akinsete and Lowe, 2008); beta-laplace (Kozubowski and Nadarajah, 2008); beta-weibull (Famoye *et al*., 2005); beta-normal (Eugene *et al*., 2002); beta-exponential (Nadarajah and Kotz, 2006) and beta-Gumbel (Nadarajah and Kotz, 2004).

This study focuses on investigating the statistical distribution properties of the Convoluted Beta-Exponential Distribution (CBED). Beta exponential distribution was defined by Nadarajah and Kotz (2006). Suppose X_1 is a random variable having the beta-exponential distribution with parameters a_1, b_1, λ_1, i.e., $X_1 \sim BE(a_1, b_1, \lambda_1)$ and X_2 has a beta-exponential distribution with parameters a_2, b_2 and λ_2, i.e., $X_2 \sim BE(a_2, b_2, \lambda_2)$ then it is desired to obtained the distribution of the of the convolution of X_1 and X_2. That is the distribution of the random variables $Z = X_1+X_2$. Researchers obtained the probability density function (pdf) and the cumulative distribution function (cdf) of the convoluted distribution. The statistical properties of the distribution that are obtained include moment, moment and characteristic generating function, hazard function. Following Sun (2011), the parameters of the distribution are obtained by the method of Maximum Likelihood Estimation (MLE).

LITERATURE REVIEW

The beta distribution is a widely applied statistical distribution in the literature. The beta distribution is used to model various kinds of data in life testing, survival analysis, telecommunication and in the study of uncertainties arising in actuarial science, economics and finance (Nadarajah, 2005). The derivation of the beta distribution as a ratio of two independent and identically distributed random variable is standard results in the literature (Akinsete, 2008). If X is a beta random variable, the probability density function (pdf) of X is given by:

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$$f(x)=\frac{\sqrt{(\alpha+\beta)}}{\sqrt{(\alpha)}\ \sqrt{(\beta)}}x^{\alpha-1}(1-x)^{\beta-1};\ 0<x<1;\ \alpha>0;\ \beta>0 \quad (1)$$

Many distributions have been generalized in recent years by either compounding them with beta distribution or as logit of the beta distribution. From Sun (2011), the cumulative distribution function (cdf) for the generalize class of distribution for the random variable X is generated by applying the inverse cdf of X to a beta distributed random variable to obtain:

$$G(x)=\frac{1}{\beta(a,\beta)}\int_0^{F(x)}t^{a-1}(1-t)^{\beta-1}dt;\ 0<\alpha,\ 0<\beta \quad (2)$$

The corresponding probability density function (pdf) from G (x) is given by:

$$g(x)=\frac{1}{B(\alpha,\beta)}\left[F(x)\right]^{\alpha-1}\left[1-F(x)\right]^{\beta-1}F^1(x) \quad (3)$$

where $F^1(x) = f(x)$ is the pdf of X

The Beta-Exponential Distribution (BED): The exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability studies. The beta-exponential distribution defined and studied by Nadarajah and Kotz (2006) is generated from the logit of a beta random variable. In the study, the researchers provide a comprehensive treatment of statistical properties of the beta-exponential distribution. The study also discusses and derives expressions for the moment generating function, characteristic function, the first four moments, variance, skewness, kurtosis, mean deviation about the mean, mean deviation about the median, Renyi entropy and the Shannon entropy.

This study proposes a generalization of the exponential distribution with the hope that it would attract wider applications in reliability. The generalization is motivated by the following general class.

If G denotes the cdf of a random variable then a generalized class of distribution can be defined by:

$$F(x)=1_{G(x)}(a,b);a>0 \text{ and } b>0 \quad (4)$$

Where:

$$1_y(a,b)=\frac{B_y(a,b)}{B(a,b)}$$

Denotes the incomplete beta function ratio and:

$$B_y(a,b)=\int_0^y w^{a-1}(1-w)^{b-1}dw \quad (5)$$

Denotes the incomplete beta function. The researcher defined the beta-exponential distribution by taking G to the cdf of an exponential distribution with parameter λ. The cdf of beta-exponential distribution then becomes:

$$F(x)=1_{1-\exp(-\lambda x)}(a,b);x>0,\ a>0,\ b>0,\ \lambda>0 \quad (6)$$

and the corresponding pdf as obtained by Nadarajah and Kotz (2006) is:

$$f(x)=\frac{\lambda}{B(a,b)}\exp(-b\lambda x)\ (1\text{-}\exp(\text{-}\lambda x))^{a\text{-}1};\quad x>0,\ a>0,\ b>0,\ \lambda>0 \quad (7)$$

This distribution is the generalization of the exponentiated exponential distribution defined by Gupta and Kundu (2003) when $b = 1$. The beta-exponential distribution reduces to the exponential distribution with parameters $b\lambda$ and $a = 1$.

Besides its mathematical simplicity when compared to other beta compounded distributions, the beta-exponential distribution can be used as an improved model for failure time data. The distribution exhibits both increasing and decreasing failure rates and the shape of the failure rate function depends on the parameter a.

THE CONVOLUTED BETA-EXPONENTIAL DISTRIBUTION (CBED)

Density and distribution function: According to Nadarajah and Kotz (2006), the beta-exponential random variable X is defined as:

$$f(x)=\frac{\lambda}{B(a,b)}e^{-b\lambda x}\left[1-e^{-\lambda x}\right]^{a-1},x>0,\ a>0,\ b>0,\ \lambda>0 \quad (8)$$

Let, X_1 be distributed as beta-exponential distribution with parameters a_1, b_1, λ_1, i.e., $X \approx BEP(a_1, b_1, \lambda_1)$ and let X_2 have beta-exponential distribution with parameters a_2, b_2, λ_2, i.e., $X \approx BEP(a_2, b_2, \lambda_2)$ then we seek the distribution of $Z = X_1 + X_2$ for $X_1 \approx BEP(a_1, b_1, \lambda_1)$:

$$f(x_1)=\frac{\lambda_1}{B(a_1,b_1)}e^{-b_1\lambda_1x_1}\left[1-e^{-\lambda_1x_1}\right]^{a_1-1} \quad (9)$$

For, $X_2 \approx BEP(a_2, b_2, \lambda_2)$:

$$f(x_2)=\frac{\lambda_2}{B(a_2,b_2)}e^{-b_2\lambda_2x_2}\left[1-e^{-\lambda_2x_2}\right]^{a_2-1} \quad (10)$$

Since, X_1 and X_2 are stochastically independent, the joint distribution of X_1 and X_2 are expressed as:

$$f(x_1x_2) = \frac{\lambda_1\lambda_2 e^{-(b_1\lambda_1x_1+b_2\lambda_2x_2)}\left(1-e^{-\lambda_1x_1}\right)^{a_1-1}\left(1-e^{-\lambda_2x_2}\right)^{a_2-1}}{B(a_1,b_1)B(a_2,b_2)} \quad (11)$$

For simplicity we let $a_1 = a_2 = 1$. Therefore:

$$f(x_1x_2) = \lambda_1\lambda_2 b_1b_2 e^{-[b_1\lambda_1x_1+b_2\lambda_2x_2]}$$

By using the concept of convolution of the two random variables, researchers may write the pdf of Z as:

$$\begin{aligned}
f(z) &= \int_0^Z f_x(Z-y)\, f_y(y)\, dy \\
&= \int_0^Z b_1b_2\, \lambda_1\lambda_2 e^{-[b_1\lambda_1(z-y)]} e^{-b_2\lambda_2 y} dy \\
&= b_1b_2\lambda_1\lambda_2\ e^{-b_1\lambda_1 z}\int_0^Z e^{-y(b_2\lambda_2-b_1\lambda_1)} \\
&\Rightarrow \frac{b_1b_2\ \lambda_1\lambda_2 e^{-b_1\lambda_1 z}}{b_1\lambda_1-b_2\lambda_2} e^{-y(b_2\lambda_2-b_1\lambda_1)}\Big|_0^Z \\
&\Rightarrow \frac{b_1b_2\ \lambda_1\lambda_2 e^{-b_1\lambda_1 z}}{b_2\lambda_2-b_1\lambda_1}\left[1-e^{-(b_2\lambda_2-b_1\lambda_1)Z}\right] \\
&= \frac{(b_1b_2\lambda_1\lambda_2)}{(b_2\lambda_2 - b_1\lambda_1)} e^{-b_2\lambda_{21}Z}\left[1-e^{-Z(b_1\lambda_1-b_2\lambda_2)}\right]; \\
& b_1>0,\ b_2>0,\ \lambda>0,\ Z>0
\end{aligned} \quad (12)$$

The above is the pdf of the convoluted beta-exponential distribution of the random variable Z. To show that f (z) is a pdf, researchers require that $\int_0^\infty f(z)dz = 1$:

$$\begin{aligned}
\int_0^\infty f_{(z)}d_Z &= \int_0^\infty \frac{b_1b_2\lambda_1\lambda_2\ e^{-b_2\lambda_2 Z}}{(b_1\lambda_1-b_2\lambda_2)}\left[1-e^{-Z(b_1\lambda_1-b_2\lambda_2)}\right]dZ \\
& \frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\int_0^\infty e^{-b_2\lambda_2 Z}\left[1-e^{-Z(b_1\lambda_1-b_2\lambda_2)}\right]dZ \\
& \lim_{t\to\infty}\frac{b_1b_2\ \lambda_1\lambda_2}{b_1\lambda_1-b_2\lambda_2}\int_0^\infty\left(\ell^{-b_2\lambda_2 Z}-\ell^{-b_1\lambda_1 Z}\right)dZ \\
& \lim_{t\to\infty}\frac{b_1b_2\ \lambda_1\lambda_2}{b_1\lambda_1-b_2\lambda_2}\left[\frac{\ell^{-b_2\lambda_2 Z}}{-b_2\lambda_2}+\frac{\ell^{-b_1\lambda_1 Z}}{-b_1\lambda_1}\right] \\
& \lim_{t\to\infty}\frac{b_1b_2\ \lambda_1\lambda_2}{b_1\lambda_1-b_2\lambda_2}\left[0-0-\left(\frac{1}{b_1\lambda_1}-\frac{1}{b_2\lambda_2}\right)\right] \\
& \frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\quad\frac{(b_1\lambda_1-b_2\lambda_2)}{b_2\lambda_2\ b_1\lambda_1} = 1
\end{aligned} \quad (13)$$

This shows that the proposed convoluted beta-exponential distribution is a real pdf (Fig. 1). The corresponding cdf of the convoluted beta-exponential distribution can be obtained using:

$$\begin{aligned}
F(Z) &= P(Z\le Z),\ \int_0^Z f(t)dt \\
&= \int_0^Z \frac{b_1b_2\ \lambda_1\lambda_2\ \ell^{-b_2\lambda_2 t}}{b_1\lambda_1 - b_2\lambda_2}\left[1-\ell^{-(b_1\lambda_1-b_2\lambda_2)t}\right]dt \\
&\Rightarrow \frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1 - b_2\lambda_2)}\int_0^Z\left(\ell^{-b_2\lambda_2 t}-\ell^{-b_1\lambda_1 t}\right)dt \\
&\Rightarrow \frac{b_1b_2\ \lambda_1\lambda_2}{b_1\lambda_1-b_2\lambda_2}\left[\frac{\ell^{-b_2\lambda_2 t}}{-b_2\lambda_2}+\frac{\ell^{-b_1\lambda_1 t}}{b_1\lambda_1}\right]_0^Z \\
&= \frac{b_1b_2\ \lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2}\left[\frac{\ell^{-b_2\lambda_2 Z}}{-b_2\lambda_2}+\frac{\ell^{-b_1\lambda_1 Z}}{b_1\lambda_1}+\frac{1}{b_2\lambda_2}-\frac{1}{b_1\lambda_1}\right] \\
&= \frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1 - b_2\lambda_2)}\left[\frac{1}{b_2\lambda_2}\left(1-\ell^{-b_2\lambda_2 Z}\right)-\frac{1}{b_1\lambda_1}\left(1-\ell^{-b_1\lambda_1 Z}\right)\right]
\end{aligned} \quad (14)$$

Researchers see immediately that:

$$\begin{aligned}
& \lim_{Z\to 0} F(Z) = 0 \ \text{and}\ \lim_{Z\to\infty} F(Z) = 1 \\
& \lim_{Z\to\infty}\frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1 - b_2\lambda_2)}\left[\frac{1}{b_2\lambda_2}\left(1-\ell^{-\infty}\right)-\frac{1}{b_1\lambda_1}\left(1-\ell^{-\infty}\right)\right] \\
& \frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1 - b_2\lambda_2)}\left[\frac{1}{b_2\lambda_2}-\frac{1}{b_1\lambda_1}\right] \\
& \frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1 - b_2\lambda_2)}\quad\frac{(b_1\lambda_1-b_2\lambda_2)}{b_1\lambda_2\ b_2\lambda_2} = 1 \\
& \lim_{Z\to\infty} F(Z) = 1
\end{aligned} \quad (15)$$

Shape of the PDF: The shape of the convoluted beta-exponential distribution is investigated as:

$$f(z) = \frac{b_1b_2\ \lambda_1\lambda_2\ \ell^{b_2\lambda_2 Z}\left[1-\ell^{-(b_1\lambda_1-b_2\lambda_2)Z}\right]}{(b_1\lambda_1 - b_2\lambda_2)} \quad (16)$$

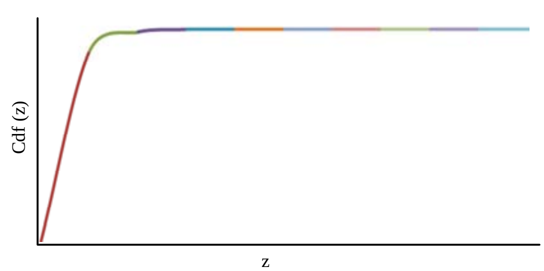

Fig. 1: The graph of cdf ($b_1 = 2$, $b_2 = 3$ and $\lambda_1 = 2$, $\lambda_2 = 4$)

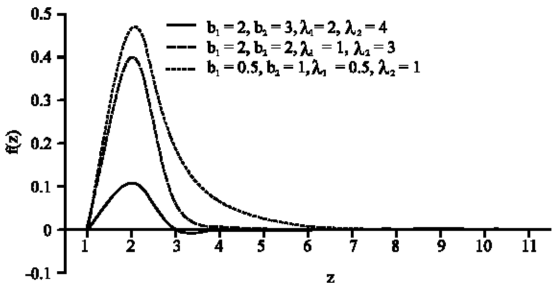

Fig. 2: The graph of pdf for different parameter values

By differentiating the above with respect to Z and set equation to zero, we have:

$$f'(z)=\frac{b_1b_2\ \lambda_1\lambda_2\ \left[(-b_2\lambda_2)\ell^{-b_2\lambda_2 Z}+b_1\lambda_1\ell^{-b_1\lambda_1 Z}\right]}{(b_1\lambda_1-b_2\lambda_2)}$$

$$f'(z)=0$$

$$\frac{b_1\lambda_1\ b_2\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\left[b_1\lambda_1\ell^{-b_1\lambda_1 Z}-b_2\lambda_2\ell^{-b_2\lambda_2 Z}\right]=0$$

Solving for Z, to give:

$$Z=\frac{1}{b_1\lambda_1-b_2\lambda_2}\ \left[\mathrm{In}\, b_1\lambda_1-\mathrm{In}\, b_2\lambda_2\right] \tag{17}$$

where, $b_1\lambda_1 \neq b_2\lambda_2$, $Z>0$. This shows that the distribution is unimodal. It can be observed that the smaller the value of b_1, b_2, the thicker the tail, respectively and the smaller the values of λ_1 and λ_2, the longer the peak (Fig. 2).

Hazard function: This is given by:

$$h(z)=\frac{f(z)}{1-F(z)}$$

$$f(z)=\frac{b_1b_2\ \lambda_1\lambda_2\ \ell^{-b_2\lambda_2 Z}\ \left[1-\ell^{-(b_1\lambda_1-b_2\lambda_2)Z}\right]}{(b_1\lambda_1\ \text{-}b_2\lambda_2)}$$

$$F(z)=\frac{b_1\lambda_1\ \left(1-\ell^{-b_2\lambda_2 Z}\right)\text{-}\,b_2\ \lambda_2\left(1-\ell^{-b_1\lambda_1 Z}\right)}{(b_1\lambda_1-b_2\lambda_2)} \tag{18}$$

$$h(z)=\frac{\dfrac{b_1b_2\lambda_1\lambda_2\ \ell^{-b_2\lambda_2 Z}\left[1-\ell^{-(b_1\lambda_1-b_2\lambda_2)Z}\right]}{(b_1\lambda_1-b_2\lambda_2)}}{1-\left[\dfrac{b_1\lambda_1\left(1-\ell^{-b_2\lambda_2 Z}\right)-b_2\lambda_2\left(1-\ell^{-b_1\lambda_1 Z}\right)}{(b_1\lambda_1-b_2\lambda_2)}\right]}$$

$$h(z)=\frac{b_1b_2\lambda_1\lambda_2\left[\ell^{-b_2\lambda_2 Z}-\ell^{-b_1\lambda_1 Z}\right]}{\left[b_1\lambda_1\ \ell^{-b_2\lambda_2 Z}-b_2\lambda_2\ell^{-b_1\lambda_1 Z}\right]}$$

The behaviour of the hazard function as Z approaches zero and as it approaches infinity is as:

$$\lim_{Z\to 0} h(z)=\frac{b_1b_2\ \lambda_1\lambda_2\ \left[\ell^{-b_2\lambda_2(0)}-\ell^{-b_1\lambda_1(0)}\right]}{b_1\lambda_1\ell^{-b_2\lambda_2(0)}-b_2\lambda_2\ell^{-b_1\lambda_1(0)}}=0$$

$$\lim_{Z\to\infty} h(z)=\lim_{Z\to\infty}\frac{b_1b_2\lambda_1\lambda_2\ \left[\ell^{-b_2\lambda_2 Z}-\ell^{-b_1\lambda_1 Z}\right]}{b_1\lambda_1\ell^{-b_2\lambda_2 Z}-b_2\lambda_2\ell^{-b_1\lambda_1 Z}}=b_2\lambda_2$$

$$h(z)\quad =\begin{cases} b_2\lambda_2 & \text{if} \quad b_2\lambda_2<b_1\lambda_1\\ b_1\lambda_1 & \text{if} \quad b_2\lambda_2>b_1\lambda_1\end{cases} \tag{19}$$

Moments and moment generating function: The moment generating function of a random variable Z is defined as:

$$M_z(t)=E\left(\ell^{tZ}\right)$$

$$M_z(t)=\frac{b_1b_2\lambda_1\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\int_0^\infty \ell^{tz}\ell^{-b_2\lambda_2 Z}\left[1-\ell^{-(b_1\lambda_1-b_2\lambda_2 Z}\right]dZ$$

$$=\frac{b_1b_2\lambda_1\lambda_2}{b_1\lambda_2-b_2\lambda_2}\int_0^\infty \ell^{-(b_2\lambda_2-t)Z}-\left[\ell^{-(b_1\lambda_1-b_2\lambda_2)Z}\right]dZ$$

$$=\frac{b_1b_2\lambda_1\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\int_0^\infty \ell^{-(b_2\lambda_2-t)Z}-\ell^{-(b_1\lambda_1-t)Z}dZ$$

$$=\frac{b_1b_2\lambda_1\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\left[\frac{\ell^{-(b_2\lambda_2-t)Z}}{-(b_2\lambda_2-t)}+\frac{\ell^{-(b_1\lambda_1-t)Z}}{-(b_1\lambda_1-t)}\right]_0^\infty$$

$$=\frac{b_1b_2\lambda_1\lambda_2}{(b_1\lambda_1-b_2\lambda_2)}\left[\begin{array}{c}\dfrac{0}{-(b_2\lambda_2-t)}+\dfrac{0}{-(b_1\lambda_1-t)}-\\ \left(\dfrac{0}{-(b_2\lambda_2-t)}+\dfrac{0}{-(b_1\lambda_1-t)}\right)\end{array}\right]$$

$$\Rightarrow\frac{b_1b_2\lambda_1\lambda_2}{b_1\lambda_2-b_2\lambda_2}\ \left[\frac{1}{b_2\lambda_2-t}-\frac{1}{b_1\lambda_1-t}\right]$$

$$M_z t=\frac{b_1b_2\ \lambda_1\lambda_2}{(b_1\lambda_1-t)\ (b_2\lambda_2-t)}$$

$$=\left(\frac{b_1\lambda_1-t}{b_1\lambda_1}\right)^{-1}\ \left(\frac{b_2\lambda_2-t}{b_2\lambda_2}\right)^{-1}$$

$$M_z t=\left(1-\frac{t}{b_1\lambda_1}\right)^{-1}\ \left(1-\frac{t}{b_2\lambda_2}\right)^{-1}$$

$$=\left[\left(1-\frac{t}{b_1\lambda_1}\right)\ \left(1-\frac{t}{b_2\lambda_{21}}\right)\right]^{-1}$$

It then follows from the above that the characteristics function of Z is$\phi_Z(t)=E(\ell^{1+Z})$:

$$\phi_z(t)=\left(1-\frac{it}{b_1\lambda_1}\right)^{-1}\ \left(1-\frac{it}{b_2\lambda_2}\right)^{-1} \tag{20}$$

From the above one may generalize that if x_1, 1 = 1, 2,... m are identically distributed random variables, each with convoluted beta-exponential distribution, the moment generating function $\phi_{sn}(t)$ of $S_n = X_1 + ... + X_m$ can be expressed as:

$$\phi_z(t) = \prod_{j=1}^{m}\left(1 - \frac{it}{b_j\lambda_j}\right)^{-1} \quad (21)$$

Moments: The moment generating function is found as:

$$M_z(t) = \left(1 - \frac{t}{b_1\lambda_1}\right)^{-1}\left(1 - \frac{t}{b_2\lambda_{21}}\right)^{-1} \quad (22)$$

By definition, the rth moment of the random variable Z is expressed as Table 1:

$$E(Z^r) = \frac{\delta^r}{\delta t^r} M_z(t)\Big|_{t=0}$$

$$M_2^1(t) = \frac{1}{b_2\lambda_2}\left(1 - \frac{t}{b_1\lambda_1}\right)^{-1}\left(1 - \frac{t}{b_2\lambda_2}\right)^{-2} + \frac{1}{b_1\lambda_1}\left(1 - \frac{t}{b_2\lambda_2}\right)^{-1}\left(1 - \frac{t}{b_1\lambda_1}\right)^{-2}$$

$$E(z) = M_2^1(0) = \frac{1}{b_2\lambda_2}\left(1 - \frac{0}{b_1\lambda_1}\right)^{-1}\left(1 - \frac{0}{b_2\lambda_2}\right)^{-2} + \frac{1}{b_1\lambda_1}\left(1 - \frac{0}{b_2\lambda_2}\right)^{-1}\left(1 - \frac{0}{b_2\lambda_2}\right)^{-2} = \frac{1}{b_2\lambda_2} + \frac{1}{b_1\lambda_1}$$

$$E(z) = \frac{b_1\lambda_1 + b_2\lambda_2}{b_1\lambda_1\, b_2\lambda_2} \quad (23)$$

The first four moments are:

$$E(z) = \frac{1}{b_1\lambda_1} + \frac{1}{b_2\lambda_2} = b_1^{-1}\lambda_1^{-1} + b_2^{-1}\lambda_2^{-1}$$

$$E(z^2) = 2\left[b_1^{-1}\lambda_1^{-1}b_2^{-1}\lambda_2^{-1} + b_1^{-2}\lambda_1^{-2}b_2^{-2}\lambda_2^{-2}\right]$$

$$E(z^3) = 6\left[b_1^{-2}\lambda_1^{-2}b_2^{-1}\lambda_2^{-1} + b_1^{-1}\lambda_1^{-1}b_2^{-2}\lambda_2^{-2} + b_1^{-3}\lambda_1^{-3} + b_2^{-3}\lambda_2^{-3}\right]$$

$$E(z^4) = 24\begin{bmatrix} b_1^{-2}\lambda_1^{-2}b_2^{-2}\lambda_2^{-2} + b_1^{-3}\lambda_1^{-3}b_2^{-1}\lambda_2^{-1} + \\ b_1^{-1}\lambda_1^{-1} + b_2^{-3}\lambda_2^{-3} + b_2^{-4}\lambda_2^{-4} + b_1^{-4}\lambda_1^{-4} \end{bmatrix}$$

Estimation of variance, skewness and kurtosis Table 2:

$$V(Z) = E(Z^2) - \left[E(Z)\right]^2$$

$$= 2\left[\frac{1}{b_1\lambda_1 b_2\lambda_2} + \frac{1}{b_1^2\lambda_1^2} + \frac{1}{b_2^2\lambda_2^2}\right] - \left(\frac{1}{b_1\lambda_1} + \frac{1}{b_2\lambda_2}\right)$$

$$V(Z) = \frac{2}{b_1\lambda_1 b_2\lambda_2} + \frac{2}{b_1^2\lambda_1^2} + \frac{2}{b_2^2\lambda_2^2}\left(\frac{1}{b_1^2\lambda_1^2} + \frac{2}{b_2^2\lambda_2^2} + \frac{2}{b_1\lambda_1 b_2\lambda_2}\right)$$

$$= \frac{1}{b_1^2\lambda_1^2} + \frac{1}{b_2^2\lambda_2^2} = \left(\frac{1}{b_1\lambda_1}\right)^2 + \left(\frac{1}{b_2\lambda_2}\right)^2 \quad (24)$$

The skewness and kurtosis will be obtained via the cumulant generating function.

Entropy and asymptotic behaviour: Entropy is a device used to quantify the uncertainty or randomizes in a system. It is defined as:

$$R(s) = \frac{1}{1-S}\ \mathrm{Log}\left[f_{(z)}^{s} dz\right], \text{ where } s>0 \text{ and } s \neq 1$$

As for the CBEP researchers have:

$$R(s) = \frac{1}{1-S}\log\left[\int_0^\infty \frac{b_1^s b_2^s \lambda_1^s \lambda_2^s}{(b_1\lambda_1 - b_2\lambda_2)^s}\left(\ell^{-b_2\lambda_2 z} - \ell^{-b_1\lambda_1 z}\right)^s dz\right]$$

$$= \frac{1}{1-s}\log\frac{b_1^s b_2^s \lambda_1^s \lambda_2^s}{(b_1\lambda_1 - b_2\lambda_2)^s}\int_0^\infty\left(\ell^{-b_2\lambda_2 z} - \ell^{-b_1\lambda_1 z}\right)^s dz$$

$$= \frac{1}{1-s}\left\{\log\left(\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2}\right)^s + \log\int_0^\infty\left(\ell^{-b_2\lambda_2 z} - \ell^{-b_1\lambda_1 z}\right)^s dz\right\}$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\left[\int_0^\infty\sum_{i=0}^{s}\binom{s}{i}(-1)^i\left(\ell^{-b_2\lambda_2 z}\right)^{s-i}\left(\ell^{-b_1\lambda_1 z}\right)\right]dz\right\}$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\left[\int_0^\infty\sum_{i=0}^{s}\binom{s}{i}(-1)^i\,\ell^{-b_2\lambda_2 z(s-i)z}\,\ell^{-b_1\lambda_1 iz}\right]dz\right\}$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\sum_{i=0}^{s}\binom{s}{i}\int_0^\infty(-1)^i\,\ell^{-(b_2\lambda_2(s-i)+b_1\lambda_2 1)z}\right\}dz$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\sum_{i=0}^{s}\binom{s}{i}\frac{\left(\ell^{-(b_2\lambda_2(s-1)+b_1\lambda_{11})z}\right)}{-(b_2\lambda_2(s-i)+b_1\lambda_1)}\Bigg|_0^\infty(-1)^{-i}\right.$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\sum_{i=0}^{s}(-1)^{-i+1}\binom{s}{i}\left[0 - \frac{1}{b_2\lambda_2(s-i)+b_1\lambda_1 i}\right]\right.$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\sum_{i=0}^{s}(-1)^i\binom{s}{i}\frac{1}{b_2\lambda_2 s - b_{2i} + b_1\lambda_1 i}\right\}$$

$$= \frac{1}{1-s}\left\{S\log\frac{b_1 b_2\lambda_1\lambda_2}{b_1\lambda_1 - b_2\lambda_2} + \log\sum_{i=0}^{s}\binom{s}{i}(-1)^i\frac{1}{b_2\lambda_2 s + (b_1\lambda_1 - b_2\lambda_2)i}\right\} \quad (25)$$

Asymptotic behaviours: Here, researchers investigate the asymptotic properties of the convoluted beta-exponential distribution. When $Z \to 0$ and $Z \to \infty$ what is the behaviour like:

Table 1: The mean of the CBED with different parameter values

Mean of Z (b_2)												
	$b_1 = 2$				$b_1 = 3$				$b_1 = 4$			
λ_2	1	2	3	4	1	2	3	4	1	2	3	4
$\lambda_1 = 1$												
1	1.50	1.00	0.83	0.75	1.33	0.83	0.67	0.58	1.25	0.75	0.58	0.50
2	1.00	0.75	0.67	0.63	0.83	0.58	0.50	0.46	0.75	0.50	0.42	0.38
3	0.83	0.67	0.61	0.58	0.67	0.50	0.44	0.42	0.58	0.42	0.36	0.33
4	0.75	0.63	0.58	0.56	0.58	0.46	0.42	0.40	0.50	0.38	0.33	0.31
$\lambda_1 = 2$												
1	1.25	0.75	0.58	0.50	1.17	0.67	0.50	0.42	1.13	0.63	0.46	0.38
2	0.75	0.50	0.42	0.38	0.67	0.42	0.33	0.29	0.63	0.38	0.29	0.25
3	0.58	0.42	0.36	0.33	0.50	0.33	0.28	0.25	0.46	0.29	0.24	0.21
4	0.50	0.38	0.33	0.31	0.42	0.29	0.25	0.23	0.38	0.25	0.21	0.19
$\lambda_1 = 3$												
1	1.17	0.67	0.50	0.42	1.11	0.61	0.44	0.36	1.08	0.58	0.42	0.33
2	0.67	0.42	0.33	0.29	0.61	0.36	0.28	0.24	0.58	0.33	0.25	0.21
3	0.50	0.33	0.28	0.25	0.44	0.28	0.22	0.19	0.42	0.25	0.19	0.17
4	0.42	0.29	0.25	0.23	0.36	0.24	0.19	0.17	0.33	0.21	0.17	0.15

Table 2: The variance of the CBED with different parameter values

Variance of Z (b_2)												
	$b_2 = 2$				$b_2 = 3$				$b_2 = 4$			
λ_2	1	2	3	4	1	2	3	4	1	2	3	4
$\lambda_1 = 1$												
1	1.25	0.50	0.36	0.31	1.11	0.36	0.22	0.17	1.06	0.31	0.17	0.13
2	0.50	0.31	0.28	0.27	0.36	0.17	0.14	0.13	0.31	0.13	0.09	0.08
3	0.36	0.28	0.26	0.26	0.22	0.14	0.12	0.12	0.17	0.09	0.07	0.07
4	0.31	0.27	0.26	0.25	0.17	0.13	0.12	0.12	0.13	0.08	0.07	0.07
$\lambda_1 = 2$												
1	1.06	0.31	0.17	0.13	1.03	0.28	0.14	0.09	1.02	0.27	0.13	0.08
2	0.31	0.13	0.09	0.08	0.28	0.09	0.06	0.04	0.27	0.08	0.04	0.03
3	0.17	0.09	0.07	0.07	0.14	0.06	0.04	0.03	0.13	0.04	0.03	0.02
4	0.13	0.08	0.07	0.07	0.09	0.04	0.03	0.03	0.08	0.03	0.02	0.02
$\lambda_1 = 3$												
1	1.03	0.28	0.14	0.09	1.01	0.26	0.12	0.07	1.01	0.26	0.12	0.07
2	0.28	0.09	0.06	0.04	0.26	0.07	0.04	0.03	0.26	0.07	0.03	0.02
3	0.14	0.06	0.04	0.03	0.12	0.04	0.02	0.02	0.12	0.03	0.02	0.01
4	0.09	0.04	0.03	0.03	0.07	0.03	0.02	0.02	0.07	0.02	0.01	0.01

$$\lim_{Z\to 0} f(z) = \operatorname*{lin}_{Z\to 0} \frac{b_1 b_2 \lambda_1 \lambda_2}{b_1\lambda_1 - b_2\lambda_2} \left(\ell^{-b_2\lambda_2 z} - \ell^{-b_1\lambda_1 z}\right)$$

$$\frac{b_1 b_2 \lambda_1 \lambda_2}{b_1\lambda_1 - b_2\lambda_2}\left(1-1\right) = 0$$

$$\lim_{Z\to\infty} f(z) = \lim_{Z\to\infty} \frac{b_1 b_2 \lambda_1 \lambda_2}{b_1\lambda_1 - b_2\lambda_2}\left(\ell^{-b_2\lambda_2(\infty)} - \ell^{-b_1\lambda_1(\infty)}\right) = 0$$

This affirms the unimodality of the distribution.

Maximum likelihood estimation of parameters: In this study, method of maximum likelihood estimation will be used. Let Z_1, Z_2,...Z_n be a random sample of n identically and independent distribution random variables each with the proposed distribution. Therefore, the likelihood function is found as:

$$L\left(Z, b_1, b_2, \lambda_1, \lambda_2\right) = \left(\frac{b_1 b_2 \lambda_1 \lambda_2}{b_1\lambda_1 - b_2\lambda_2}\right)^n \prod_{i=1}^{n}\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)$$

By taking the ln of both side:

$$\ln L\left(Z, b_1, b_2, \lambda_1, \lambda_2\right) = \text{In}\left(\frac{b_1 b_2 \lambda_1 \lambda_2}{b_1\lambda_1 - b_2\lambda_2}\right)^n + \text{In} \prod_{i=1}^{n}\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)$$

Taking the partial derivative with respect to b_1, b_2, λ_1 and λ_2 researchers have:

$$\frac{\delta L}{\delta b_1} = \frac{n\, b_2 \lambda_1 \lambda_2}{b_1 b_2 \lambda_1 \lambda_2} - \frac{n \lambda_1}{\left(b_1\lambda_1 - b_2\lambda_2\right)} + \frac{\sum_{i=1}^{n}\left(\lambda_1 Z_i\, \ell^{-b_1\lambda_1 Z_i}\right)}{\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}}$$

$$\frac{\delta L}{\delta b_1} = n\left[\frac{1}{b_1} - \frac{\lambda_1}{b_1\lambda_1 b_2\lambda_2}\right] + \sum_{i=1}^{n}\frac{Z_i\ \ell^{-b_1\lambda_1 Z_i}}{\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}}$$

$$= \frac{n\, b_2\lambda_1\lambda_2}{b_1 b_2\lambda_1\lambda_2} - \frac{n\lambda_1}{\left(b_1\lambda_1 - b_2\lambda_2\right)} + \frac{\sum_{i=1}^{n}\left(\lambda_1 Z_i\ \ell^{-b_1\lambda_1 Z_i}\right)}{\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}}\frac{n}{b_2} + \frac{n\lambda_2}{b_1\lambda_1 b_2\lambda_2} - \lambda_2\sum_{i=1}^{n}\frac{-Z_i\ \ell^{-b_2\lambda_2 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

$$\frac{\delta L}{\delta b_2} + n\left[\frac{1}{b_2} + \frac{\lambda_2}{b_1\lambda_1 b_2\lambda_2}\right] - \lambda_2\sum_{i=1}^{n}\frac{Z_i\ \ell^{-b_2\lambda_2 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

$$\frac{\delta L}{\delta \lambda_2} = \frac{n b_1 b_2\lambda_2}{b_1 b_2\lambda_1\lambda_2} - \frac{n b_1}{\left(b_1\lambda_1 - b_2\lambda_2\right)} + \sum_{i=1}^{n}\frac{b_1 Z_i\ \ell^{-b_1\lambda_1 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)} = n\left(\frac{1}{\lambda_1} - \frac{b_1}{b_1\lambda_2 - b_2\lambda_2}\right) + \sum_{i=1}^{n}\frac{Z_i\ \ell^{-b_1\lambda_1 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

$$\frac{\delta L}{\delta \lambda_1} = n\left[\left(\frac{1}{\lambda_1} - \frac{b_1}{b_1\lambda_1 - b_2\lambda_2}\right)\right] + b_1\sum_{i=1}^{n}\frac{Z_i\ \ell^{-b_1\lambda_1 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

$$\frac{\delta L}{\delta \lambda_2} = \frac{n b_1 b_2\lambda_1}{b_1 b_2\lambda_1\lambda_2} - \frac{n(-b_2)}{\left(b_1\lambda_1 - b_2\lambda_2\right)} + \sum_{i=1}^{n}\frac{-b_2 Z_i\ \ell^{-b_2\lambda_2 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)} = n\left[\frac{1}{\lambda_2} + \frac{b_2}{\left(b_1\lambda_1 b_2\lambda_2\right)}\right] - b_2\sum_{i=1}^{n}\frac{Z_i\ \ell^{-b_2\lambda_2 Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

The maximum likelihood estimates of b_1, b_2, λ_1 and λ_2 can be obtained by solving the resulting equations numerically.

Internal estimation and tests of hypotheses: For interval estimation of the parameters $\left(b_1, b_2, \lambda_1 \text{ and } \lambda_2\right)$ as well as the tests of hypotheses we use the Fisher information matrix to obtain:

$$n\begin{pmatrix} b_1, b_2, \\ \lambda_1, \lambda_2 \end{pmatrix} = \begin{bmatrix} -E\left(\frac{\delta L}{\delta^2 b_1}\right) - E\left(\frac{\delta^2 L}{\delta b_1 \delta b_2}\right) - E\left(\frac{\delta^2 L}{\delta b_1 \delta \lambda_1}\right) - E\left(\frac{\delta^2 L}{\delta b_1 \delta \lambda_2}\right) - E\left(\frac{\delta^2 L}{\delta b_1 \delta b_2}\right) - E\left(\frac{\delta^2 L}{\delta^2 b_2}\right) - E\left(\frac{\delta^2 L}{\delta b_2 \delta \lambda_1}\right) - E\left(\frac{\delta^2 L}{\delta b_2 \delta \lambda_2}\right) \\ -E\left(\frac{\delta^2 L}{\delta b_1 \delta \lambda_1}\right) - E\left(\frac{\delta^2 L}{\delta b_2 \delta \lambda_1}\right) - E\left(\frac{\delta^2 L}{\delta^2 \lambda_1}\right) - E\left(\frac{\delta^2 L}{\delta \lambda_1 \delta \lambda_2}\right) - E\left(\frac{\delta^2 L}{\delta b_1 \delta \lambda_2}\right) - E\left(\frac{\delta^2 L}{\delta b_2 \delta \lambda_2}\right) - E\left(\frac{\delta^2 L}{\delta \lambda_1 \delta \lambda_2}\right) - E\left(\frac{\delta^2 L}{\delta^2 \lambda_2}\right) \end{bmatrix} \tag{26}$$

Elements in this matrix can be obtained by connecting from the first derivatives earlier obtained:

$$\frac{\delta^2 L}{\delta b_1} = -n b_1^{-2} - n\lambda_1^2\left(b_1\lambda_1 - b_2\lambda_2\right)^{-2} - \lambda_1^2\sum_{i=1}^{n}\frac{Z_i^2\ell^{-\left(b_1\lambda_1 + b_2\lambda_2 Z_i\right)}}{\left(\ell^{-(b_2)_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

$$\frac{\delta^2 L}{\delta b_2} = -n b_2^{-2} - n\lambda_2^2\left(b_1\lambda_1 - b_2\lambda_2\right)^{-2} - \lambda_2^2\sum_{i=1}^{n}\frac{Z_i^2\ell^{-\left(b_1\lambda_1 + b_2\lambda_2\right)Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)^2}$$

$$\frac{\delta^2 L}{\delta \lambda_1} = -n\lambda_1^{-2} - n b_1^2\left(b_1\lambda_1 - b_2\lambda_2\right) - b_1^2\sum_{i=1}^{n}\frac{Z_i^2\ell^{\left(b_1\lambda_1 + b_2\lambda_2\right)Z_i}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

$$\frac{\delta^2 L}{\delta \lambda_2} = -n\lambda_2^{-2} + n b_2^2\left(b_1\lambda_1 - b_2\lambda_2\right)^{-2} - b_2^2\sum_{i=1}^{n}\frac{Z_i^2\ell^{-\left(b_1\lambda_1 + b_2\lambda_2\right)Z_i}}{\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}}$$

$$\frac{\delta^2 L}{\delta b_1 b_2} = -n\lambda_1\lambda_2\left(b_1\lambda_1 b_2 b_2\right)^{-2} - \lambda_1\lambda_2\sum_{i=1}^{n}\frac{Z_i^1\ell^{-\left(b_1\lambda_1 + b_2\lambda_2 Z_i\right)}}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)^2}$$

$$\frac{\delta^2 L}{\delta b_1 \delta \lambda_1} n\left(b_1\lambda_1 - b_2 b_2\right)^{-1} + b_1\lambda_1\frac{\left[\sum_{i=1}^{n} Z_i\ \ell^{-2b_1\lambda_1 Z_i} - \sum_{i=1}^{n} Z_i^2\ \ell^{-b_1\lambda_1 Z_i}\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)^2 + \sum_{i=1}^{n} Z_i\ \ell^{-b_1\lambda_1 Z_i}\right]}{\left(\ell^{-b_2\lambda_2 Z_i} - \ell^{-b_1\lambda_1 Z_i}\right)}$$

Cumulant generating function: The cumulant generating function for the distribution is:

$$K_z(t) = \ln\left(1-\frac{t}{b_1\lambda_1}\right)^{-1}\left(1-\frac{t}{b_2\lambda_2}\right)^{-1} = \sum_{n=1}^{\infty}(n-1)!\left[\left(\frac{1}{b_1\lambda_1}\right)^n+\left(\frac{1}{b_2\lambda_2}\right)^n\right]\frac{t^n}{n!}$$

Note that cumulant K_n are the coefficient of $t^n/n_!$:

$$K_1 = \frac{1}{b_1\lambda_1}+\frac{1}{b_2\lambda_2},\ n=1;\ K_2 = \left(\frac{1}{b_1\lambda_1}\right)^2+\left(\frac{1}{b_2\lambda_2}\right)^2,\ n=2;$$

$$K_3 = 2\left[\left(\frac{1}{b_1\lambda_1}\right)^3+\left(\frac{1}{b_2\lambda_2}\right)^3\right],\ n=3;\ K_4 = 6\left[\left(\frac{1}{b_1\lambda_1}\right)^4+\left(\frac{1}{b_2\lambda_2}\right)^4\right],\ n=4$$

Skewness and kurtosis: The skewness and Kurtosis will be obtained using the first four cumulant Table 3 and 4.

$$\text{Skewness} = \frac{\mu_4}{\mu_2^2} = \frac{K_4+3K_2^2}{K_2^2};\ \text{Kurtosis} = \frac{K_4}{K_2^2}$$

Table 3: The skewness of the CBED with different parameter values

Skewness of z (b_2)

	$b_1 = 2$				$b_1 = 3$				$b_1 = 4$			
λ_2	1	2	3	4	1	2	3	4	1	2	3	4
λ_1 = 1												
1	14.76	9.16	17.95	11.07	10.40	16.29	16.41	17.26	11.18	12.09	17.62	16.84
2	18.00	9.43	17.50	11.27	10.52	16.24	16.49	17.22	11.15	12.09	17.63	16.83
3	16.67	10.09	17.92	11.45	10.50	16.13	16.52	17.18	11.13	12.10	17.63	16.83
4	14.76	11.29	17.83	11.98	10.64	15.92	16.68	17.06	11.07	12.10	17.65	16.83
λ_1 = 2												
1	10.99	9.29	17.51	11.21	10.50	16.26	16.47	17.23	11.16	12.09	17.62	16.83
2	14.76	9.64	18.00	11.25	10.44	16.21	16.46	17.22	11.16	12.09	17.62	16.84
3	17.29	10.02	17.83	11.44	10.51	16.14	16.53	17.18	11.13	12.10	17.63	16.83
4	18.00	10.61	17.87	11.68	10.56	16.03	16.60	17.13	11.11	12.10	17.64	16.83
λ_1 = 3												
1	9.95	9.36	17.01	11.35	10.62	16.26	16.55	17.21	11.14	12.09	17.63	16.83
2	12.24	9.91	17.67	11.43	10.53	16.16	16.53	17.18	11.13	12.09	17.63	16.83
3	14.76	10.37	17.98	11.55	10.51	16.08	16.55	17.15	11.12	12.10	17.64	16.83
4	14.76	9.16	17.95	11.07	10.40	16.29	16.41	17.26	11.18	12.09	17.62	16.84

Table 4: The kurtosis of the CBED with different parameter values

Kurtosis of z (b_2)

	$b_1 = 2$				$b_1 = 3$				$b_1 = 4$			
λ_2	1	2	3	4	1	2	3	4	1	2	3	4
λ_1 = 1												
1	20.40	24.00	44.77	81.60	49.20	44.77	54.00	80.88	90.71	408.00	80.88	96.00
2	24.00	81.60	196.80	362.82	44.77	80.88	183.60	343.32	81.60	192.00	179.08	326.40
3	44.77	196.80	464.26	841.30	54.00	183.60	442.80	816.35	80.88	258.67	421.67	787.20
4	81.60	362.82	841.30	1512.74	80.88	343.32	816.35	1485.67	96.00	408.00	787.20	1451.29
λ_1 = 2												
1	90.71	81.60	80.88	96.00	210.32	196.80	183.60	179.08	378.18	6168.00	343.32	326.40
2	81.60	96.00	179.08	326.40	196.80	179.08	216.00	323.52	362.82	1632.00	323.52	384.00
3	80.88	179.08	421.67	787.20	183.60	216.00	402.92	734.40	343.32	898.67	440.98	716.31
4	96.00	326.40	787.20	1451.29	179.08	323.52	734.40	1373.26	326.40	768.00	716.31	1305.60
λ_1 = 3												
1	210.32	196.80	183.60	179.08	480.15	464.26	442.80	421.67	858.08	31128.00	816.35	787.20
2	196.80	179.08	216.00	323.52	464.26	421.67	402.92	440.98	841.30	7872.00	734.40	716.31
3	183.60	216.00	402.92	734.40	442.80	402.92	486.00	727.92	816.35	3672.00	727.92	864.00
4	179.08	323.52	734.40	1373.26	421.67	440.98	727.92	1283.63	787.20	2328.00	864.00	1294.08

$$\text{Skewness} = \frac{6\left[\left(\frac{1}{b_1\lambda_1}\right)^4 + \left(\frac{1}{b_2\lambda_2}\right)^4\right] + 3\left(\frac{1}{b_1\lambda_1}\right)^2 + \left(\frac{1}{b_2\lambda_2}\right)^2}{\left[\left(\frac{1}{b_1\lambda_1}\right)^2 + \left(\frac{1}{b_2\lambda_2}\right)^2\right]^2}$$

$$= \frac{9\left(\frac{1}{b_1\lambda_1}\right)^4 + 9\left(\frac{1}{b_2\lambda_2}\right)^4 + 6\left(\frac{1}{b_1\lambda_1}\right)^2\left(\frac{1}{b_2\lambda_2}\right)^2}{\left(\frac{1}{b_1\lambda_1}\right)^4 + \left(\frac{1}{b_2\lambda_2}\right)^4 + 2\left(\frac{1}{b_1\lambda_1}\right)^2\left(\frac{1}{b_2\lambda_2}\right)^2}$$

$$\text{Kurtosis} = \frac{6\left[\left(\frac{1}{b_1\lambda_1}\right)^4 + \left(\frac{1}{b_2\lambda_2}\right)^4\right]}{\left[2\left(\frac{1}{b_1\lambda_1}\right)^2 + \left(\frac{1}{b_2\lambda_2}\right)^2\right]^2}$$

$$= \frac{6\left[\left(b_2\lambda_2\right)^4 + \left(b_1\lambda_1\right)^4\right]}{\left[\left(b_1\lambda_1\right)^2 + \left(b_2\lambda_2\right)^2\right]^2}$$

CONCLUSION

In this study, researchers provided the mathematical treatment of the properties of the Convoluted Beta-Exponential Distribution (CBED) proposed by Nadarajah and Kotz (2006). The properties obtained among others include moment, moment and characteristic generating function, the entropy, the hazard function and the cumulant generating function. The estimation of parameters is approached by the method of maximum likelihood. The properties were compared with that of the Convoluted Beta-Weibull Distribution (CBED) introduced by Sun (2011). From Table 1 and Table 2, the mean of the CBEP decreases when λ_1 and λ_2 increase while the mean of the CBWD deceases as the b_1 and b_2 increases.

The computed variances of CBED are all less than that of CBWD except at $\lambda_1 = 1$. The skewness of CBED are larger than that of CBWD. This implies a fatter tail of the former than the latter.

Similarly in Table 4, CBED possesses larger kurtosis value than the CBWD. This confirms the leptokurtic tendency of the distribution. These findings establishes that like the exponential power distribution of Gray and French (1990) and in line with the findings of Peters (1991) the CBED which displays high skewness with fat tails and leptokurtic provides a reasonably good fit for stock return data.

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